

# Optical Waveguide Theory (E)



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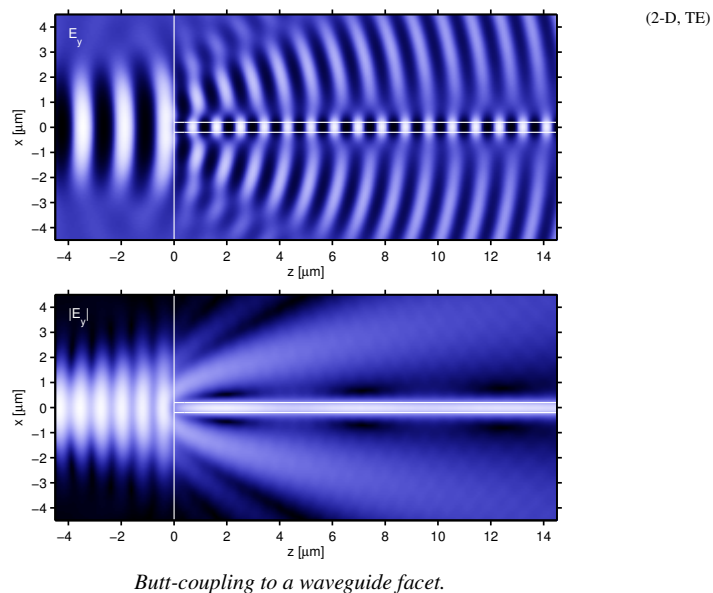
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## Course overview

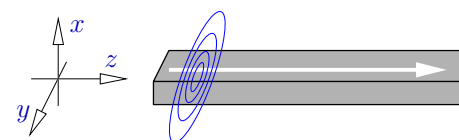
### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

### Context: Relevance of guided modes



### Waveguides: Mode problems



$$\mu = 1, \epsilon = n^2, \sim \exp(i\omega t) \quad (\text{FD})$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H},$$

$$\nabla \times \mathbf{H} = i\omega\epsilon_0\epsilon\mathbf{E}.$$

- **Waveguide:** a system that is homogeneous along its axis  $z$ ,  
 $\partial_z \epsilon = 0, \partial_z n = 0.$

- Look for solutions (**modes**) that vary harmonically with  $z$ :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z},$$

mode profile  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ ,  
propagation constant  $\beta$ ,  
effective index  $n_{\text{eff}} = \beta/k.$

$$\partial_z \longrightarrow -i\beta, \quad (\& \text{ boundary conditions})$$

⚡ **Eigenvalue** problem with eigenvalue  $\beta$ , eigenfunction  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ ,  
“ $\mathbf{M}(\beta) \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} = 0$ ”.

## Mode equations

(drop  $\bar{\phantom{x}}$ )

$$\hookrightarrow \begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express  $E_x, E_y, E_z, H_z$  through principal components  $H_x, H_y$ :

$$\hookrightarrow \begin{aligned} \partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x &= 0, \\ \epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y &= 0, \end{aligned}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{1}{\omega\epsilon_0\epsilon} \begin{pmatrix} \beta H_y - \beta^{-1}(\partial_{yx} H_x + \partial_y^2 H_y) \\ -\beta H_x + \beta^{-1}(\partial_{xy} H_y + \partial_x^2 H_x) \\ -i(\partial_x H_y - \partial_y H_x) \end{pmatrix}, \quad \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ -i\beta^{-1}(\partial_x H_x + \partial_y H_y) \end{pmatrix}.$$

( $H_x, H_y$  are continuous for all  $x, y$ .)

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## Mode equations

(drop  $\bar{\phantom{x}}$ )

$$\hookrightarrow \begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express  $E_x, E_y, H_x, H_y$  through principal components  $E_z, H_z$ :

$$\hookrightarrow (\dots).$$

( $E_z, H_z$  are usually small components.)

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## Mode equations

(drop  $\bar{\phantom{x}}$ )

$$\hookrightarrow \begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express  $H_x, H_y, H_z, E_z$  through principal components  $E_x, E_y$ :

$$\hookrightarrow (\dots).$$

( $E_x, E_y$  are discontinuous at specific interfaces.)

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## Plane mode profiles

- Modes are eigenfunctions  
 $\longleftrightarrow$  profiles are determined up to a complex constant only.

