

# *Optical Waveguide Theory (D)*



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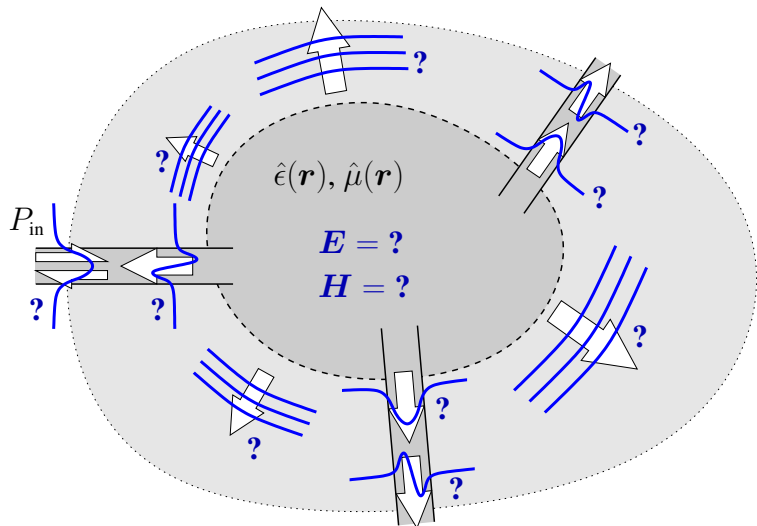
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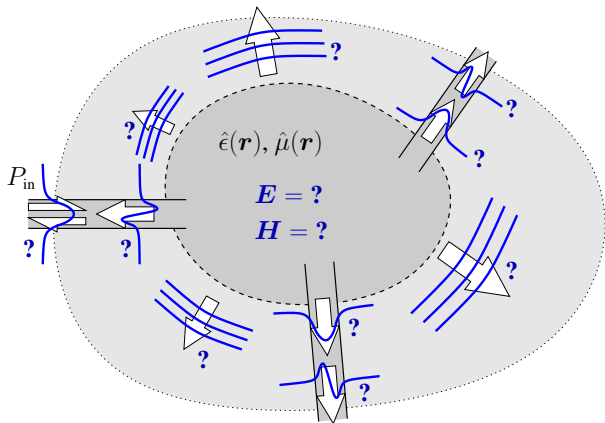
### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

## Guided wave scattering problems, schematically



## Guided wave scattering problems, schematically



Given  $\hat{\epsilon}(\mathbf{r})$ ,  $\hat{\mu}(\mathbf{r})$  & external excitation (incoming guided mode),  
determine  $\mathbf{E}$ ,  $\mathbf{H}$  within the computational domain  
& determine the optical power carried by outgoing waves.

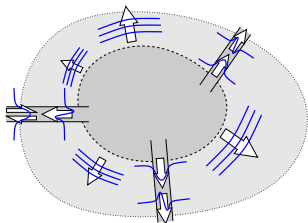
## Scattering problems, time domain

(TD)

$$\mathbf{E}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t),$$

$$\nabla \times \mathbf{E} = -\mu_0 \hat{\mu} \dot{\mathbf{H}},$$

$$\nabla \times \mathbf{H} = \epsilon_0 \hat{\epsilon} \dot{\mathbf{E}}.$$



## Scattering problems, time domain

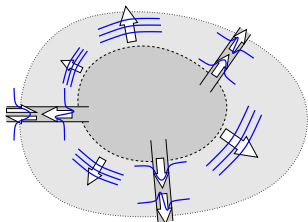
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- $\begin{pmatrix} 3\text{-D} \\ 2\text{-D} \\ 1\text{-D} \end{pmatrix}$  computational domain  $\times$  time interval.
- Initial & boundary conditions  $\longleftrightarrow$  incident waves.
- “Local” time-explicit iterative schemes possible (e.g. FDTD).
- Time evolution available; direct modeling of pulse propagation.
- Dispersion (...?).
- Guided wave excitation (...?).
- Fourier transform  $\longrightarrow$  spectral information.



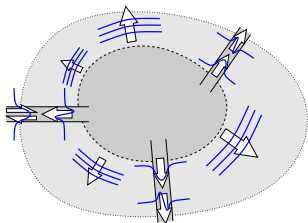
## Scattering problems, frequency domain

(FD)

$$\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}), \sim \exp(i\omega t),$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H},$$

$$\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}.$$



## Scattering problems, frequency domain

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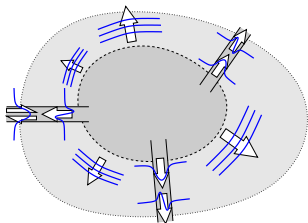
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- $\begin{pmatrix} 3\text{-D} \\ 2\text{-D} \\ 1\text{-D} \end{pmatrix}$  computational domain.

