1. Starting from the representation theorem for functions in \( \mathbb{R}^2 \) (cf. homework exercise IV.2),

\[
    u(x) = \frac{1}{2\pi} \oint_{\partial \Omega} \left\{ \frac{\partial u}{\partial n}(y) \ln \frac{1}{|y - x|} - u(y) \frac{\partial}{\partial n} \ln \frac{1}{|y - x|} \right\} \, ds_y - \frac{1}{2\pi} \int_{\Omega} \Delta u(y) \ln \frac{1}{|y - x|} \, dA_y , \tag{1}
\]

derive the mean value property for harmonic functions in \( \mathbb{R}^2 \):

\[
    u(x) = \frac{1}{2\pi r} \oint_{\partial B(x,r)} u(y) \, ds_y . \tag{2}
\]

Here \( u \) is a function that is harmonic in a bounded domain \( \Gamma \subset \mathbb{R}^2 \). The integral extends over the boundary \( \partial B(x,r) \) of a disk \( B(x,r) \) centered at \( x \in \Gamma \) with a sufficiently small radius \( r \), such that \( B(x,r) \subset \Gamma \).

2. Consider the Dirichlet problem for the Poisson equation on a bounded domain \( \Omega \) in \( \mathbb{R}^2 \), with inhomogeneous boundary conditions:

\[
    \Delta u(x) = F(x) \quad \text{for} \quad x \in \Omega , \quad u(x) = f(x) \quad \text{for} \quad x \in \partial \Omega , \tag{3}
\]

where \( F \) and \( f \) are given continuous functions. Show that a formal solution of the problem (3) can be stated as

\[
    u(x) = - \iint_{\Omega} F(y) G(x,y) \, dA_y - \oint_{\partial \Omega} f(y) \frac{\partial}{\partial n} G(x,y) \, ds_y . \tag{4}
\]

Here \( G \) is the Greens function for the Dirichlet problem of the Laplace equation on \( \Omega \), i.e. \( G \) is given as

\[
    G(x,y) = v(x,y) + \frac{1}{2\pi} \ln \frac{1}{|y - x|} \tag{5}
\]

where for each \( x \in \Omega \) the function \( v \) satisfies the conditions

\[
    \Delta_y v(x,y) = 0 \quad \text{for} \quad y \in \Omega , \quad v(x,y) = - \frac{1}{2\pi} \ln \frac{1}{|y - x|} \quad \text{for} \quad y \in \partial \Omega . \tag{6}
\]

Program: Assume that for the respective domain the functions \( v \) and consequently also \( G \) are known. Start with the representation theorem (1), together with Greens second identity for \( \mathbb{R}^2 \) (homework exercise IV.2), evaluated for \( u \) and \( v \) as above, where integrals and derivatives are meant with respect to \( y \), for fixed \( x \). Then apply a reasoning analogous to the \( \mathbb{R}^3 \)-case (lecture) to derive the solution (4).

3. Consider the Dirichlet problem for a disk \( B(0,\rho) \subset \mathbb{R}^2 \) of radius \( \rho \) around the origin:

\[
    \Delta u(x) = 0 \quad \text{for} \quad x \in B(0,\rho) , \quad u(x) = f(x) \quad \text{for} \quad x \in \partial B(0,\rho) , \tag{7}
\]

with \( f \) a given function on the boundary of the disk. Show that the solution can be written in the form

\[
    u(x) = - \oint_{\partial B(0,\rho)} f(y) \frac{\partial}{\partial n} G(x,y) \, ds_y \tag{8}
\]

where \( G \) is the Greens function for the disk:

\[
    G(x,y) = \frac{1}{2\pi} \ln \frac{1}{|y - x|} - \frac{1}{2\pi} \ln \frac{\rho}{|y - \rho x/y|} . \tag{9}
\]

In order to show that the \( v \)-part of \( G \) (cf. equation (5)) satisfies the Laplace equation inside the disk, you might wish to rewrite certain terms by using the identity \((\rho y/|y| - |y|x/\rho)^2 = (\rho x/|x| - |x|y/\rho)^2\).

Good luck!