- Propagating modes,  $\beta \in \mathbb{R}$ , lossless structures,  $\epsilon \in \mathbb{R}$ :

$$E_z := iE'_z, \quad H_z := iH'_z \rightsquigarrow \text{real PDE for } E_x, E_y, E'_z, H_x, H_y, H'_z:$$

$$\begin{pmatrix} \partial_y E'_z + \beta E_y \\ -\beta E_x - \partial_x E'_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ -H'_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H'_z + \beta H_y \\ -\beta H_x - \partial_x H'_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ -E'_z \end{pmatrix};$$

it is possible to choose a phase such that

$E_x, E_y, H_x, H_y$  are real,

$E_z, H_z$  are imaginary

$\longleftrightarrow$  plane mode profiles.

(It makes sense to prepare real plots of mode profile components.)  
 (That requires a suitable adjustment of the global phase.)

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## Guided modes

- Guided modes: profiles located “around” the waveguide core

↔ discrete  $\beta \in \mathbb{R}$ ,  $\iint S_z dx dz < \infty$ .

- In general: Hybrid modes, all six field components present. Planar-like waveguides ↔ adapt 2-D naming scheme; “TE-like” / “TM-like” modes.

(↔ 5-component semivectorial approximations, plane  $\perp$  x-axis:  
 quasi-TE: tiny  $E_x$ , dominant  $E_y$ , small  $E_z$ ; major  $H_x$ , small  $H_y$ , minor  $H_z$ ,  
 quasi-TM: tiny  $H_x$ , dominant  $H_y$ , small  $H_z$ ; major  $E_x$ , small  $E_y$ , minor  $E_z$ .)

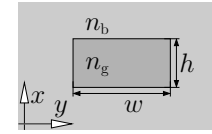
- Mode indices mostly relate to numbers of nodal lines in the dominant electric or magnetic field component.

(Naming schemes are highly context dependent.)



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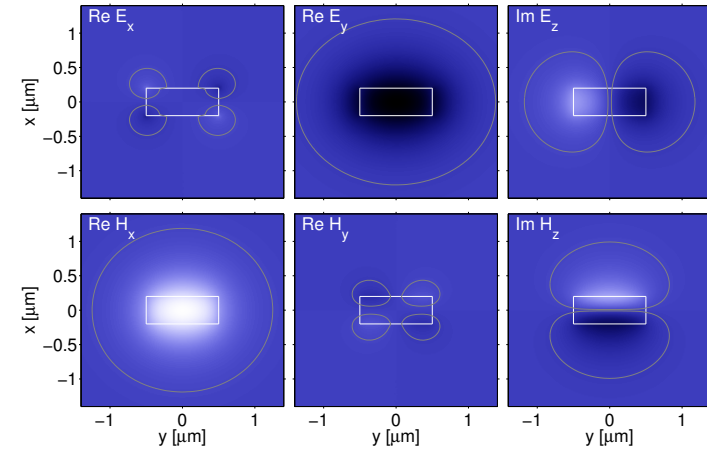
## A rectangular strip waveguide, fundamental mode profiles



$\lambda = 1.55 \mu\text{m}$ ,  
 $n_b = 1.45$ ,  
 $n_g = 1.99$ ,  
 $w = 1.0 \mu\text{m}$ ,  
 $h = 0.4 \mu\text{m}$ ;

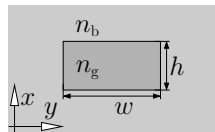
$x \in [-2, 2] \mu\text{m}$ ,  
 $y \in [-2, 2] \mu\text{m}$ ;  
 $n_{\text{eff}} = 1.63554$   
 [JCMwave].

