Optical Waveguide Theory (C)

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Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

. . . ?

“This concerns time harmonic fields . . . with angular frequency . . . , for vacuum wavenumber . . . , speed of light . . . , and wavelength . . . .”

“The problem is governed by the Maxwell curl equations in the frequency domain for the electric field . . . and magnetic field . . . , for (lossless) uncharged dielectric, nonmagnetic linear (isotropic) media with (piecewise constant) relative permittivity . . . :

. . . (.) ”

[M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016)]

Maxwell equations, Fourier transform

$$\nabla\cdot\boldsymbol{D}=\rho_{\rm f},\quad \nabla\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{H}=\boldsymbol{J}_{\rm f}+\dot{\boldsymbol{D}}$$

$$\&\quad \boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,\quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\frac{1}{\sqrt{2\pi}}\int\boldsymbol{F}(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

$$\boldsymbol{E}(\boldsymbol{r},t),\ \boldsymbol{D}(\boldsymbol{r},t),\ \boldsymbol{B}(\boldsymbol{r},t),\ \boldsymbol{H}(\boldsymbol{r},t),\ \rho_{\rm f}(\boldsymbol{r},t),\ \boldsymbol{J}_{\rm f}(\boldsymbol{r},t)$$

$$\leftrightsquigarrow\ \tilde{\boldsymbol{E}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{D}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{H}}(\boldsymbol{r},\omega),\ \tilde{\rho}_{\rm f}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{J}}_{\rm f}(\boldsymbol{r},\omega),$$

$$\nabla\cdot\tilde{\boldsymbol{D}}=\tilde{\rho}_{\rm f},\quad \nabla\times\tilde{\boldsymbol{E}}=-\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \nabla\cdot\tilde{\boldsymbol{B}}=0,\quad \nabla\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\rm f}+\mathrm{i}\omega\tilde{\boldsymbol{D}}$$

(Caution: arbitrary choice of $\sim\exp(\pm\mathrm{i}\,\omega t)$!).

Maxwell equations, frequency domain

$$\nabla\cdot\tilde{\boldsymbol{D}} = \tilde{\rho}_\mathrm{f}, \quad \nabla\times\tilde{\boldsymbol{E}} = -\mathrm{i}\omega\tilde{\boldsymbol{B}}, \quad \nabla\cdot\tilde{\boldsymbol{B}} = 0, \quad \nabla\times\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}}_\mathrm{f} + \mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

$$\boldsymbol{F}(\boldsymbol{r},t)\in\mathbb{R} \quad \rightsquigarrow \quad \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega) = (\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^*$$

"at frequency ω_0": $\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega) = \sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\delta(\omega-\omega_0) + \sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\delta(\omega+\omega_0)$

$$\boldsymbol{F}(\boldsymbol{r},t) = \frac{1}{2}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t} + \bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\right\},$$

$$\boldsymbol{F}(\boldsymbol{r},t) = \mathrm{Re}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}\right\},$$

$$\text{“}\boldsymbol{F}(\boldsymbol{r},t) = \frac{1}{2}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t} + \text{c.c.”}.$$

$$\bar{\boldsymbol{E}}(\boldsymbol{r}),\ \bar{\boldsymbol{D}}(\boldsymbol{r}),\ \bar{\boldsymbol{B}}(\boldsymbol{r}),\ \bar{\boldsymbol{H}}(\boldsymbol{r}),\ \bar{\rho}_\mathrm{f}(\boldsymbol{r}),\ \bar{\boldsymbol{J}}_\mathrm{f}(\boldsymbol{r}),\ \sim \exp(\mathrm{i}\omega_0 t),$$

$$\nabla\cdot\bar{\boldsymbol{D}} = \bar{\rho}_\mathrm{f}, \quad \nabla\times\bar{\boldsymbol{E}} = -\mathrm{i}\omega_0\bar{\boldsymbol{B}}, \quad \nabla\cdot\bar{\boldsymbol{B}} = 0, \quad \nabla\times\bar{\boldsymbol{H}} = \bar{\boldsymbol{J}}_\mathrm{f} + \mathrm{i}\omega_0\bar{\boldsymbol{D}}.$$

Caution: Decorations $\tilde{}$, $\bar{}$, $_0$ are ususally omitted; context determines interpretation of symbols.

