

# *Optical Waveguide Theory (C)*



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### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

...?

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“This concerns time harmonic fields ... with angular frequency ..., for vacuum wavenumber ..., speed of light ..., and wavelength ...”

“The problem is governed by the Maxwell curl equations in the frequency domain for the electric field ... and magnetic field ..., for (lossless) uncharged dielectric, nonmagnetic linear (isotropic) media with (piecewise constant) relative permittivity ... :

...

(.)”

[ M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016) ]

## Maxwell equations, Fourier transform

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}$$

$$\& \quad \mathbf{F}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\mathbf{F}}(\mathbf{r}, \omega) e^{i\omega t} d\omega, \quad \tilde{\mathbf{F}}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int \mathbf{F}(\mathbf{r}, t) e^{-i\omega t} dt$$

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$$\mathbf{E}(\mathbf{r}, t), \mathbf{D}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t), \rho_f(\mathbf{r}, t), \mathbf{J}_f(\mathbf{r}, t)$$

$$\longleftrightarrow \tilde{\mathbf{E}}(\mathbf{r}, \omega), \tilde{\mathbf{D}}(\mathbf{r}, \omega), \tilde{\mathbf{B}}(\mathbf{r}, \omega), \tilde{\mathbf{H}}(\mathbf{r}, \omega), \tilde{\rho}_f(\mathbf{r}, \omega), \tilde{\mathbf{J}}_f(\mathbf{r}, \omega),$$

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_f, \quad \nabla \times \tilde{\mathbf{E}} = -i\omega \tilde{\mathbf{B}}, \quad \nabla \cdot \tilde{\mathbf{B}} = 0, \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}_f + i\omega \tilde{\mathbf{D}}$$

(Caution: arbitrary choice of  $\sim \exp(\pm i\omega t)$ !).

## Maxwell equations, frequency domain

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$$\mathbf{F}(\mathbf{r}, t) \in \mathbb{R} \quad \rightsquigarrow \quad \tilde{\mathbf{F}}(\mathbf{r}, -\omega) = (\tilde{\mathbf{F}}(\mathbf{r}, \omega))^*$$

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“at frequency  $\omega_0$ ”: 
$$\tilde{\mathbf{F}}(\mathbf{r}, \omega) = \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}(\mathbf{r}) \delta(\omega - \omega_0) + \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}^*(\mathbf{r}) \delta(\omega + \omega_0)$$

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↪  $\mathbf{F}(\mathbf{r}, t) = \frac{1}{2} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} + \bar{\mathbf{F}}^*(\mathbf{r}) e^{-i\omega_0 t} \right\},$

$$\mathbf{F}(\mathbf{r}, t) = \text{Re} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} \right\},$$

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## Maxwell equations, frequency domain

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$$\hookrightarrow \bar{\mathbf{E}}(\mathbf{r}), \bar{\mathbf{D}}(\mathbf{r}), \bar{\mathbf{B}}(\mathbf{r}), \bar{\mathbf{H}}(\mathbf{r}), \tilde{\rho}_f(\mathbf{r}), \bar{\mathbf{J}}_f(\mathbf{r}), \sim \exp(i\omega_0 t),$$

$$\nabla \cdot \bar{\mathbf{D}} = \bar{\rho}_f, \quad \nabla \times \bar{\mathbf{E}} = -i\omega_0 \bar{\mathbf{B}}, \quad \nabla \cdot \bar{\mathbf{B}} = 0, \quad \nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}}_f + i\omega_0 \bar{\mathbf{D}}.$$

Caution: Decorations  $\bar{\cdot}, \tilde{\cdot}, \_0$  are usually omitted; context determines interpretation of symbols.

## ***Polarization***

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$\tilde{\mathbf{P}}$ : density of electric dipole moment (bound charges).

$$\tilde{\mathbf{D}} = \epsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}, \quad [\tilde{\mathbf{D}}] = [\tilde{\mathbf{P}}] = \frac{\text{As m}}{\text{m}^3}, \quad [\tilde{\mathbf{E}}] = \frac{\text{V}}{\text{m}},$$

▶ vacuum permittivity  $\epsilon_0 = 8.854187817 \dots \cdot 10^{-12} \left[ \frac{\text{F}}{\text{m}} = \frac{\text{As}}{\text{Vm}} \right]$ .