(q-)TE<sub>00</sub>



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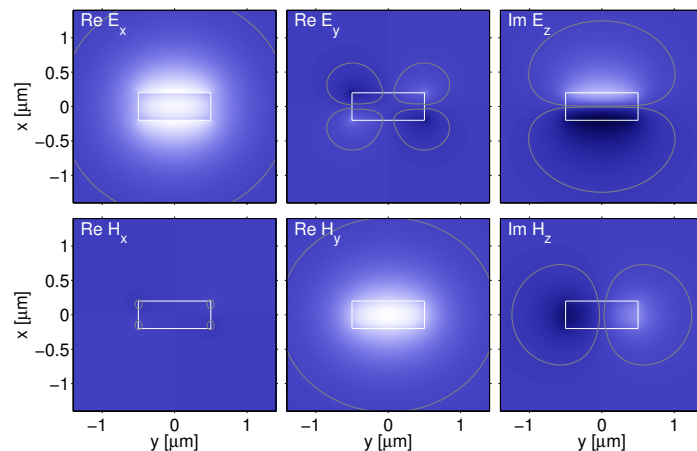
## A rectangular strip waveguide, fundamental mode profiles



$\lambda = 1.55 \mu\text{m}$ ,  
 $n_b = 1.45$ ,  
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 $w = 1.0 \mu\text{m}$ ,  
 $h = 0.4 \mu\text{m}$ ;

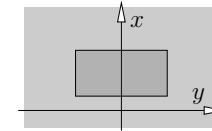
$x \in [-2, 2] \mu\text{m}$ ,  
 $y \in [-2, 2] \mu\text{m}$ ;  
 $n_{\text{eff}} = 1.56809$

(q-)TM<sub>00</sub> [JCMwave].



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## Symmetric waveguides



Waveguide with mirror symmetry  $y \rightarrow -y$ :  
 ↔ modes have a definite parity.

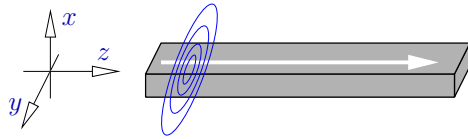
$$\begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

↪ Equal parity of  $H_x, E_y, H_z$ , reversed parity of  $E_x, H_y, E_z$ .



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## Directional modes



(FD)  $\sim \exp(i\omega t)$

Longitudinally homogeneous waveguide: mirror symmetry  $z \rightarrow -z$ .

$$\begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

forward:  $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^f \\ \bar{\mathbf{H}}^f \end{pmatrix}(x, y) e^{-i\beta z}, \quad \begin{matrix} \bar{\mathbf{E}}^f = (E_x, E_y, E_z), \\ \bar{\mathbf{H}}^f = (H_x, H_y, H_z), \end{matrix}$

backward:  $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^b \\ \bar{\mathbf{H}}^b \end{pmatrix}(x, y) e^{+i\beta z}, \quad \begin{matrix} \bar{\mathbf{E}}^b = (E_x, E_y, -E_z), \\ \bar{\mathbf{H}}^b = (-H_x, -H_y, H_z). \end{matrix}$

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## Mode orthogonality

- A set of guided modes of the same waveguide ( $\epsilon$ ):

$\beta \in \mathbb{R}$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z}, \quad \begin{matrix} \nabla \times \mathbf{E}_m = -i\omega\mu_0 \mathbf{H}_m, \\ \nabla \times \mathbf{H}_m = i\omega\epsilon_0 \mathbf{E}_m, \\ \beta_l \neq \beta_m, \text{ if } l \neq m. \end{matrix}$$

- $P_m = \frac{1}{4} \iint (E_{mx}^* H_{my} - E_{my}^* H_{mx} + E_{mx} H_{my}^* - E_{my} H_{mx}^*) dx dy$ .
- $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .
- $\nabla \cdot (\mathbf{E}_l^* \times \mathbf{H}_m + \mathbf{E}_m \times \mathbf{H}_l^*) = 0$  for all  $l, m$

$$0 = i(\beta_l - \beta_m) \left\{ \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m + \bar{\mathbf{E}}_m \times \bar{\mathbf{H}}_l^*)_z dx dy \right\} e^{i(\beta_l - \beta_m)z},$$

$$(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \iint (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx dy$$

$$(\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m) = \begin{cases} 0, & \text{if } l \neq m, \\ P_m, & \text{otherwise.} \end{cases}$$

(The modes are "power orthogonal".)  
(Statements hold for propagating guided modes.)  
( $(\cdot, \cdot; \cdot, \cdot)$  is frequently used for mode normalization.)

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## Modal power

- E.m. power density:  $\mathbf{S} = \frac{1}{2} \text{Re} (\mathbf{E}^* \times \mathbf{H})$ .

