1. *Spherical Pendulum*
   $\approx$ exercise 61, page 73 of the lecture notes.

   Consider a spherical pendulum: A mass at the end of a weightless bar of length l, allowed to move freely under the influence of gravitation. The position $\boldsymbol{r} = l(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, 1-\cos\theta)$ of the mass is prescribed in spherical coordinates, where the 3-axis is oriented in the vertical direction, the angle θ gives the tilt of the bar with respect to the vertical direction, and ϕ is the azimuth angle with respect to the vertical axis. Then the kinetic energy T and the potential energy V of the pendulum are given by

   $$T = \frac{1}{2} m\dot{\boldsymbol{r}}^2, \quad V = mgl(1-\cos\theta)\,,$$

   with g being the acceleration due to gravity and the dot indicating the time derivative.

   (a) Draw a sketch to clarify the geometry.

   (b) The pendulum is a finite dimensional Lagrangian system with Lagrange function $L = T - V$ and generalized coordinates θ, ϕ. Write L in terms of θ, ϕ, $\dot{\theta}$, $\dot{\phi}$ and derive the equations of motion for the pendulum.

   (c) Alternatively, the pendulum can be described as a finite dimensional Hamiltonian system with Hamilton function

   $$H(\theta,\phi,p_\theta,p_\phi) = \frac{1}{2ml^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mgl(1-\cos\theta)\,.$$

   Show that this viewpoint leads to the same equations of motion.

   (d) Show that time independent (equilibrium) solutions are given as $\mathrm{Crit}\{H(\boldsymbol{u})\}$, where $\boldsymbol{u}$ collects the four arguments of H. Derive these solutions and give a physical interpretation.

   (e) Observe that, except from H (which is conserved for the autonomous Hamiltonian system), there is another quantity I that is conserved in time, an additional first integral, the angular momentum related to the azimuth angle. Show this in both the Lagrangian and the Hamiltonian description.

   (f) Reduce the dynamics by prescribing a value γ for I, i.e. substitute the constant into the equations of motion (then one of the angles disappears from the equations), and state the equation for equilibrium solutions of the reduced dynamics (these are time independent for the angle that remains in the equations).

   (g) Show that the equilibria of the reduced dynamics are relative equilibria, i.e. constraint critical points of $H(\theta,\phi,p_\theta,p_\phi)$ at given $I(\theta,\phi,p_\theta,p_\phi) = \gamma$ (viewed as a finite dimensional stationarity problem).

   (h) Give a physical interpretation for the motion of the relative equilibrium solutions in space. Relate the angular velocity to the derivative of the value function of the constraint variational problem above.

2. *Integral operator as inverse of a differential operator*
$\approx$ exercise 72, page 85 of the lecture notes.

Consider the boundary value problem

$$-u_{xx} = f\,, \quad u(0) = u(\pi) = 0\,,$$

that can be viewed as the operator equation

$$Lu = f\,, \quad \text{with} \quad Lu := -u_{xx}\,, \quad u \in \mathcal{U}_0 := \left\{ u \in C^2([0,\pi]) \,\middle|\, u(0) = u(\pi) = 0 \right\}.$$

(a) One can solve this equation in an elementary way, the solution is given by $u = Kf$, where K is the integral operator defined by

$$(Kv)(x) := \int_0^\pi g(x,y)\,v(y)\,\mathrm{d}y\,, \text{ with } g(x,y) = \begin{cases} (1-x/\pi)y & \text{for } 0 \le y \le x, \\ (1-y/\pi)x & \text{for } x \le y \le \pi, \end{cases} \quad x \in [0,\pi]\,.$$

Verify by a direct calculation that this is the solution.

(b) Compute the eigenvalues λ and the normalized eigenfunctions φ of L, i.e. nonvanishing solutions of the problem $L\varphi = \lambda\varphi$ with $\varphi \in \mathcal{U}_0$ and $\int_0^\pi \varphi^2(x)\,\mathrm{d}x = 1$. Show that the eigenfunctions of L are also eigenfunctions of K with eigenvalues $1/\lambda$, i.e.

$$L\varphi = \lambda\varphi \quad \Rightarrow \quad K\varphi = \mu\varphi \quad \text{for} \quad \mu = 1/\lambda\,.$$

(c) Observe that a Fourier-sine expansion of a function $v \in \mathcal{U}_0$ can be viewed as an expansion of v into eigenfunctions of L and K, respectively. State the effect of L and K on v if v is represented in this way. Assuming that all summations converge as necessary: What is the effect of L^n and K^n on v (where the natural exponent indicates repeated application)? How do you consequently define the operator $F(L)$ for a function F that is represented by a Taylor series expansion?

3. *Eigenfrequencies of a square membrane*
Page 107 of the lecture notes, section 6.5, exercise 3.

Restrict the exercise to parts (a) and (b).

Solutions are to be handed in until Wednesday, June 15, 09:00.