

# *Optical Waveguide Theory (B)*



Manfred Hammer\*

Theoretical Electrical Engineering  
Paderborn University, Germany

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*\* Theoretical Electrical Engineering, Paderborn University  
Warburger Straße 100, 33098 Paderborn, Germany*

*Phone: +49(0)5251/60-3560  
E-mail: manfred.hammer@uni-paderborn.de*

### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B [Brush up on mathematical tools.](#)
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

## ***Vector calculus, keywords***

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Ingredients:

(here: Cartesian coordinates)

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- Time derivatives:  $\frac{\partial \phi}{\partial t}, \partial_t \phi, \dot{\phi}, \nabla_T \phi$ .

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- Laplacian:  $\Delta = \nabla \cdot \nabla = \nabla^2$ ,

$$\Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi, \quad \Delta\mathbf{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$$

## ***Dirac delta***

---

A linear functional  
that extracts the value of a function at one point:



$$\mathbf{1-D:} \quad \int_a^b f(x) \delta(x - x_0) dx = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

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$$3\text{-D: } \int_{\mathcal{V}} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) d\mathcal{V} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

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Implications: manifold.

## ***Fourier transform, 1-D***

---

**1-D:** A function  $f(x) \in \mathbb{C}$  of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

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- $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$ .

## ***Fourier transform***

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3-D: A field  $\phi(\mathbf{r})$ :

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4-D: A field  $\phi(\mathbf{r}, t)$ :

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iint \tilde{\phi}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3k d\omega,$$
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## Directionally constant systems

A **linear** PDE in two unknowns

$$(A \partial_{xx} + B \partial_{yy} + C \partial_{xy} + D \partial_x + E \partial_y + F) \psi(x, y) = 0,$$

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If the system is **constant in  $x$** ,  $\partial_x A = \dots = \partial_x F = 0$ ,

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↪  $\int (B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) e^{ikx} dk = 0,$

↪  $(B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) = 0, \text{ (for all } k),$

... a set of DEs in one unknown.

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(& boundary conditions, ...)

## **General solution of the wave equation**

$$\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\mathbf{r}, t) = 0, \quad \psi(\mathbf{r}, 0) = \psi_0(\mathbf{r}), \quad \partial_t \psi(\mathbf{r}, 0) = \phi_0(\mathbf{r}),$$

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## ***A touch of variational calculus***

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- **Functional:** 
$$\begin{aligned}\mathcal{L} : U &\longrightarrow \mathbb{R}, \mathbb{C}, \\ u &\longrightarrow \mathcal{L}(u),\end{aligned}$$

a map from a space  $U$  of functions to real / complex numbers.

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$$\left. \frac{d}{ds} \mathcal{L}(u + s v) \right|_{s=0} = 0 \quad \text{for all } v,$$

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- **Restriction of a functional:**

... to a parametrized family of functions;

↔ extremization with respect to these parameters,

↔ approximations of stationary points of the functional.

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*Example:*

$$U = \{u : [0, \pi] \rightarrow \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \rightarrow \mathbb{R},$$

( ... )

$$\mathcal{L}(u) = \frac{\int_0^\pi (\partial_x u)^2 dx}{\int_0^\pi u^2 dx}.$$

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$u$  satisfies DE & b.c.,

$$\begin{aligned} \partial_x^2 u &= -\lambda u, \quad \lambda = \mathcal{L}(u), \\ u(0) &= u(\pi) = 0. \end{aligned}$$



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$\downarrow$  Restrict  $\mathcal{L}$ ,  $L(\mathbf{a}) = \mathcal{L}(u|\mathbf{a})$ .

$$L \text{ stationary at } \mathbf{a}, \quad \nabla_{\mathbf{a}} L = 0. \quad \longleftrightarrow$$

$$\mathcal{L}(u) = \frac{\int_0^\pi (\partial_x u)^2 dx}{\int_0^\pi u^2 dx}.$$

$u$  satisfies DE & b.c.,

$$\begin{aligned} \partial_x^2 u &= -\lambda u, \quad \lambda = \mathcal{L}(u), \\ u(0) &= u(\pi) = 0. \end{aligned}$$

Approximate solution  
of DE / eigenproblem.

## Upcoming

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Next lectures:

- Maxwell equations, different formulations, interfaces, energy and power flow.
- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.

