1. Show that the function
\[ u(x, y) = \ln((x - x_0)^2 + (y - y_0)^2) \] (1)
satisfies the Laplace equation
\[ \Delta u = u_{xx} + u_{yy} = 0 \] (2)
everywhere in the \(x\)-\(y\)-plane with the exception of the point \((x_0, y_0)\).

2. Consider the 1-D wave equation
\[ u_{tt} = c^2 u_{xx}, \quad \text{for} \quad t \geq 0, \quad -\infty < x < \infty, \] (3)
with initial conditions
\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \] (4)
where \(c\) is a positive constant and \(\varphi, \psi\) are given (smooth) functions.

(a) Verify that the function
\[ u(x, t) = \frac{1}{2} \varphi(x + ct) + \frac{1}{2} \varphi(x - ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds \] (5)
satisfies the initial conditions (4).

(b) Show that, for two arbitrary smooth functions \(f\) and \(b\) of one variable, the expression
\[ u(x, t) = f(x - ct) + b(x + ct) \] (6)
corresponds to a solution of the wave equation (3).

(c) Use the result 2b to show that the function (5) is a solution to equation (3).

3. Consider the first order partial differential equation
\[ u_x + e^x u_y = 1. \] (7)

(a) Apply the transformation procedure for linear first order equations as outlined in the lecture:
   i. Write down the characteristic equation, and determine an explicit parameterized expression for the characteristic lines. Here “characteristics” are meant as curves in the \(x\)-\(y\)-plane.
   ii. Apply a transformation induced by the characteristics, and try to find a general solution of the transformed equation that includes an arbitrary smooth function \(f\). After transforming that result back to the original variables, verify explicitly that your solution satisfies equation (7).

(b) Investigate equation (7) a second time, now by what we called the method of characteristics.
   i. Determine the characteristics of (7), here meant as curves in \(x\)-\(y\)-\(u\)-space.
   ii. Solve the equation (7) around the \(x\)-axis, for Cauchy data \(u(x, 0) = 1\) given on the \(x\)-axis. You might wish to use the results from 3(b)i.
   iii. Find two different solutions of equation (7) for Cauchy data \(u(x, e^x) = x\) on the curve \(y = e^x\).
   iv. Show that there is no solution of equation (7) for Cauchy data \(u(x, e^x) = 1\) (here it is sufficient to use directly the expression for the Cauchy data and the partial differential equation).

\(\text{Good luck!}\)