- “ $\mathbf{M}(\overrightarrow{\text{field}}) = \overrightarrow{\text{excitation}}$ ”;  
matrix needs to be determined, stored; system needs to be solved.
- Spectral information directly available.
- Dispersion — straightforward.
- Guided wave excitation — straightforward.
- Fourier transform  $\longrightarrow$  time evolution / pulse propagation.





## Open problems

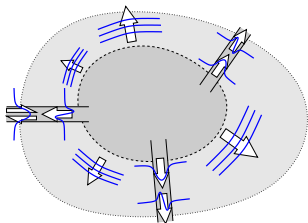
(TD & FD)

“Open” spatial computational domain

~> boundary conditions need to

- permit outgoing radiated fields  
& outgoing (reflected) guided modes to exit the domain,
- launch the incoming external excitation.

~<> simulate a nonexistent boundary, an unlimited domain.



## Open problems

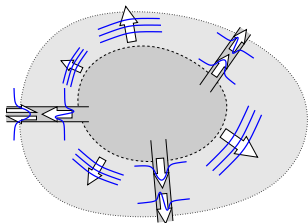
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“Open” spatial computational domain

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- permit outgoing radiated fields & outgoing (reflected) guided modes to exit the domain,
  - launch the incoming external excitation.
- ~<> simulate a nonexistent boundary, an unlimited domain.



- Keywords:
- transparent-influx boundary conditions,
  - absorbing boundary conditions,
  - perfectly matched layers (PMLs).



## 2-D problems

---

$$\hat{\epsilon} = \epsilon \hat{1}, \quad \hat{\mu} = \mu \hat{1}, \quad \sim \exp(i\omega t) \quad (\text{FD})$$

$$\begin{pmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

## 2-D problems

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$$\hat{\epsilon} = \epsilon \hat{1}, \quad \hat{\mu} = \mu \hat{1}, \quad \sim \exp(i\omega t) \quad (\text{FD})$$

Assume  $\partial_y \epsilon = 0$ ,  $\partial_y \mu = 0$ ; consider solutions  $\partial_y \mathbf{E} = 0$ ,  $\partial_y \mathbf{H} = 0$ :

$$\begin{pmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{pmatrix} = -i\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

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↪ Two decoupled sets of equations:

- $\{E_y, H_x, H_z\}$ : **transverse electric (TE)** fields,  $\mathbf{E} \perp x\text{-}z\text{-plane}$ .
- $\{H_y, E_x, E_z\}$ : **transverse magnetic (TM)** fields,  $\mathbf{H} \perp x\text{-}z\text{-plane}$ .

(Different conventions on the use of TE, TM.)

(Applies also to the TD.)

## 2-D TE waves

---

$$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0 \quad (\text{FD})$$

- Principal component  $E_y$ ,

$$H_x = \frac{-i}{\omega\mu_0\mu} \partial_z E_y, \quad H_z = \frac{i}{\omega\mu_0\mu} \partial_x E_y, \quad i\omega\epsilon_0\epsilon E_y = \partial_z H_x - \partial_x H_z$$

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$$\curvearrowright \quad \partial_x \frac{1}{\mu} \partial_x E_y + \partial_z \frac{1}{\mu} \partial_z E_y + k^2 \epsilon E_y = 0. \quad (*)$$

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- If  $\mu = 1$ :  $\epsilon(x, z)$  (!)

$$\hookrightarrow \partial_x^2 E_y + \partial_z^2 E_y + k^2 \epsilon E_y = 0, \quad (**)$$

scalar 2-D (TE) Helmholtz equation ( $E_y$ ,  $\partial_n E_y$  continuous).

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(Reflection / transmission problems: s-polarized waves satisfy (\*), (\*\*).)

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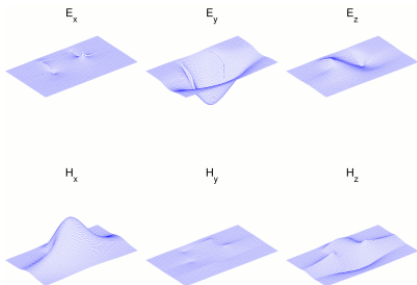
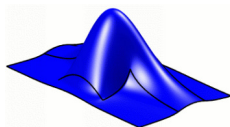
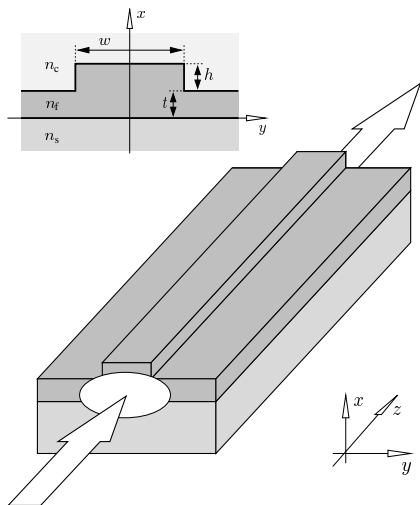
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(Reflection / transmission problems: p-polarized waves satisfy (\*), (\*\*).)

