1. Consider the linear second order PDE with parameter $\alpha$

$$u_{xx} + \alpha u_{xy} + 7 u_{yy} = 0.$$ \hfill (1)

(a) Depending on the value of $\alpha$, determine the type of the PDE (hyperbolic, parabolic, or elliptic).

(b) For $\alpha = -8$,

i. state the characteristic equations of (1),

ii. solve these for the characteristic curves, and sketch some of the characteristics,

iii. apply a change of variables and transform the PDE (1) to its canonical form,

iv. find a general solution of the transformed equation that includes two free functions,

v. transform the solution back to the original variables $x, y$, and

vi. verify that the general solution you’ve found in this way satisfies the original equation (1).

(c) Follow the procedure of part 1b for equation Eq. (1) with $\alpha = \sqrt{28}$.

(d) For $\alpha = \sqrt{3}$, devise a change of variables that transforms (1) to its canonical form.

2. An alternative canonical form for hyperbolic equations:

(a) Find a change of variables $\{x, y; u(x, y)\} \longrightarrow \{\xi, \eta; w(\xi, \eta)\}$ that transforms the hyperbolic equation in standard form

$$u_{xy} + \Phi(u, u_x, u_y, x, y) = 0$$ \hfill (2)

into a PDE of the form

$$w_{\xi\xi} - w_{\eta\eta} + \Psi(w, w_\xi, w_\eta, \xi, \eta) = 0.$$ \hfill (3)

Hint: Try linear superpositions with constant coefficients of the original coordinates for the transformation rules $\{x, y\} \longrightarrow \{\xi, \eta\}$.

(b) Transform the equation

$$v_{rr} - 8 v_{rs} + 7 v_{ss} = 0$$ \hfill (4)

to the alternative canonical form (3).

3. Consider a linear hyperbolic equation with constant coefficients in alternative canonical form, i.e. an equation

$$u_{xx} - u_{yy} + a u_x + b u_y + c u = 0,$$ \hfill (5)

where $a, b,$ and $c$ are constants. Determine constants $\alpha, \beta,$ and $h$ such that the function

$$v(x, y) = e^{\alpha x + \beta y} w(x, y)$$ \hfill (6)

satisfies the following PDE without first order derivatives:

$$v_{xx} - v_{yy} + h v = 0.$$ \hfill (7)

Good luck!