Polarization

$\tilde{\boldsymbol{P}}$: density of electric dipole moment (bound charges).

$$\tilde{\boldsymbol{D}} = \epsilon_0\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{P}}, \qquad [\tilde{\boldsymbol{D}}] = [\tilde{\boldsymbol{P}}] = \frac{\mathrm{As\,m}}{\mathrm{m}^3},\ [\tilde{\boldsymbol{E}}] = \frac{\mathrm{V}}{\mathrm{m}},$$

vacuum permittivity $\epsilon_0 = 8.854187817\ldots\cdot 10^{-12}\left[\frac{\mathrm{F}}{\mathrm{m}} = \frac{\mathrm{As}}{\mathrm{Vm}}\right]$.

- Local dipoles induced by $\tilde{\boldsymbol{E}}$ $\rightsquigarrow$ $\tilde{\boldsymbol{P}}(\tilde{\boldsymbol{E}})$.
- Linear dielectrica:

$\tilde{\boldsymbol{P}} = \epsilon_0\hat{\chi}_\mathrm{e}\tilde{\boldsymbol{E}}$, $\quad\hat{\chi}_\mathrm{e}$: dielectric susceptibility, $[\hat{\chi}_\mathrm{e}] = \hat{1}$.

$\tilde{\boldsymbol{D}} = \epsilon_0(\hat{1}+\hat{\chi}_\mathrm{e})\tilde{\boldsymbol{E}} = \epsilon_0\hat{\epsilon}\tilde{\boldsymbol{E}}$, $\quad\hat{\epsilon}$: relative permittivity, $[\hat{\epsilon}] = \hat{1}$.

- $\hat{\chi}_\mathrm{e}(\boldsymbol{r},\omega)$, $\hat{\epsilon}(\boldsymbol{r},\omega)$ are determined in the frequency domain.
- Complications: $\mathrm{Im}\,\epsilon$, $\hat{\epsilon}(T)$, $\hat{\epsilon}(\boldsymbol{F})$, $\chi^{(2)}_{jkl}E_kE_l$, $\chi^{(3)}_{jklm}E_kE_lE_m$, ...
- Simpler cases: $\hat{\epsilon}(\not{\boldsymbol{r}})$, $\hat{\epsilon} = \epsilon\hat{1}$.

Magnetization

$\tilde{\boldsymbol{M}}$: density of magnetic dipole moments (bound currents).

$$\tilde{\boldsymbol{H}} = \frac{1}{\mu_0}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{M}}, \qquad [\tilde{\boldsymbol{H}}] = [\boldsymbol{M}] = \frac{\mathrm{A\,m}^2}{\mathrm{m}^3},\ [\tilde{\boldsymbol{B}}] = \mathrm{T} = \frac{\mathrm{Vs}}{\mathrm{m}^2},$$

vacuum permeability $\mu_0 = 4\pi\cdot 10^{-7}\left[\frac{\mathrm{N}}{\mathrm{A}^2} = \frac{\mathrm{Vs}}{\mathrm{Am}}\right]$.

- Local dipoles induced by $\tilde{\boldsymbol{H}}$ $\rightsquigarrow$ $\tilde{\boldsymbol{M}}(\tilde{\boldsymbol{H}})$.
- Linear magnetic media:

$\tilde{\boldsymbol{M}} = \hat{\chi}_\mathrm{m}\tilde{\boldsymbol{H}}$, $\quad\hat{\chi}_\mathrm{m}$: magnetic susceptibility, $[\hat{\chi}_\mathrm{m}] = \hat{1}$.