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• Local dipoles induced by  $\tilde{\mathbf{E}}$    $\tilde{\mathbf{P}}(\tilde{\mathbf{E}})$ .

• Linear dielectrics:

$$\tilde{\mathbf{P}} = \epsilon_0 \hat{\chi}_e \tilde{\mathbf{E}}, \quad \hat{\chi}_e: \text{dielectric susceptibility}, \quad [\hat{\chi}_e] = \hat{1}.$$

  $\tilde{\mathbf{D}} = \epsilon_0 (\hat{1} + \hat{\chi}_e) \tilde{\mathbf{E}} = \epsilon_0 \hat{\epsilon} \tilde{\mathbf{E}}, \quad \hat{\epsilon}: \text{relative permittivity}, \quad [\hat{\epsilon}] = \hat{1}.$


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- $\hat{\chi}_e(\mathbf{r}, \omega)$ ,  $\hat{\epsilon}(\mathbf{r}, \omega)$  are determined in the frequency domain.
- Complications:  $\text{Im } \epsilon$ ,  $\hat{\epsilon}(T)$ ,  $\hat{\epsilon}(\mathbf{F})$ ,  $\chi_{jkl}^{(2)} E_k E_l$ ,  $\chi_{jklm}^{(3)} E_k E_l E_m, \dots$
- Simpler cases:  $\hat{\epsilon}(\mathbf{r})$ ,  $\hat{\epsilon} = \epsilon \hat{1}$ .

## Magnetization

$\tilde{\mathbf{M}}$ : density of magnetic dipole moments (bound currents).

$$\tilde{\mathbf{H}} = \frac{1}{\mu_0} \tilde{\mathbf{B}} - \tilde{\mathbf{M}}, \quad [\tilde{\mathbf{H}}] = [\mathbf{M}] = \frac{\text{A m}^2}{\text{m}^3}, \quad [\tilde{\mathbf{B}}] = \text{T} = \frac{\text{Vs}}{\text{m}^2},$$


▶ vacuum permeability  $\mu_0 = 4\pi \cdot 10^{-7} \left[ \frac{\text{N}}{\text{A}^2} = \frac{\text{Vs}}{\text{Am}} \right]$ .

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• Linear magnetic media:

$\tilde{\mathbf{M}} = \hat{\chi}_m \tilde{\mathbf{H}}, \quad \hat{\chi}_m$ : magnetic susceptibility,  $[\hat{\chi}_m] = \hat{1}$ .  
↪  $\tilde{\mathbf{B}} = \mu_0(\hat{1} + \hat{\chi}_m)\tilde{\mathbf{H}} = \mu_0\hat{\mu}\tilde{\mathbf{H}}, \quad \hat{\mu}$ : relative permeability,  $[\hat{\mu}] = \hat{1}$ .



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$$\begin{aligned} \tilde{\mathbf{M}} &= \hat{\chi}_m \tilde{\mathbf{H}}, & \hat{\chi}_m: \text{magnetic susceptibility, } [\hat{\chi}_m] &= \hat{1}. \\ \tilde{\mathbf{B}} &= \mu_0 (\hat{1} + \hat{\chi}_m) \tilde{\mathbf{H}} = \mu_0 \hat{\mu} \tilde{\mathbf{H}}, & \hat{\mu}: \text{relative permeability, } [\hat{\mu}] &= \hat{1}. \end{aligned}$$

- $\hat{\chi}_m(\mathbf{r}, \omega)$ ,  $\hat{\mu}(\mathbf{r}, \omega)$  are determined in the frequency domain.
- Complications: manifold.
- Traditional integrated optics (frequencies, media):  $\hat{\mu}(\mathbf{r}) = \hat{1}$ .

