Optical Waveguide Theory (B)

Manfred Hammer[*]

Theoretical Electrical Engineering
Paderborn University, Germany

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[*] *Theoretical Electrical Engineering, Paderborn University*
Warburger Straße 100, 33098 Paderborn, Germany
Phone: +49(0)5251/60-3560
E-mail: manfred.hammer@uni-paderborn.de

Course overview

Optical waveguide theory

Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z),\ \ t.$
- Scalar and vector fields: $\phi(\boldsymbol{r}, t),\ \boldsymbol{A}(\boldsymbol{r}, t), \qquad \boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$
- Inner product: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z.$
- Vector product: $\boldsymbol{A} \times \boldsymbol{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}.$
- Time derivatives: $\dfrac{\partial \phi}{\partial t},\ \partial_t \phi,\ \dot{\phi},\ \cancel{\nabla_T \phi}.$

Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$
- Gradient: $\mathrm{grad}\phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix}.$
- Divergence: $\mathrm{div}\boldsymbol{A} = \boldsymbol{\nabla} \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z.$
- Curl: $\mathrm{curl}\boldsymbol{A} = \mathrm{rot}\boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}.$
- Laplacian: $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2,$

$$\Delta\phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_x^2 \phi, \qquad \Delta\boldsymbol{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$$

Dirac delta

A linear functional
that extracts the value of a function at one point:

1-D: $$\int_a^b f(x)\,\delta(x-x_0)\,\mathrm{d}x = \begin{cases} f(x_0), & \text{if } a<x_0<b,\\ 0 & \text{otherwise;}\end{cases}$$

$$\delta(x-x_0)=0,\ \text{if } x\neq x_0.$$

3-D: $$\int_{\mathcal{V}} f(\boldsymbol{r})\,\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}\mathcal{V} = \begin{cases} f(\boldsymbol{r}_0), & \text{if } \boldsymbol{r}_0\in\mathcal{V},\\ 0 & \text{otherwise;}\end{cases}$$

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=0,\ \text{if } \boldsymbol{r}\neq\boldsymbol{r}_0.$$

Implications: manifold.

Fourier transform, 1-D

1-D: A function $f(x)\in\mathbb{C}$ of one variable:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde f(k)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k,\qquad \tilde f(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1+\beta f_2}=\alpha\tilde f_1+\beta\tilde f_2$.
- $f(x)=f(-x) \rightsquigarrow \tilde f(k)=\tilde f(-k)$.
- $f(x)=-f(-x) \rightsquigarrow \tilde f(k)=-\tilde f(-k)$.
- $f\in\mathbb{R} \rightsquigarrow \tilde f(-k)=\tilde f^*(k)$.
- $\delta(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$

Fourier transform

3-D: A field $\phi(\boldsymbol{r})$:

$$\phi(\boldsymbol{r})=\frac{1}{\sqrt{2\pi}^3}\int\tilde\phi(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3k,\qquad \tilde\phi(\boldsymbol{k})=\frac{1}{\sqrt{2\pi}^3}\int\phi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3r.$$

4-D: A field $\phi(\boldsymbol{r},t)$:

$$\phi(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}^4}\iint\tilde\phi(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\mathrm{d}^3k\,\mathrm{d}\omega,$$

$$\tilde\phi(\boldsymbol{k},\omega)=\frac{1}{\sqrt{2\pi}^4}\iint\phi(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\mathrm{d}^3r\,\mathrm{d}t.$$

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx}+B\,\partial_{yy}+C\,\partial_{xy}+D\,\partial_x+E\,\partial_y+F)\,\psi(x,y)=0,$$

coefficients $A(x,y),\ldots,F(x,y)$.

