1. Consider the Cauchy problem for the 1-D wave equation:

\[ u_{tt} = c^2 u_{xx} \quad \text{for} \quad -\infty < x < \infty, \ t > 0, \]  
\[ u(x, 0) = \varphi(x), \ u_t(x, 0) = \psi(x), \ \text{for} \quad -\infty < x < \infty. \]  

(a) Evaluate the d’Alembert solution for \( c = 7 \) and given functions \( \varphi(x) = \cos(3x), \ \psi(x) = x. \)

(b) Draw (qualitative) graphs of the solution for \( c = 1 \) and initial values \( \psi = 0, \)

\[ \varphi(x) = \begin{cases} 
\sin(2x) & \text{for} \ -\pi < x < \pi, \\
0 & \text{for} \ |x| > \pi,
\end{cases} \]  

at times \( t = 0, t = \pi/2, t = \pi, \) and \( t = 2\pi. \)

(c) Write the solution for \( c = 2 \) and initial data \( \psi = 0, \ \varphi(x) = \sin(\pi x), \) and draw (qualitative) graphs of the solution at times \( t = 0, t = 1/4, t = 1/2, t = 3/4, t = 1, t = 5/4, \) and \( t = 3/2. \)

2. Duhamel’s principle:

Assume that the functions \( w(x, t; T) \) are known solutions of the homogeneous 1-D wave equation, corresponding to initial values \( f \) given at time \( T:\)

\[ w_{xx} - w_{tt} = 0, \ \text{for} \quad -\infty < x < \infty, \ t > T, \quad w(x, T; T) = 0, \quad w_t(x, T; T) = -f(x; T). \]  

Hence we have an entire series of solutions to initial value problems, with additional parameter \( T, \) where the functions \( w(x, t; T) \) are available for all parameter values \( T > 0. \)

Show that the function \( u \) defined by

\[ u(x, t) = \int_0^t w(x, t; T) \, dT \]  

solves the Cauchy problem for an inhomogeneous wave equation with homogeneous initial conditions:

\[ u_{xx}(x, t) - u_{tt}(x, t) = f(x, t), \ \text{for} \quad -\infty < x < \infty, \ t > 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0. \]  

Hint: It is not necessary to write out \( w \) explicitly; just exploit the properties of the functions as stated in (4).
3. Consider an initial-boundary-value problem for the 1-D wave equation with inhomogeneous boundary condition:

\[ u_{tt} = c^2 u_{xx}, \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad u(0,t) = \chi(t), \quad \text{for } x > 0, \ t > 0. \]  

(7)

Here \( \varphi, \psi, \) and \( \chi \) are given functions. We’ll proceed stepwise towards a solution.

(a) For given solutions \( v, w \) of the two problems with either inhomogeneous initial conditions

\[ v_{tt} = c^2 v_{xx}, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad v(0,t) = 0, \quad \text{for } x > 0, \ t > 0, \]  

(8)

or an inhomogeneous boundary condition

\[ w_{tt} = c^2 w_{xx}, \quad w(x,0) = 0, \quad w_t(x,0) = 0, \quad w(0,t) = \chi(t), \quad \text{for } x > 0, \ t > 0, \]  

(9)

check that the function \( u = v + w \) satisfies all constraints of problem (7).

(b) Recall the solution of problem (8), as discussed in the lecture.

(c) To establish a solution of problem (9), try a reasoning in terms of the characteristic triangle: Nonzero contributions to the solution originate only from the source at \( x = 0 \); in propagating along the characteristics the perturbations can only reach points \((x,t)\) in the first quadrant of the \(x-t\)-plane with \( x < ct \). Clarify this by means of an appropriate sketch.

Then choose an ansatz \( w \) of the form

\[ w(x,t) = f(x - ct), \]  

(10)

and fix the (piecewise defined) function \( f \) such that the solution satisfies the initial- and boundary conditions of problem (9).

(d) Assemble the solution \( u \) of problem (7), and compare with the expression given in the textbook (section 4.5).

\emph{Good luck!}