## Rib waveguide

... variant of an integrated optical waveguide with 2-D confinement





## Waveguides: Mode problems

---

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mu\mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0\epsilon\mathbf{E}. \quad \sim \exp(i\omega t) \quad (\text{FD})$$

- **Waveguide:** a system that is homogeneous along its axis  $z$ ,  
 $\partial_z\epsilon = 0, \partial_z\mu = 0, \partial_z n = 0$ .

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- Look for solutions (**modes**) that vary harmonically with  $z$ :

$$\mathbf{E}(x, y, z) = \bar{\mathbf{E}}(x, y) e^{-i\beta z}, \quad \mathbf{H}(x, y, z) = \bar{\mathbf{H}}(x, y) e^{-i\beta z},$$

mode profile  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ , propagation constant  $\beta$ .

(drop  $\bar{\phantom{x}}$ )

## Waveguides: Mode problems

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vectorial mode equations, variants.

(...)

## Waveguides: Mode equations

---

- Where  $\epsilon(\mathbf{r})$ ,  $\mu(\mathbf{r})$ :  $\sim \exp(i\omega t)$  (FD)

$$\Delta\tilde{\mathbf{E}} + k^2\epsilon\mu\tilde{\mathbf{E}} = 0, \quad \Delta\tilde{\mathbf{H}} + k^2\epsilon\mu\tilde{\mathbf{H}} = 0$$

↪

$$\partial_x^2 \mathbf{E} + \partial_y^2 \mathbf{E} + (k^2\epsilon\mu - \beta^2)\mathbf{E} = 0,$$
$$\partial_x^2 \mathbf{H} + \partial_y^2 \mathbf{H} + (k^2\epsilon\mu - \beta^2)\mathbf{H} = 0,$$

scalar **mode equation**, valid for all components of  $\mathbf{E}$ ,  $\mathbf{H}$ ,  
to be supplemented by suitable **boundary** and **interface conditions**.

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scalar **mode equation**, valid for all components of  $\mathbf{E}$ ,  $\mathbf{H}$ ,  
to be supplemented by suitable **boundary** and **interface conditions**.

- ↔ **Eigenvalue** problem with eigenvalue  $\beta$ , eigenfunction  $\mathbf{E}$ ,  $\mathbf{H}$ ,  
“ $\mathbf{M}(\beta) \overrightarrow{(\text{profile})} = 0$ ”.

## Waveguides: Mode equations

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$$\Delta \tilde{\mathbf{E}} + k^2 \epsilon \mu \tilde{\mathbf{E}} = 0, \quad \Delta \tilde{\mathbf{H}} + k^2 \epsilon \mu \tilde{\mathbf{H}} = 0$$

$$\begin{aligned} \hookrightarrow \quad \partial_x^2 \mathbf{E} + \partial_y^2 \mathbf{E} + (k^2 \epsilon \mu - \beta^2) \mathbf{E} &= 0, \\ \partial_x^2 \mathbf{H} + \partial_y^2 \mathbf{H} + (k^2 \epsilon \mu - \beta^2) \mathbf{H} &= 0, \end{aligned}$$

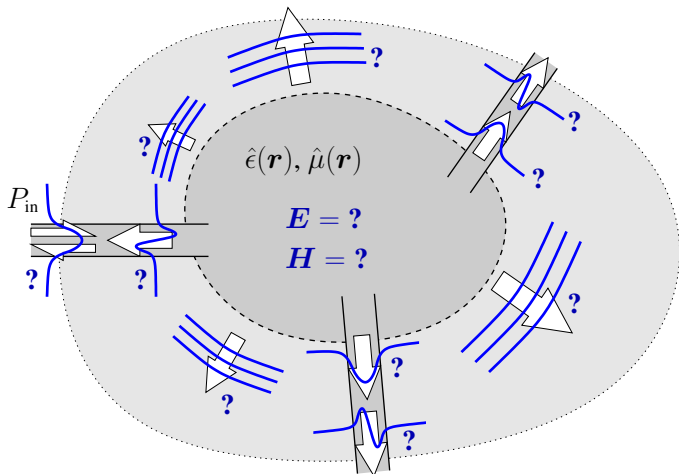
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to be supplemented by suitable **boundary** and **interface conditions**.

- $\leftrightarrow$  **Eigenvalue** problem with eigenvalue  $\beta$ , eigenfunction  $\mathbf{E}$ ,  $\mathbf{H}$ ,  
“ $\mathbf{M}(\beta) (\overrightarrow{\text{profile}}) = 0$ ”.