If the system is constant in x, $\partial_x A=\ldots=\partial_x F=0$,

- write ψ as $\quad\psi(x,y)=\displaystyle\int\tilde\psi(k,y)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\,.$

$$\hookrightarrow\ \int\big(B\,\partial_{yy}+(E+\mathrm{i}kC)\partial_y+(F+\mathrm{i}kD-k^2A)\big)\,\tilde\psi(k,y)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k=0,$$

$$\hookrightarrow\ \big(B\,\partial_{yy}+(E+\mathrm{i}kC)\partial_y+(F+\mathrm{i}kD-k^2A)\big)\,\tilde\psi(k,y)=0,\ \text{(for all } k\text{)},$$

… a set of DEs in one unknown.

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y), \ldots, F(x,y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,,$

- use an ansatz $\qquad \psi(x,y) = \tilde\psi(y)\;\mathrm{e}^{\mathrm{i}kx}\,.$

$\hookrightarrow \quad \big(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2A)\big)\,\tilde\psi(y) = 0,$

... a DE in one unknown, with parameter k.

(& boundary conditions, ...)

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

$$\& \qquad \psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde\psi(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

$$\hookrightarrow \quad \left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}^2}\right)\tilde\psi(\boldsymbol{k},\omega) = 0,$$

$$\hookrightarrow \quad \tilde\psi(\boldsymbol{k},\omega) = a_{\mathrm{f}}(\boldsymbol{k})\,\delta(\omega - \omega_k) + a_{\mathrm{b}}(\boldsymbol{k})\,\delta(\omega + \omega_k), \qquad \omega_k = \mathrm{c}\,|\boldsymbol{k}|,$$

$$\hookrightarrow \quad \psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int\Big(a_{\mathrm{f}}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} + a_{\mathrm{b}}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} + \omega_k t)}\Big)\,\mathrm{d}^3k,$$

- $\psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}),\ \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}) \rightsquigarrow \ldots \rightsquigarrow a_{\mathrm{f}}(\boldsymbol{k}), a_{\mathrm{b}}(\boldsymbol{k}).$

A touch of variational calculus

- Functional: $\qquad \mathcal{L}: \begin{array}{ccc} U & \longrightarrow & \mathbb{R}, \mathbb{C}, \\ u & \longrightarrow & \mathcal{L}(u), \end{array}$

  a map from a space U of functions to real / complex numbers.

- Stationary functional: $\qquad \dfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(u + s\,v)\Big|_{s=0} = 0 \quad$ for all v,

  the variation of $\mathcal{L}$ at u vanishes for arbitrary directions v.

- Restriction of a functional:

  ... to a parametrized family of functions;

  $\leftrightsquigarrow$ extremization with respect to these parameters,
  $\leftrightsquigarrow$ approximations of stationary points of the functional.

A touch of variational calculus

Example:

$U = \{u : [0,\pi] \to \mathbb{R} \mid u(0) = u(\pi) = 0\}$,

$\mathcal{L} : U \to \mathbb{R}$,

$$\mathcal{L}(u) = \frac{\displaystyle\int_0^\pi (\partial_x u)^2 \mathrm{d}x}{\displaystyle\int_0^\pi u^2 \mathrm{d}x}.$$

(...)

$\mathcal{L}$ stationary at u, $\quad \dfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(u + s\,v)\Big|_{s=0} = 0 \quad \forall\, v. \qquad \leftrightsquigarrow \qquad$ u satisfies DE & b.c., $\ \partial_x^2 u = -\lambda u,\ \ \lambda = \mathcal{L}(u),\ \ u(0) = u(\pi) = 0.$

$\downarrow$ Restrict $\mathcal{L}$, $\quad L(\boldsymbol{a}) = \mathcal{L}(u|_{\boldsymbol{a}})$.

L stationary at $\boldsymbol{a}$, $\quad \boldsymbol{\nabla}_{\boldsymbol{a}} L = 0. \qquad \leftrightsquigarrow \qquad$ Approximate solution of DE / eigenproblem.

Upcoming

Next lectures:

- Maxwell equations, different formulations, interfaces, energy and power flow.
- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.