## Maxwell equations, dispersion

(Material) **dispersion**:  $\hat{\epsilon}(\mathbf{r}, \omega)$ ,  $\hat{\mu}(\mathbf{r}, \omega)$  are frequency dependent.

$$\tilde{\mathbf{D}}(\mathbf{r}, \omega) = \epsilon_0 \hat{\epsilon}(\mathbf{r}, \omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega), \quad \tilde{\mathbf{B}}(\mathbf{r}, \omega) = \mu_0 \hat{\mu}(\mathbf{r}, \omega) \tilde{\mathbf{H}}(\mathbf{r}, \omega)$$

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$$\curvearrowright \mathbf{D}(\mathbf{r}, t) = \epsilon_0 \int \hat{\epsilon}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t') dt',$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \int \hat{\mu}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{H}(\mathbf{r}, t') dt'.$$

## Helmholtz equations


Linear dielectric media without free charges or currents,  
time dependence  $\sim \exp(i\omega t)$ , fields  $\mathbf{E}(\mathbf{r})$ ,  $\mathbf{D}(\mathbf{r})$ ,  $\mathbf{B}(\mathbf{r})$ ,  $\mathbf{H}(\mathbf{r})$ ,  
material properties  $\hat{\epsilon}(\mathbf{r})$ ,  $\hat{\mu}(\mathbf{r})$ :

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -i\omega\mathbf{B}, & \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= i\omega\mathbf{D}, \\ \mathbf{D} &= \epsilon_0\hat{\epsilon}\mathbf{E}, & \mathbf{B} &= \mu_0\hat{\mu}\mathbf{H}.\end{aligned}$$

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$$\hookrightarrow \nabla \times \mathbf{E} = -i\omega \mu_0 \hat{\mu} \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega \epsilon_0 \hat{\epsilon} \mathbf{E}, \quad \nabla \cdot \hat{\epsilon} \mathbf{E} = 0, \quad \nabla \cdot \hat{\mu} \mathbf{H} = 0.$$

$$\hookrightarrow \nabla \times (\hat{\mu}^{-1} \nabla \times \mathbf{E}) = \omega^2 \epsilon_0 \mu_0 \hat{\epsilon} \mathbf{E} \quad \text{or} \quad \nabla \times (\hat{\epsilon}^{-1} \nabla \times \mathbf{H}) = \omega^2 \epsilon_0 \mu_0 \hat{\mu} \mathbf{H}.$$

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Linear dielectric media without free charges or currents,  
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Where  $\hat{\epsilon} = \epsilon \hat{1}$ ,  $\nabla \epsilon = 0$ ,  $\hat{\mu} = \mu \hat{1}$ ,  $\nabla \mu = 0$ : (!)

$$\hookrightarrow \Delta \mathbf{E} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{E} = 0 \quad \text{or} \quad \Delta \mathbf{H} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{H} = 0, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

## Plane harmonic waves

Where  $\hat{\epsilon} = \epsilon \hat{1}$ ,  $\nabla \epsilon = 0$ ,  $\hat{\mu} = \mu \hat{1}$ ,  $\nabla \mu = 0$ : (!)

Components of  $\mathbf{E}$ ,  $\mathbf{H}$  satisfy 
$$\Delta \psi + \frac{\omega^2}{c^2} \epsilon \mu \psi = 0.$$

↪ 
$$\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k}_m \cdot \mathbf{r} - \omega t)}, \quad -k_m^2 + \frac{\omega^2}{c^2} \epsilon \mu = 0.$$

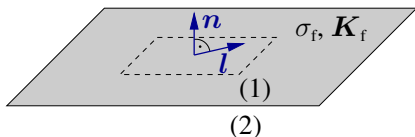
(Mixture of TD and FD expressions;  $\nabla \cdot \mathbf{E} = \rho_{ext}$ ,  $\nabla \cdot \mathbf{H} = \mathbf{j}_{ext}$ ,  $\text{Re}$ ,  $1/2$ , c.c. omitted; sloppy, but common.)