(FD)  $\sim \exp(i\omega t)$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix}(x, y) e^{-i\beta z}, \quad \begin{matrix} \bar{\mathbf{E}} = a(\bar{E}_x, \bar{E}_y, i\bar{E}_z'), \\ \bar{\mathbf{H}} = a(\bar{H}_x, \bar{H}_y, i\bar{H}_z'), \\ a \in \mathbb{C}, \bar{E}_x, \dots, \bar{H}_z' \in \mathbb{R}, \\ \text{a guided mode, } \beta \in \mathbb{R}. \end{matrix}$$

$$\mathbf{S} = \frac{|a|^2}{2} \begin{pmatrix} 0 \\ 0 \\ \bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x \end{pmatrix},$$

or  $S_x = 0, S_y = 0, S_z = \frac{1}{2} \text{Re} (E_x^* H_y - E_y^* H_x)$ . ( $S_z(x, y)$ )

- Power carried by the mode:

$$P = \iint S_z dx dy = \frac{1}{4} \iint (E_x^* H_y - E_y^* H_x + E_x H_y^* - E_y H_x^*) dx dy$$

(backward mode.  $E_x \rightarrow E_x, E_y \rightarrow E_y, H_x \rightarrow -H_x, H_y \rightarrow -H_y; P \rightarrow -P$ )

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## Power transport by a mode superposition

- A set of guided modes of the same waveguide ( $\epsilon$ ):

$\beta \in \mathbb{R}$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z}, \quad P_m = (\mathbf{E}_m, \mathbf{H}_m; \mathbf{E}_m, \mathbf{H}_m).$$

- Superposition with amplitudes  $a_m \in \mathbb{C}$ :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \sum_m a_m \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \sum_m a_m \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z}.$$

Power flow along the waveguide:

$$\begin{aligned} \iint S_z dx dy &= (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) \\ &= \sum_l \sum_m a_l^* a_m (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m) \\ &= \sum_m |a_m|^2 P_m. \end{aligned}$$

(Forward / backward modes:  $P \geq 0$ .)

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## Mode interference

- Two modes  $m = 1, 2$ :

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z}.$$

$\beta \in \mathbb{R}$

- Superposition with amplitudes  $a_1, a_2$ :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = a_1 \begin{pmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{H}}_1 \end{pmatrix}(x, y) e^{-i\beta_1 z} + a_2 \begin{pmatrix} \bar{\mathbf{E}}_2 \\ \bar{\mathbf{H}}_2 \end{pmatrix}(x, y) e^{-i\beta_2 z}.$$

- Fix a position  $x, y$  and component  $F$ :

$$F(z) = a_1 \bar{F}_1 e^{-i\beta_1 z} + a_2 \bar{F}_2 e^{-i\beta_2 z}, \quad r e^{-i\phi} := a_1^* a_2 \bar{F}_1^* \bar{F}_2, \quad \text{Omit } (x, y).$$

$$\hookrightarrow |F|^2(z) = |a_1|^2 |\bar{F}_1|^2 + |a_2|^2 |\bar{F}_2|^2 + 2r \cos((\beta_1 - \beta_2)z + \phi).$$

Periodic beating pattern with half-beat-length  $L_c = \frac{\pi}{|\beta_1 - \beta_2|}$ .

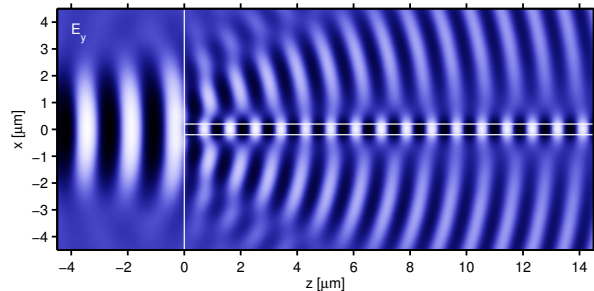
(Supermodes ) (Evanescent coupling )

(“Coupling length”  $L_c$ .)

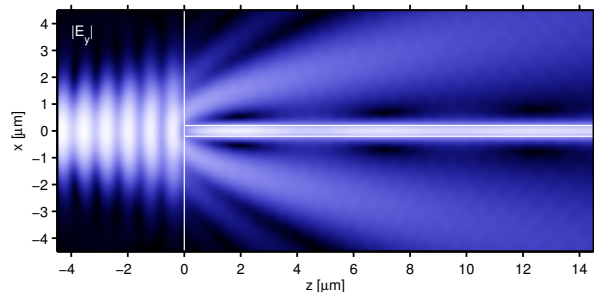
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## What about non-guided fields?



