1. **Plane harmonic electromagnetic waves**

Consider an electromagnetic wave with an electric field of the form

\[ E(r, t) = E_0 e^{i(k \cdot r - \omega t)} , \]

with frequency \( \omega \), wave vector \( k \), and amplitude vector \( E_0 \) (remember the conventions for complex notation of time harmonic fields). The wave is assumed to propagate through a homogeneous linear medium without free charges or currents, characterized by the relative permittivity \( \varepsilon \) and relative permeability \( \mu \).

(a) Write out the remaining parts \( D, B, H \) of the electromagnetic field in terms of the quantities introduced above. State conditions such that all Maxwell equations are satisfied.

(b) Show that, for this wave, the time averaged energy density \( \bar{\tau} \) and the time-averaged Poynting vector \( \mathbf{S} \) take the following alternative forms:

\[ \bar{\tau} = \frac{1}{2} \varepsilon_0 \varepsilon |E_0|^2 = \frac{1}{2} \mu_0 \mu |H_0|^2 , \]

\[ \mathbf{S} = \frac{1}{2} \sqrt{\frac{\varepsilon_0 \varepsilon}{\mu_0 \mu}} |E_0|^2 \frac{k}{k} = \frac{1}{2} \sqrt{\frac{\mu_0 \mu}{\varepsilon_0 \varepsilon}} |H_0|^2 \frac{k}{k} = \frac{\varepsilon_m}{\mu} \frac{k}{k} . \]

Here \( H_0 \) is the amplitude vector associated with the magnetic part of the wave field.

2. **Polarization**

Consider plane harmonic electromagnetic waves with frequency \( \omega \) that propagate with wavenumber \( k \) along the Cartesian \( z \)-axis. Specifically, we look at waves with electric fields of the form

\[ E(r, t) = E_{0x} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{i(kz - \omega t)} , \]  \hspace{1cm} (1)

\[ E(r, t) = E_{0y} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{i(kz - \omega t)} , \]  \hspace{1cm} (2)

\[ E(r, t) = E_{1} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{i(kz - \omega t)} , \]  \hspace{1cm} (3)

\[ E(r, t) = E_{t} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{i(kz - \omega t)} . \]  \hspace{1cm} (4)

Here \( E_{0x}, E_{0y}, E_{1}, \) and \( E_{t} \) are constant complex wave amplitudes.

(a) Write out the physical electric field for each of the waves, using the conventions for the complex notation of time-harmonic fields. Where convenient, split the complex amplitudes \( E_0 = |E_0| \exp(i\varphi) \) into their absolute value \( |E_0| \) and argument \( \varphi \).

(b) Draw rough schematics of these physical fields, for each of the three cases (1), (2), (3), (4):

i. a snapshot at time \( t_0 \) of the electric components \( E_x(z, t_0), E_y(z, t_0) \) vs. \( z \). Mark the wavelength \( \lambda = 2\pi/k \) in your plots.

ii. a plot of the electric components \( E_x(z_0, t), E_y(z_0, t) \) vs. time \( t \), for fixed position \( z_0 \). Mark the period \( T = 2\pi/\omega \) in your plots.

iii. the trace of \( E(z_0, t) \) in the \( E_x - E_y \)-plane with time for fixed position \( z_0 \).

iv. the trace of \( E(z, t_0) \) in the \( E_x - E_y \)-plane with position for fixed time \( t_0 \).

Choose convenient values for \( t_0, z_0, \) and for the absolute values \( |E_0| \) and arguments \( \varphi \) of the wave amplitudes. You might wish to combine different cases into single drawings.
(c) Show that a wave with general polarization

\[ E(r, t) = \begin{pmatrix} E_{0x} \\ E_{0y} \\ 0 \end{pmatrix} e^{i(kz - \omega t)}, \]

(5)
can always be written

i. as the sum of two orthogonal linearly polarized waves, here an \( x \)-polarized wave \( \text{\textbullet\textcircled{1}} \) and a \( y \)-polarized wave \( \text{\textbullet\textcircled{2}} \),

ii. as the sum of a left circularly polarized wave \( \text{\textbullet\textcircled{3}} \) and a right circularly polarized wave \( \text{\textbullet\textcircled{4}} \).

3. **Spherical waves**

Show that the spherically symmetric field

\[ \psi(r, t) = \frac{1}{r} a(kr - \omega t) + \frac{1}{r} b(kr + \omega t) \]
solves the scalar wave equation

\[ \left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(r, t) = 0 \]

for arbitrary (smooth) functions \( a \) and \( b \) of one variable, provided that the relation \( \omega^2 = k^2 c^2 \) between frequency, wavenumber, and phase velocity is satisfied. Use the representation of the Laplace-operator in spherical coordinates.

*Hand in your solutions until Wednesday, October 02, 10:45. Good luck!*