$\tilde{\boldsymbol{B}} = \mu_0(\hat{1}+\hat{\chi}_\mathrm{m})\tilde{\boldsymbol{H}} = \mu_0\hat{\mu}\tilde{\boldsymbol{H}}$, $\hat{\mu}$: relative permeability, $[\hat{\mu}] = \hat{1}$.

- $\hat{\chi}_\mathrm{m}(\boldsymbol{r},\omega)$, $\hat{\mu}(\boldsymbol{r},\omega)$ are determined in the frequency domain.
- Complications: manifold.
- Traditional integrated optics (frequencies, media): $\hat{\mu}(\not{\boldsymbol{r}}) = \hat{1}$.

Maxwell equations, dispersion

(Material) dispersion: $\hat{\epsilon}(\boldsymbol{r},\omega)$, $\hat{\mu}(\boldsymbol{r},\omega)$ are frequency dependent.

$$\tilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \epsilon_0\hat{\epsilon}(\boldsymbol{r},\omega)\tilde{\boldsymbol{E}}(\boldsymbol{r},\omega), \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega) = \mu_0\hat{\mu}(\boldsymbol{r},\omega)\tilde{\boldsymbol{H}}(\boldsymbol{r},\omega)$$

$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0\int\hat{\epsilon}_\mathrm{TD}(\boldsymbol{r},t-t')\boldsymbol{E}(\boldsymbol{r},t')\,\mathrm{d}t',$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\int\hat{\mu}_\mathrm{TD}(\boldsymbol{r},t-t')\boldsymbol{H}(\boldsymbol{r},t')\,\mathrm{d}t'.$$

Helmholtz equations

Linear dielectric media without free charges or currents, time dependence $\sim \exp(\mathrm{i}\omega t)$, fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$, material properties $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\mu}(\boldsymbol{r})$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\mathrm{i}\omega\boldsymbol{B}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \mathrm{i}\omega\boldsymbol{D},$$
$$\boldsymbol{D} = \epsilon_0\hat{\epsilon}\,\boldsymbol{E}, \quad \boldsymbol{B} = \mu_0\hat{\mu}\,\boldsymbol{H}.$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}, \qquad \boldsymbol{\nabla}\cdot\hat{\epsilon}\,\boldsymbol{E} = 0, \quad \boldsymbol{\nabla}\cdot\hat{\mu}\,\boldsymbol{H} = 0.$$

$$\boldsymbol{\nabla}\times(\hat{\mu}^{-1}\boldsymbol{\nabla}\times\boldsymbol{E}) = \omega^2\epsilon_0\mu_0\hat{\epsilon}\,\boldsymbol{E} \quad \text{or} \quad \boldsymbol{\nabla}\times(\hat{\epsilon}^{-1}\boldsymbol{\nabla}\times\boldsymbol{H}) = \omega^2\epsilon_0\mu_0\hat{\mu}\,\boldsymbol{H}.$$

Where $\hat{\epsilon} = \epsilon\hat{1}$, $\boldsymbol{\nabla}\epsilon = 0$, $\hat{\mu} = \mu\hat{1}$, $\boldsymbol{\nabla}\mu = 0$: (!)

$$\Delta\boldsymbol{E} + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\,\boldsymbol{E} = 0 \quad \text{or} \quad \Delta\boldsymbol{H} + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\,\boldsymbol{H} = 0, \quad \mathrm{c} = \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$

Plane harmonic waves

Where $\hat{\epsilon} = \epsilon\hat{1}$, $\boldsymbol{\nabla}\epsilon = 0$, $\hat{\mu} = \mu\hat{1}$, $\boldsymbol{\nabla}\mu = 0$: (!)
Components of $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy $\quad \Delta\psi + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\,\psi = 0.$

$$\psi(\boldsymbol{r},t) = \psi_0\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_\mathrm{m}\cdot\boldsymbol{r} - \omega t)}, \qquad -\boldsymbol{k}_\mathrm{m}^2 + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu = 0.$$

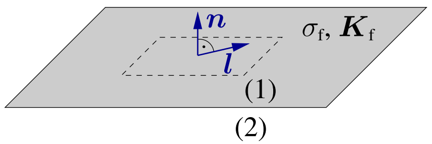
(Mixture of TD and FD expressions; $\tilde{\ }$, $\bar{\ }$, Re, $1/2$, c.c. omitted; sloppy, but common.)