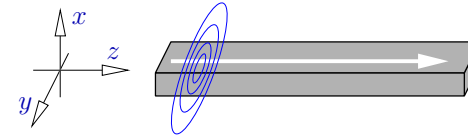
(2-D, TE)



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## Polarization of a guided wave field



Unidirectional guided waves in a “long” dielectric channel that supports fundamental TE- and TM-like modes only:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = a_{\text{TE}} \begin{pmatrix} \bar{\mathbf{E}}_{\text{TE}} \\ \bar{\mathbf{H}}_{\text{TE}} \end{pmatrix}(x, y) e^{-i\beta_{\text{TE}} z} + a_{\text{TM}} \begin{pmatrix} \bar{\mathbf{E}}_{\text{TM}} \\ \bar{\mathbf{H}}_{\text{TM}} \end{pmatrix}(x, y) e^{-i\beta_{\text{TM}} z},$$

amplitudes  $a_{\text{TE}}, a_{\text{TM}} \in \mathbb{C}$ .

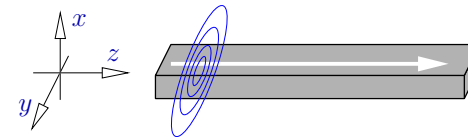
- $E_{\text{TE}z} \neq 0, E_{\text{TM}z} \neq 0$ .
- $\bar{\mathbf{E}}_{\text{TE}}(x, y) \neq \bar{\mathbf{E}}_{\text{TM}}(x, y)$ .
- At  $(x, y)$ : adjust  $\mathbf{E}/|\mathbf{E}|$  via  $a_{\text{TE}}, a_{\text{TM}}$ .
- $a_{\text{TE}}, a_{\text{TM}}$  fixed:  $(\mathbf{E}/|\mathbf{E}|)(x, y)$  varies.

“Polarization” frequently indicates the presence of only one mode.

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## Normal modes: real mode problems



- lossless waveguide,  $\epsilon \in \mathbb{R}$ ,
- “real” boundary conditions at  $x, y$  “far away” from the core,
- “real” vectorial mode equations:

$$\partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x = 0,$$

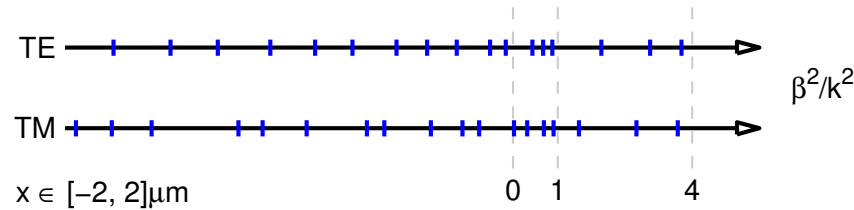
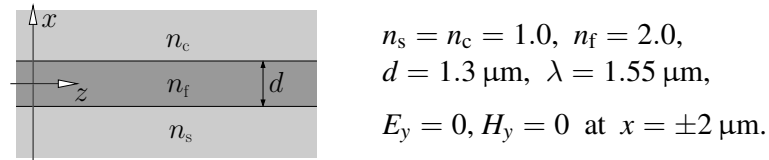
$$\epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y = 0,$$

$\hookrightarrow$  real principal components  $H_x(x, y), H_y(x, y), \beta^2 \in \mathbb{R}$ .