- **Guided modes**: discrete  $\beta \in \mathbb{R}$ ,  $\iint S_z \, dx dz < \infty$ .
- **Radiation modes**: continuum of  $\beta^2 \in \mathbb{R}$ , oscillating external fields.
- **Leaky modes**: discrete  $\beta \in \mathbb{C}$ , outgoing wave boundary conditions.
- (...)

## Guided wave scattering problems

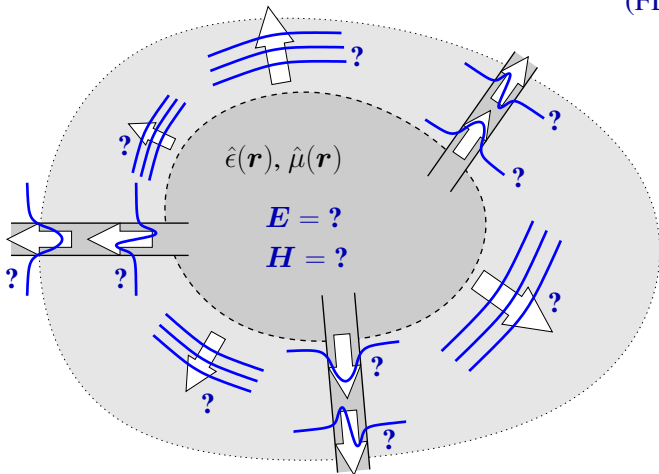
(FD)



Given external excitation  $\sim \exp(i\omega t)$ ,  $\omega \in \mathbb{R}$ .

## Resonance problems

(FD ...)



Omit excitation, look for nonzero solutions that decay in time.



## Resonance problems

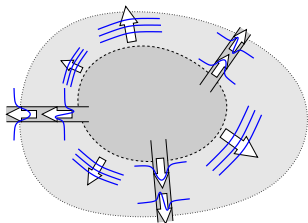
(FD ...)

$$\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}), \sim \exp(i\omega t), \omega = ?$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H},$$

$$\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E},$$

& outgoing wave boundary conditions.



## Resonance problems

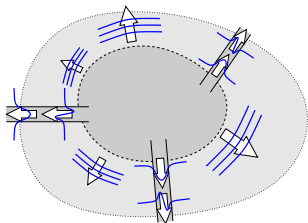
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$$\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E},$$

& outgoing wave boundary conditions.



- Look for nonzero solutions with  $\omega \in \mathbb{C}$  that oscillate and decay (slowly ...) in time.
- “ $\mathbf{M}(\omega) (\overrightarrow{\text{field}}) = 0$ ”, **eigenvalue problem**.
- Solutions: discrete eigenfrequencies  $\omega$ , resonant mode profiles.

## Scalar approximation

Linear, isotropic, nonmagnetic media,  $\epsilon = n^2$  ;  
a structure with “small” variations in  $\epsilon$  :

A **scalar approximation** may be adequate,

$$\nabla \cdot (\epsilon \mathbf{E}) \approx \epsilon \nabla \cdot \mathbf{E}$$

$$\curvearrowright \Delta \psi - \frac{1}{c^2} \epsilon \ddot{\psi} = 0, \quad \text{(TD)}$$

$$\Delta \psi + k^2 \epsilon \psi = 0, \quad \text{(FD)}$$

satisfied by all components  $\psi$  of  $\mathbf{E}$ ,  $\mathbf{H}$ .

(Applicable to basically all types of problems.)

## ***Beam propagation method***

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- Starting point:  $\Delta\psi + k^2\epsilon\psi = 0$ ,  $\sim \exp(i\omega t)$  (FD)  
“small” changes in  $\epsilon = n^2$  along a propagation coordinate  $z$ .

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- Ansatz:  $\psi(x, y, z) = \psi_0(x, y, z) e^{-ikn_r z}$ ,  
reference effective index  $n_r$ ,  
assume that  $\psi_0$  varies “slowly” along  $z$   $\longleftrightarrow$  neglect  $\partial_z^2\psi_0$ .

$\hookrightarrow -i2kn_r\partial_z\psi_0 + (\partial_x^2 + \partial_y^2)\psi_0 + k^2(\epsilon - n_r^2)\psi_0 = 0,$

PDE of first order in  $z$ , solved as an initial value problem.

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- Restriction to unidirectional propagation, reflections are neglected.
- Paraxial propagation, errors for waves with effective indices  $\neq n_r$ .

(Many variants (vectorial, wide-angle, bi-directional, ...) have been proposed.)

(Other ways of motivating the approximation exist.)

(Term “BPM” in use also for other types of methods.)

- Keywords: Paraxial approximation,  
Slowly-varying-envelope approximation (SVEA),  
Beam propagation method (BPM).

## Upcoming

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Next lectures:

- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.
- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.