- Medium: refractive index: $n = \sqrt{\epsilon\mu}$
- Periodicity in time:
  - angular frequency: ω,
  - frequency: $f = \omega/(2\pi)$,
  - period: $T = 1/f = 2\pi/\omega$,
- Spatial periodicity:
  - wave vector: $\boldsymbol{k}_\mathrm{m}$, $k_\mathrm{m} = |\boldsymbol{k}_\mathrm{m}|$,
  - wavenumber: $k_\mathrm{m} = \omega/\mathrm{c}_\mathrm{m} = (\omega/\mathrm{c})n = k\,n$,
  - vacuum wavenumber: $k = \omega/\mathrm{c}$,
  - vacuum wavelength: $\lambda = 2\pi/k = 2\pi\mathrm{c}/\omega$,
  - wavelength in the medium: $\lambda_\mathrm{m} = 2\pi/k_\mathrm{m} = 2\pi/(kn) = \lambda/n$.
- Phase velocity:
  - speed of light in vacuum: $\mathrm{c} = 1/\sqrt{\epsilon_0\mu_0} = \lambda f$,
  - in the medium: $\mathrm{c}_\mathrm{m} = \mathrm{c}/n = \lambda_\mathrm{m} f$.

(Use of symbols depends highly on context.)

Electromagnetic spectrum

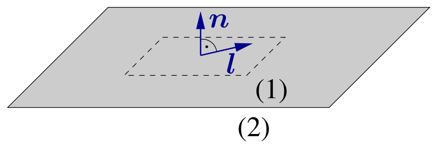
Interface conditions



Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface charge density σ_f, surface current density $\boldsymbol{K}_\mathrm{f}$:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma_\mathrm{f}, \quad \boldsymbol{l}\cdot(\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0,$$
$$\boldsymbol{n}\cdot(\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \quad \boldsymbol{l}\cdot(\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{l}\cdot(\boldsymbol{K}_\mathrm{f}\times\boldsymbol{n}).$$

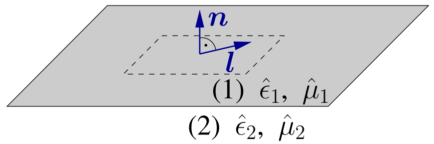
Interface conditions



Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface without free charges or currents:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1 - \boldsymbol{D}_2) = 0, \quad \boldsymbol{l}\cdot(\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0,$$
$$\boldsymbol{n}\cdot(\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \quad \boldsymbol{l}\cdot(\boldsymbol{H}_1 - \boldsymbol{H}_2) = 0\,.$$

Interface conditions

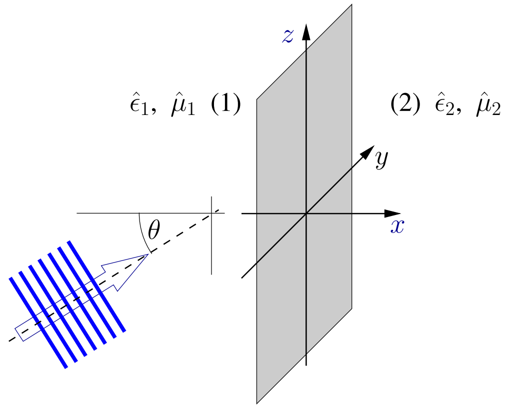


Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, linear media with permittivities $\hat{\epsilon}_1$, $\hat{\epsilon}_2$, and permeabilities $\hat{\mu}_1$, $\hat{\mu}_2$:

$$\boldsymbol{n}\cdot(\hat{\epsilon}_1\boldsymbol{E}_1-\hat{\epsilon}_2\boldsymbol{E}_2)=0,\qquad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$
$$\boldsymbol{n}\cdot(\hat{\mu}_1\boldsymbol{H}_1-\hat{\mu}_2\boldsymbol{H}_2)=0,\qquad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=0.$$