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## 2-D slab waveguide, normal mode spectrum

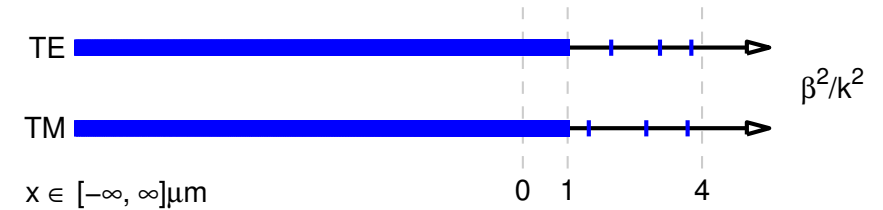
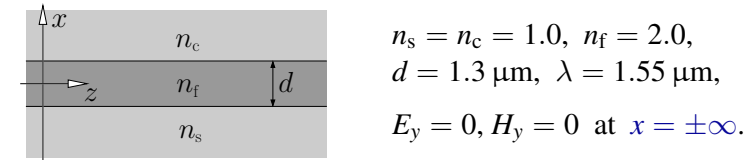


- $n_f^2 < \beta^2/k^2$  : no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$  : **guided modes**.
- $0 < \beta^2/k^2 < n_s^2$  : **propagating radiation modes**.
- $\beta^2/k^2 < 0$  : **evanescent radiation modes**.

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## 2-D slab waveguide, normal mode spectrum



- $n_f^2 < \beta^2/k^2$  : no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$  : **guided modes (discrete spectrum)**.
- $0 < \beta^2/k^2 < n_s^2$  : **propagating radiation modes (continuous spec.)**.
- $\beta^2/k^2 < 0$  : **evanescent radiation modes (continuous spec.)**.

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## Propagating & evanescent modes

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^{f,b} \\ \bar{\mathbf{H}}^{f,b} \end{pmatrix} (x, y) e^{\mp i\beta z}. \quad \sim \exp(i\omega t) \text{ (FD)}$$

- $\beta^2 > 0 \iff \beta = \sqrt{\beta^2}, \beta \in \mathbb{R}, \beta > 0,$   
 $\sim e^{\mp i\beta z}$ , a forward/backward **propagating mode**.  
(Physical relevance of individual modes.)
- $\beta^2 < 0 \iff \beta = -i\sqrt{|\beta^2|} = -i\alpha, \alpha = \sqrt{|\beta^2|} \in \mathbb{R}, \alpha > 0,$   
 $\sim e^{\mp \alpha z}$ , a forward/backward traveling **evanescent mode**.  
 “forward”:  $\sim e^{-\alpha z}$ , field decays with  $z$ ,  
 “backward”:  $\sim e^{+\alpha z}$ , field grows with  $z$ .  
(Relevant for purposes of field expansions.)
- {forward & backward, propagating & evanescent modes}  
 = the set of **normal modes**.

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## Evanescent modes

$$\beta = -i\alpha, \alpha \in \mathbb{R} \quad \epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- **Plane mode profiles**: real PDE for  $E_x, E_y, E_z, iH_x, iH_y, iH_z$ ;  
 common phase with real  $E_x, E_y, E_z$ , imaginary  $H_x, H_y, H_z$ .
- **Directional evanescent modes**:  
 $\{E_x, E_y, E_z, H_x, H_y, H_z; \alpha\}^f \rightsquigarrow \{E_x, E_y, -E_z, -H_x, -H_y, H_z; -\alpha\}^b$ .
- **Modal power**:  
 $\bar{\mathbf{E}} = a(E'_x, E'_y, E'_z),$   
 $\bar{\mathbf{H}} = ia(H'_x, H'_y, H'_z),$   
 $E'_x, \dots, H'_z \in \mathbb{R}, a \in \mathbb{C}$   
 $\hookrightarrow S_z = \frac{1}{2} \text{Re} (E_x^* H_y - E_y^* H_x) = 0, \quad \iint S_z dx dy = 0.$

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## Completeness of normal modes



$$\epsilon \in \mathbb{R}, \sim \exp(i\omega t) \text{ (FD)}$$

A lossless,  $z$ -homogeneous waveguide configuration; **general solution** of the Maxwell equations between cross sectional planes A and B:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_{m \in \mathcal{N}} F_m \begin{pmatrix} \bar{\mathbf{E}}_m^f \\ \bar{\mathbf{H}}_m^f \end{pmatrix} (x, y) e^{-i\beta_m z} + \sum_{m \in \mathcal{N}} B_m \begin{pmatrix} \bar{\mathbf{E}}_m^b \\ \bar{\mathbf{H}}_m^b \end{pmatrix} (x, y) e^{+i\beta_m z}, \quad \Sigma \rightarrow \not\subseteq$$

$\mathcal{N}$ : the set of forward **normal modes** supported by the waveguide.

