1. *Calculus for variational derivatives* (Exercise 1, section 2.6, page 31)

   Functionals that map functions to real numbers can be added and multiplied.
   By applying definitions 17, 18, and 25,

   (a) verify that the following rules of calculation hold for the first variation:

   $$\delta(\mathcal{L}_1 + \mathcal{L}_2) = \delta\mathcal{L}_1 + \delta\mathcal{L}_2\,,$$
   $$\delta(\mathcal{L}_1 \cdot \mathcal{L}_2) = \mathcal{L}_2\delta\mathcal{L}_1 + \mathcal{L}_1\delta\mathcal{L}_2\,,$$
   $$\delta\frac{\mathcal{L}_1}{\mathcal{L}_2} = \frac{\mathcal{L}_2\delta\mathcal{L}_1 - \mathcal{L}_1\delta\mathcal{L}_2}{{\mathcal{L}_2}^2}\,,$$
   $$\delta g(\mathcal{L}) = g'(\mathcal{L})\delta\mathcal{L}\,, \quad \text{for functions} \quad g:\mathbb{R}\to\mathbb{R}\,.$$

   (b) Show that the above rules apply for the variational derivative as well, and

   (c) derive the corresponding expressions for the second variation.

2. *Variational formulation of boundary value problems* (Exercise 4, section 2.6, page 33)

   Find the variational formulation of each of the following boundary value problems:

   (a) $-u_{xx} = \sin(u) + \mathrm{e}^x u^2, \quad u(0) = 0, \quad u_x(1) = 7,$

   (b) $-\frac{1}{r}\partial_r(r\partial_r u) = f(r), \quad u_r(0) = 0, \quad u(1) = 0,$

   (c) $-\mathrm{div}\,(\sigma(x,y)\boldsymbol{\nabla} u(x,y)) + u(x,y) = 0, \quad u(x,0) = u(x,1) = 0, \quad u_x(0,y) = u_x(1,y) = 0.$

   To some degree this is a matter of trial and error: Try a reasonable functional (look for inspirations in the lecture notes), derive the corresponding Euler-Lagrange equation, and modify the functional, if you do not arrive at the required differential equation. Show that the functions at which your functionals become stationary (the critical points) satisfy the required boundary conditions; it may be necessary to consider additional boundary functionals for the variational formulations.

3. *Energy functional with general boundary conditions* (Exercise 35, page 40)

   Consider the variational problem

   $$\mathrm{Crit}\Big\{ \int_\Omega \big(\frac{1}{2}|\boldsymbol{\nabla}\phi|^2 - \rho\,\phi\big)\,\mathrm{d}^3x - \int_{\partial\Omega_2} \psi_2\,\phi\,\mathrm{d}^2x \;\Big|\; \phi(\boldsymbol{x}) = \psi_1(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\partial\Omega_1 \Big\}.$$

   Here the boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ of the domain $\Omega \subset \mathbb{R}^3$ is split into the two parts $\partial\Omega_1$ and $\partial\Omega_2$, with $\psi_1$ a given function on $\partial\Omega_1$ and $\psi_2$ a given function on $\partial\Omega_2$. The function $\rho$ is defined inside $\Omega$ (see pages 39, 40 of the lecture notes for an interpretation of these quantities in terms of electrostatics).

   (a) Show that a critical point of the functional satisfies the following equations:

   $$-\Delta\phi(\boldsymbol{x}) = \rho(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\Omega, \quad \phi(\boldsymbol{x}) = \psi_1(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\partial\Omega_1, \quad \partial_n\phi(\boldsymbol{x}) = \psi_2(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\partial\Omega_2.$$

   (b) Show that there exists at most one critical point, and that, if it exists, it is in fact a minimizer.

   (c) When $\partial\Omega_1$ is empty, a solution exists only if a certain relation between $\rho$ and $\psi_2$ is satisfied. Derive this condition (integrate the Euler-Lagrange equation over the domain).

*Solutions are to be handed in until Wednesday, May 18, 09:00.*