Reflection and transmission of plane waves at dielectric interfaces

(FD)



- $\boldsymbol{\nabla}\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}$,
  $\boldsymbol{\nabla}\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}$
- $\hat{\epsilon}(\boldsymbol{r})$ and $\hat{\mu}(\boldsymbol{r})$ are constant along y, z

$$\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{E}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)},\quad \boldsymbol{H}(\boldsymbol{r})=\boldsymbol{H}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)}$$

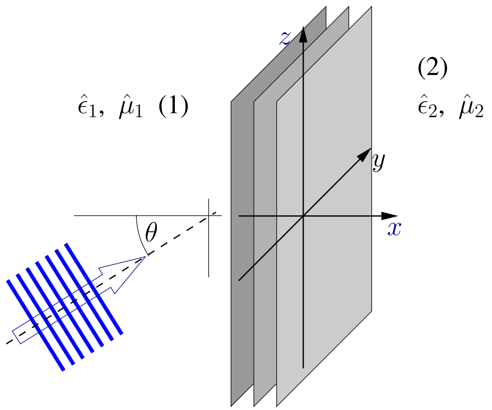
1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

(incoming plane wave at angle θ)
(orient coordinates ($k_y=0$), plane of incidence, distinguish polarizations)
(write ansatz functions for incoming, reflected, and transmitted waves)
(interface conditions determine the amplitudes)

Fresnel equations.

Dielectric multilayer structures

(FD)



- $\boldsymbol{\nabla}\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}$,
  $\boldsymbol{\nabla}\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}$
- $\hat{\epsilon}(\boldsymbol{r})$ and $\hat{\mu}(\boldsymbol{r})$ are constant along y, z

$$\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{E}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)},\quad \boldsymbol{H}(\boldsymbol{r})=\boldsymbol{H}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)}$$

1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

(. . .)
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Reflectance and transmittance properties.

Energy of electromagnetic fields

(TD)

- Force on a particle with charge q, velocity $\boldsymbol{v}$, in a field $\boldsymbol{E},\boldsymbol{B}$:
  $\boldsymbol{F}=q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$,
- work for shifting the particle by $\mathrm{d}\boldsymbol{r}=\boldsymbol{v}\,\mathrm{d}t$:
  $\mathrm{d}W=\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}=q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v}\,\mathrm{d}t=q\boldsymbol{E}\cdot\boldsymbol{v}\,\mathrm{d}t$,
- respective power: $\quad \dfrac{\mathrm{d}W}{\mathrm{d}t}=q\boldsymbol{E}\cdot\boldsymbol{v}$.

For a charge density $\rho_{\mathrm{f}}(\boldsymbol{r},t)$:

force density $\boldsymbol{f}=\rho_{\mathrm{f}}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$,

power density $\boldsymbol{f}\cdot\boldsymbol{v}=\rho_{\mathrm{f}}\boldsymbol{E}\cdot\boldsymbol{v}=\boldsymbol{J}_{\mathrm{f}}\cdot\boldsymbol{E}$,

total work per time unit done in $\mathcal{V}$: $\quad \dfrac{\mathrm{d}W_{\mathcal{V}}}{\mathrm{d}t}=\displaystyle\int_{\mathcal{V}}\boldsymbol{J}_{\mathrm{f}}\cdot\boldsymbol{E}\,\mathrm{d}\mathcal{V}$.