- Medium: refractive index:  $n = \sqrt{\epsilon \mu}$
- Periodicity in time: angular frequency:  $\omega$ ,  
frequency:  $f = \omega / (2\pi)$ ,  
period:  $T = 1/f = 2\pi / \omega$ ,
- Spatial periodicity: wave vector:  $\mathbf{k}_m$ ,  $k_m = |\mathbf{k}_m|$ ,  
wavenumber:  $k_m = \omega / c_m = (\omega / c) n = k n$ ,  
vacuum wavenumber:  $k = \omega / c$ ,  
vacuum wavelength:  $\lambda = 2\pi / k = 2\pi c / \omega$ ,  
wavelength in the medium:  $\lambda_m = 2\pi / k_m = 2\pi / (k n) = \lambda / n$ .
- Phase velocity: speed of light in vacuum:  $c = 1 / \sqrt{\epsilon_0 \mu_0} = \lambda f$ ,  
in the medium:  $c_m = c / n = \lambda_m f$ .

(Use of symbols depends highly on context.)



## Interface conditions

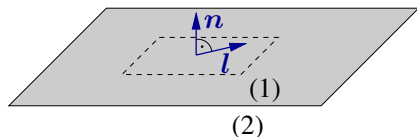


Surface between media (1) and (2), surface normal  $\mathbf{n}$ , tangents  $\mathbf{l}$ , surface charge density  $\sigma_f$ , surface current density  $\mathbf{K}_f$  :

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{l} \cdot (\mathbf{K}_f \times \mathbf{n}).$$

## Interface conditions



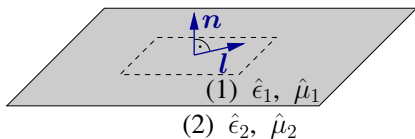
Surface between media (1) and (2), surface normal  $\mathbf{n}$ , tangents  $\mathbf{l}$ , surface without free charges or currents:

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

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## Interface conditions

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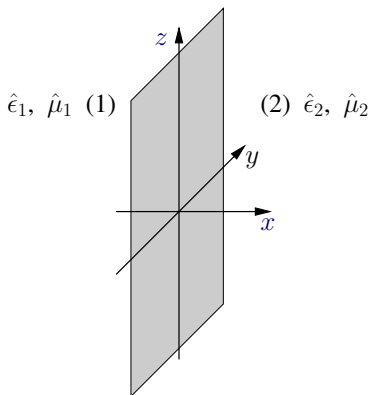
Surface between media (1) and (2), surface normal  $\mathbf{n}$ , tangents  $\mathbf{l}$ , linear media with permittivities  $\hat{\epsilon}_1$ ,  $\hat{\epsilon}_2$ , and permeabilities  $\hat{\mu}_1$ ,  $\hat{\mu}_2$ :

$$\mathbf{n} \cdot (\hat{\epsilon}_1 \mathbf{E}_1 - \hat{\epsilon}_2 \mathbf{E}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\hat{\mu}_1 \mathbf{H}_1 - \hat{\mu}_2 \mathbf{H}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = 0.$$

## Reflection and transmission of plane waves at dielectric interfaces

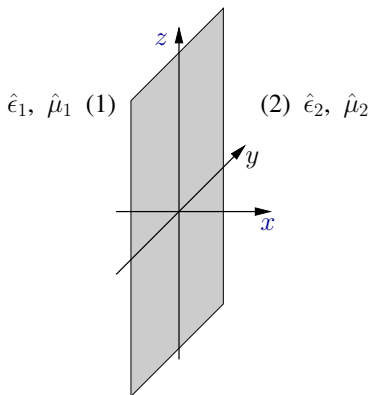
(FD)



- $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$ ,  
 $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$
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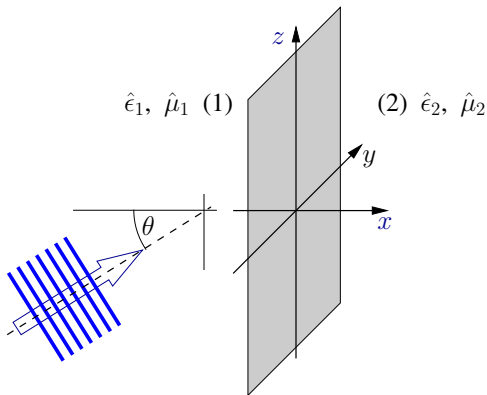
↪  $\mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}$ ,  $\mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$

---

1-D problem for  $\mathbf{E}'$ ,  $\mathbf{H}'$ .