("Solution": obvious; "general": without proof.)



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## Orthogonality of normal modes

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}. \quad \sim \exp(i\omega t) \text{ (FD)}$$

	$\bar{\mathbf{E}}$	$\bar{\mathbf{H}}$	$\beta$
[prop., f]	$(E'_x, E'_y, iE'_z)$	$(H'_x, H'_y, iH'_z)$	$\beta > 0$
[prop., b]	$(E'_x, E'_y, -iE'_z)$	$(-H'_x, -H'_y, iH'_z)$	$\beta < 0$
[evan., f]	$(E'_x, E'_y, E'_z)$	$(iH'_x, iH'_y, iH'_z)$	$\beta = -i\alpha, \alpha > 0$
[evan., b]	$(E'_x, E'_y, -E'_z)$	$(-iH'_x, -iH'_y, iH'_z)$	$\beta = i\alpha, \alpha > 0$

individual  $E'_x, \dots, H'_z \in \mathbb{R}$ .

$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) := \frac{1}{4} \iint (E_{ax}^* H_{by} - E_{ay}^* H_{bx} + H_{ay}^* E_{bx} - H_{ax}^* E_{by}) dx dy$$

$$\begin{pmatrix} \mathbf{E}_{1,2} \\ \mathbf{H}_{1,2} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_{1,2} \\ \bar{\mathbf{H}}_{1,2} \end{pmatrix} (x, y) e^{-i\beta_{1,2} z}, \quad \begin{aligned} \nabla \times \mathbf{E}_{1,2} &= -i\omega\mu_0 \mathbf{H}_{1,2}, \\ \nabla \times \mathbf{H}_{1,2} &= i\omega\epsilon_0 \mathbf{E}_{1,2}, \end{aligned}$$

$$\nabla \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) = 0 \rightsquigarrow 0 = (\beta_1^* - \beta_2) (\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2).$$



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## Completeness of normal modes

Stronger statement:

"any" transverse 2-component field on a cross sectional plane can be expanded into alternatively

- the transverse electric components of forward normal modes,
- the transverse magnetic components of forward normal modes,
- the transverse electric components of backward normal modes,
- the transverse magnetic components of backward normal modes.

## Orthogonality of normal modes

Nondegenerate directional normal modes of the same waveguide ( $\epsilon$ ):

$$\begin{pmatrix} \mathbf{E}_m^{f,b} \\ \mathbf{H}_m^{f,b} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m^{f,b} \\ \bar{\mathbf{H}}_m^{f,b} \end{pmatrix} (x, y) e^{-i\beta_m^{f,b} z}, \quad \begin{aligned} \nabla \times \mathbf{E}_m &= -i\omega\mu_0 \mathbf{H}_m, \\ \nabla \times \mathbf{H}_m &= i\omega\epsilon_0 \mathbf{E}_m, \end{aligned}$$

$$\beta_l \neq \beta_m, \text{ if } l \neq m.$$

- A propagating mode  $m$ :

$$\begin{aligned} (\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f) &=: P_m, \quad (\bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b; \bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b) = -P_m, \quad P_m \in \mathbb{R}, \\ (\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b) &= (\bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b; \bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f) = 0, \\ (\bar{\mathbf{E}}_m^d, \bar{\mathbf{H}}_m^d; \bar{\mathbf{E}}_l^r, \bar{\mathbf{H}}_l^r) &= (\bar{\mathbf{E}}_l^r, \bar{\mathbf{H}}_l^r; \bar{\mathbf{E}}_m^d, \bar{\mathbf{H}}_m^d) = 0 \text{ for all } l \neq m, \text{ d,r = f,b.} \end{aligned}$$

- An evanescent mode  $m$ :

$$\begin{aligned} (\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f) &= (\bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b; \bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b) = 0, \\ (\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b) &=: P_m, \quad (\bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b; \bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f) = -P_m, \quad P_m \notin \mathbb{R}, \\ (\bar{\mathbf{E}}_m^d, \bar{\mathbf{H}}_m^d; \bar{\mathbf{E}}_l^r, \bar{\mathbf{H}}_l^r) &= (\bar{\mathbf{E}}_l^r, \bar{\mathbf{H}}_l^r; \bar{\mathbf{E}}_m^d, \bar{\mathbf{H}}_m^d) = 0 \text{ for all } l \neq m, \text{ d,r = f,b.} \end{aligned}$$

(This implies orthogonality of propagating and evanescent modes.)