Power & energy density, Poynting theorem

(TD)

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}},$$

$$\hookrightarrow \frac{\mathrm{d}}{\mathrm{d}t} W^\mathrm{mech}_\mathcal{V} = \int_\mathcal{V} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}\, \mathrm{d}\mathcal{V} = -\int_\mathcal{V} (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})\, \mathrm{d}\mathcal{V} - \int_\mathcal{V} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H})\, \mathrm{d}\mathcal{V},$$

- Poynting vector: $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, (energy flux density, power density)
- energy density: $w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})$, $W^\mathrm{field}_\mathcal{V} = \int_\mathcal{V} w\, \mathrm{d}\mathcal{V}$,
- $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\epsilon}\,\cancel{(\omega)}$, $\boldsymbol{D} = \epsilon_0 \hat{\epsilon} \boldsymbol{E}$, $\hat{\mu}^\dagger = \hat{\mu}$, $\hat{\mu}\,\cancel{(\omega)}$, $\boldsymbol{B} = \mu_0 \hat{\mu} \boldsymbol{H}$ (!)
  $\rightsquigarrow$ $\dot{w} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$

$\mathcal{V}$ arbitrary

$$\hookrightarrow \dot{w} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(W^\mathrm{mech}_\mathcal{V} + W^\mathrm{field}_\mathcal{V}\right) = -\oint_{\partial\mathcal{V}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}.$$

Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (. . .) linear media:

$$w = \frac{1}{2}(\epsilon_0 \boldsymbol{E} \cdot \hat{\epsilon} \boldsymbol{E} + \mu_0 \boldsymbol{H} \cdot \hat{\mu} \boldsymbol{H}), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \quad \dot{w} + \nabla \cdot \boldsymbol{S} = 0,$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\, \tilde{\boldsymbol{E}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{H}(\boldsymbol{r}, t) = \mathrm{Re}\, \tilde{\boldsymbol{H}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}$$

$\hookrightarrow$ $\boldsymbol{S}$, w oscillate in time.

(FD)

Consider time-averaged quantities: $\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t')\, \mathrm{d}t'$

(exercise)

$$\hookrightarrow \bar{w} = \frac{1}{4} \mathrm{Re}\left(\epsilon_0 \tilde{\boldsymbol{E}}^* \cdot \hat{\epsilon} \tilde{\boldsymbol{E}} + \mu_0 \tilde{\boldsymbol{H}}^* \cdot \hat{\mu} \tilde{\boldsymbol{H}}\right), \quad \bar{\boldsymbol{S}} = \frac{1}{2} \mathrm{Re}\left(\tilde{\boldsymbol{E}}^* \times \tilde{\boldsymbol{H}}\right).$$

$$\dot{\bar{w}} = \bar{\dot{w}} = 0, \quad \overline{\nabla \cdot \boldsymbol{S}} = \nabla \cdot \bar{\boldsymbol{S}} \quad \rightsquigarrow \quad \nabla \cdot \bar{\boldsymbol{S}} = 0, \quad \oint_\mathcal{V} \bar{\boldsymbol{S}} \cdot \mathrm{d}\boldsymbol{a} = 0;$$

"power balance", conservation of energy.

Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:
Ohm's law $\boldsymbol{J}_\mathrm{f} = \sigma \boldsymbol{E}$, σ: conductivity of the material.

$$\nabla \cdot \boldsymbol{D} = \rho_\mathrm{f}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \dot{\boldsymbol{D}}.$$

$$\hookrightarrow \dot{\rho}_\mathrm{f} = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_\mathrm{f}, \quad \rho_\mathrm{f}(\boldsymbol{r}, t) = \rho_\mathrm{f}(\boldsymbol{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0)\right),$$

assume $\rho_\mathrm{f}(\boldsymbol{r}, t_0) = 0 \rightsquigarrow \rho_\mathrm{f}(\boldsymbol{r}, t) = 0 \quad \forall t.$

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon_0 \epsilon \dot{\boldsymbol{E}}.$$

Telegrapher equation

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon_0 \epsilon \dot{\boldsymbol{E}}$$

$$\hookrightarrow \Delta \boldsymbol{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\boldsymbol{E}} - \mu_0 \mu \sigma \dot{\boldsymbol{E}} = 0, \quad \Delta \boldsymbol{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\boldsymbol{H}} - \mu_0 \mu \sigma \dot{\boldsymbol{H}} = 0,$$

Telegrapher equation.