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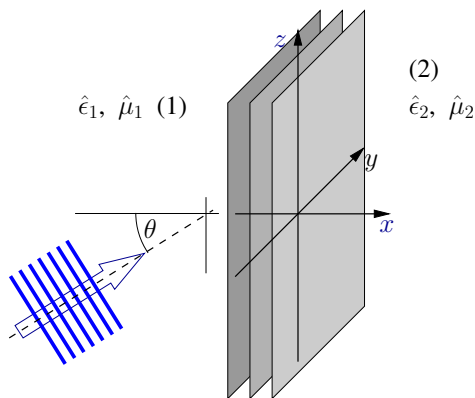
1-D problem for  $\mathbf{E}'$ ,  $\mathbf{H}'$ .

(incoming plane wave at angle  $\theta$ )  
(orient coordinates ( $k_y = 0$ ), plane of incidence, distinguish polarizations)  
(write ansatz functions for incoming, reflected, and transmitted waves)  
(interface conditions determine the amplitudes)

↪ Fresnel equations.

## Dielectric multilayer structures

(FD)



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1-D problem for  $\mathbf{E}'$ ,  $\mathbf{H}'$ .

(...)  
(...)  
(...)  
(...)

↪ Reflectance and transmittance properties.

## Energy of electromagnetic fields

(TD)

- Force on a particle with charge  $q$ , velocity  $\mathbf{v}$ , in a field  $\mathbf{E}$ ,  $\mathbf{B}$ :

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$



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For a charge density  $\rho_f(\mathbf{r}, t)$ :

force density  $\mathbf{f} = \rho_f(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ ,

↪ power density  $\mathbf{f} \cdot \mathbf{v} = \rho_f \mathbf{E} \cdot \mathbf{v} = \mathbf{J}_f \cdot \mathbf{E}$ ,

total work per time unit done in  $\mathcal{V}$ :

$$\frac{dW_{\mathcal{V}}}{dt} = \int_{\mathcal{V}} \mathbf{J}_f \cdot \mathbf{E} d\mathcal{V}.$$

## Power & energy density, Poynting theorem

(TD)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$

$$\hookrightarrow \frac{d}{dt} W_{\mathcal{V}}^{\text{mech}} = \int_{\mathcal{V}} \mathbf{J}_f \cdot \mathbf{E} \, d\mathcal{V} = - \int_{\mathcal{V}} (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) \, d\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\mathbf{E} \times \mathbf{H}) \, d\mathcal{V},$$

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$\mathcal{V}$  arbitrary

$$\hookrightarrow \dot{w} + \nabla \cdot \mathbf{S} = -\mathbf{J}_f \cdot \mathbf{E}, \quad \frac{d}{dt} (W_{\mathcal{V}}^{\text{mech}} + W_{\mathcal{V}}^{\text{field}}) = - \oint_{\partial \mathcal{V}} \mathbf{S} \cdot d\mathbf{a}.$$



## Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (. . .) linear media:

$$w = \frac{1}{2}(\epsilon_0 \mathbf{E} \cdot \hat{\epsilon} \mathbf{E} + \mu_0 \mathbf{H} \cdot \hat{\mu} \mathbf{H}), \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad \dot{w} + \nabla \cdot \mathbf{S} = 0,$$

$$\mathbf{E}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{H}}(\mathbf{r}) e^{i\omega t}$$

↪  $\mathbf{S}$ ,  $w$  oscillate in time.

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---

Consider time-averaged quantities:  $\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t') dt'$  (FD)

↪  $\bar{w} = \frac{1}{4} \text{Re} \left( \epsilon_0 \tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \tilde{\mathbf{E}} + \mu_0 \tilde{\mathbf{H}}^* \cdot \hat{\mu} \tilde{\mathbf{H}} \right), \quad \bar{\mathbf{S}} = \frac{1}{2} \text{Re} \left( \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}} \right).$  (exercise)

$$\bar{\dot{w}} = \dot{\bar{w}} = 0, \quad \overline{\nabla \cdot \mathbf{S}} = \nabla \cdot \bar{\mathbf{S}} \quad \rightsquigarrow \quad \nabla \cdot \bar{\mathbf{S}} = 0, \quad \oint_{\mathcal{V}} \bar{\mathbf{S}} \cdot d\mathbf{a} = 0;$$

“power balance”, conservation of energy.

## Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law  $\mathbf{J}_f = \sigma \mathbf{E}$ ,  $\sigma$ : conductivity of the material.

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$$\curvearrowright \dot{\rho}_f = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_f, \quad \rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0)\right),$$

$$\text{assume } \rho_f(\mathbf{r}, t_0) = 0 \rightsquigarrow \rho_f(\mathbf{r}, t) = 0 \quad \forall t.$$

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---

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}.$$

## Telegrapher equation

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}$$

$$\rightarrow \Delta \mathbf{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{E}} - \mu_0 \mu \sigma \dot{\mathbf{E}} = 0, \quad \Delta \mathbf{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{H}} - \mu_0 \mu \sigma \dot{\mathbf{H}} = 0,$$

Telegrapher equation.

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Telegrapher equation.

---

Frequency domain:  $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}$ ,

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma \right) \tilde{\mathbf{E}} = 0.$$

$$\text{Nonconducting media } \sigma = 0, \quad \Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0.$$



## Telegrapher equation

---

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, & \nabla \times \mathbf{E} &= -\mu_0 \mu \dot{\mathbf{H}}, & \nabla \cdot \mathbf{H} &= 0, & \nabla \times \mathbf{H} &= \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}} \\ \hookrightarrow \Delta \mathbf{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{E}} - \mu_0 \mu \sigma \dot{\mathbf{E}} &= 0, & \Delta \mathbf{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{H}} - \mu_0 \mu \sigma \dot{\mathbf{H}} &= 0, \end{aligned}$$

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Nonconducting media  $\sigma = 0$ ,  $\Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0$ .

Define  $\bar{\epsilon}$  such that  $\frac{\omega^2}{c^2} \bar{\epsilon} \mu = \frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma$ , i.e.  $\bar{\epsilon} = \epsilon - i \frac{\sigma}{\epsilon_0 \omega}$

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + k^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \text{Helmholtz equation, } \bar{\epsilon} \in \mathbb{C}, \quad k = \frac{\omega}{c}.$$

## Wave attenuation

---

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↪ solutions  $\sim e^{-i(k\bar{n}z - \omega t)}$

with refractive index  $\bar{n} = n' - i n'' = \sqrt{\bar{\epsilon} \mu} \in \mathbb{C}$ , (!)

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$$e^{-i(k\bar{n}z - \omega t)} = e^{-i(kn'z - \omega t)} e^{-kn''z},$$

damped plane wave solutions

↔ sign of  $n''$ , choice of  $\exp(\pm i\omega)$ .

Issues:

- penetration depth,
- $S$  and  $w$  decay with  $z$ ,
- still transverse waves,
- $E, H$  no longer in phase,
- notions of wavenumber, wavelength, phase velocity  $\in \mathbb{C}$ .

## Simulations in integrated optics

A typical setting:

- “uncharged dielectric medium”:  $\rho_f = 0, \mathbf{J}_f = 0$ .
- “linear medium”:  $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}, \mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$ .
- “isotropic medium”:  $\hat{\epsilon} = \epsilon \hat{\mathbf{1}}, \hat{\mu} = \mu \hat{\mathbf{1}}$ .
- “nonmagnetic medium”:  $\hat{\mu} = \hat{\mathbf{1}}$ .
- “lossless medium”:  $\hat{\epsilon}^\dagger = \hat{\epsilon}, \hat{\mu}^\dagger = \hat{\mu}, (\epsilon, \mu \in \mathbb{R})$ .
- “piecewise constant”  $\rightarrow$  “dependent on position”.
- “electric and magnetic field”: eliminate  $\mathbf{D}$  and  $\mathbf{B}$ , retain  $\mathbf{E}$  and  $\mathbf{H}$ .
- “governed by the curl equations”: divergence eqns. are satisfied.
- “frequency domain, time harmonic fields, frequency, wavelength”:  
... as discussed.

## Upcoming

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Next lectures:

- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.