( $1/\sqrt{|P_m|}$  is frequently used for mode normalization.)



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## Power flow associated with a normal mode expansion

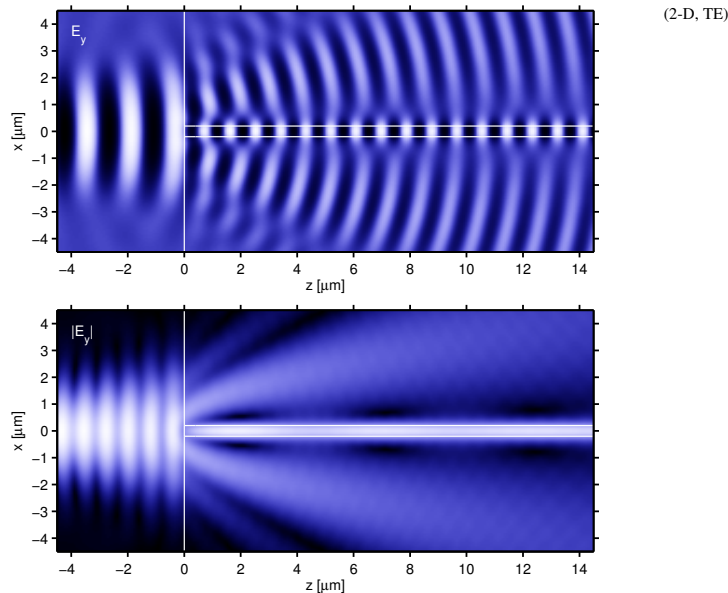
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_{m \in \mathcal{N}} \left\{ F_m \begin{pmatrix} \bar{\mathbf{E}}_m^f \\ \bar{\mathbf{H}}_m^f \end{pmatrix} (x, y) e^{-i\beta_m z} + B_m \begin{pmatrix} \bar{\mathbf{E}}_m^b \\ \bar{\mathbf{H}}_m^b \end{pmatrix} (x, y) e^{+i\beta_m z} \right\}$$

Power carried along  $z$ :

$$\begin{aligned} P &= \iint S_z dx dy = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) \\ &= \sum_{m \text{ propag.}} (|F_m|^2 - |B_m|^2) P_m + \sum_{m \text{ evanes.}} (F_m^* B_m - B_m^* F_m) P_m. \end{aligned}$$

- $P$  is independent of  $z$ .
- Individual contributions from forward and backward propagating modes.
- Contributions from evanescent modes require forward and backward fields to be present.
- Unidirectional field (forward:  $B_m = 0$  for all  $m$ ): Only propagating modes carry power.

## Waveguide facet: Port definition



## Projection onto normal modes



$\mathbf{E}, \mathbf{H}$ : a solution of the Maxwell equations for the  $z$ -homogeneous waveguide between two cross sectional planes A and B.

Extract local mode amplitudes by **projection onto normal modes**:

- A propagating mode  $m$ ,  $\beta_m > 0$ :

$$(\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \mathbf{E}, \mathbf{H}) = F_m P_m e^{-i\beta_m z},$$

$$F_m e^{-i\beta_m z} = \frac{(\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \mathbf{E}, \mathbf{H})}{(\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f)}$$

$$(\bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b; \mathbf{E}, \mathbf{H}) = -B_m P_m e^{i\beta_m z}.$$

- An evanescent mode  $m$ ,  $\beta_m = -i\alpha_m$ ,  $\alpha_m > 0$ :

$$(\bar{\mathbf{E}}_m^f, \bar{\mathbf{H}}_m^f; \mathbf{E}, \mathbf{H}) = B_m P_m e^{\alpha_m z}, \quad (\bar{\mathbf{E}}_m^b, \bar{\mathbf{H}}_m^b; \mathbf{E}, \mathbf{H}) = -F_m P_m e^{-\alpha_m z}.$$

Ports of a photonic integrated circuit.

## Upcoming

Next lectures:

- Examples for dielectric optical waveguides.
- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- Bent optical waveguides; whispering gallery resonances; circular microresonators.