Frequency domain: $\boldsymbol{E}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}$,

$$\hookrightarrow \Delta \tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2} \epsilon \mu - \mathrm{i}\omega \mu_0 \mu \sigma\right) \tilde{\boldsymbol{E}} = 0.$$

Nonconducting media $\sigma = 0$, $\Delta \tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2} \epsilon \mu\right) \tilde{\boldsymbol{E}} = 0.$

Define $\bar{\epsilon}$ such that $\frac{\omega^2}{\mathrm{c}^2} \bar{\epsilon} \mu = \frac{\omega^2}{\mathrm{c}^2} \epsilon \mu - \mathrm{i}\omega \mu_0 \mu \sigma$, i.e. $\bar{\epsilon} = \epsilon - \mathrm{i} \frac{\sigma}{\epsilon_0 \omega}$

$$\hookrightarrow \Delta \tilde{\boldsymbol{E}} + k^2 \bar{\epsilon} \mu \tilde{\boldsymbol{E}} = 0,$$ Helmholtz equation, $\bar{\epsilon} \in \mathbb{C}$, $k = \frac{\omega}{\mathrm{c}}$.

Wave attenuation

$$\Delta\tilde{\boldsymbol{E}} + k^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}} = 0, \quad \bar{\epsilon} \in \mathbb{C}$$

solutions $\sim \mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)}$

with refractive index $\bar{n} = n' - \mathrm{i}n'' = \sqrt{\bar{\epsilon}\mu} \in \mathbb{C}$, (!)

$$\mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)} = \mathrm{e}^{-\mathrm{i}(kn'z - \omega t)}\,\mathrm{e}^{-kn''z},$$

damped plane wave solutions

$\leftrightsquigarrow$ sign of n'', choice of $\exp(\pm\mathrm{i}\omega)$.

Issues:

- penetration depth,
- $\boldsymbol{S}$ and w decay with z,
- still transverse waves,
- $\boldsymbol{E}$, $\boldsymbol{H}$ no longer in phase,
- notions of wavenumber, wavelength, phase velocity $\in \mathbb{C}$.

Simulations in integrated optics

A typical setting:

- "uncharged dielectric medium": $\cancel{q_{\mathrm{f}}}$, $\cancel{\boldsymbol{J}_{\mathrm{f}}}$.
- "linear medium": $\boldsymbol{D} = \epsilon_0\hat{\epsilon}\boldsymbol{E}$, $\boldsymbol{B} = \mu_0\hat{\mu}\boldsymbol{H}$.
- "isotropic medium": $\hat{\epsilon} = \epsilon\,\hat{1}$, $\hat{\mu} = \mu\,\hat{1}$.
- "nonmagnetic medium": $\hat{\mu} = \hat{1}$.
- "lossless medium": $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\mu}^\dagger = \hat{\mu}$, $(\epsilon, \mu \in \mathbb{R})$.
- "piecewise constant" $\rightarrow$ "dependent on position".
- "electric and magnetic field": eliminate $\boldsymbol{D}$ and $\boldsymbol{B}$, retain $\boldsymbol{E}$ and $\boldsymbol{H}$.
- "governed by the curl equations": divergence eqns. are satisfied.
- "frequency domain, time harmonic fields, frequency, wavelength": . . . as discussed.

Upcoming

Next lectures:

- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.