# *Guided wave interaction in photonic integrated circuits*

M. Hammer*

Integrated Optical MicroSystems
MESA$^+$ Institute for Nanotechnology
University of Twente, The Netherlands

*International Conference on Applied Mathematics — Modeling, Analysis & Computation*
*City University of Hong Kong, Hong Kong, May 28 – June 01, 2012*

* *Department of Electrical Engineering, University of Twente* *P.O. Box 217, 7500 AE Enschede, The Netherlands*
*Phone: +31/53/489-3448* *Fax: +31/53/489-3996* *E-mail: m.hammer@utwente.nl*

# *Photonics*

"… photonics includes the generation, emission, transmission, modulation, signal processing, switching, amplification, detection and sensing of light

… closely related to optics. …"

*www.wikipedia.org*

# *Photonics*

"… photonics includes the generation, emission, transmission, modulation, signal processing, switching, amplification, detection and sensing of light … closely related to optics. …"

*www.wikipedia.org*

Basis: the Maxwell equations of classical electrodynamics (here).

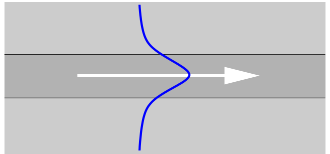
# *Photonic integrated circuits*

"A photonic integrated circuit (PIC) or integrated optical circuit . . .
integrates multiple photonic functions

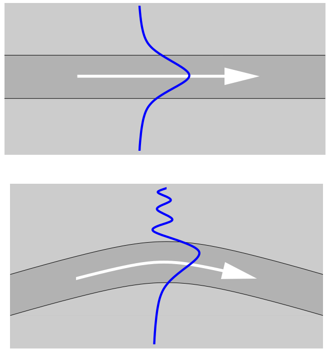
. . . more compact . . . higher performance . . ."

*www.wikipedia.org*

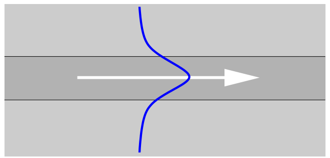
# *Guided wave interaction in photonic integrated circuits*

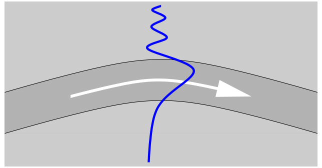


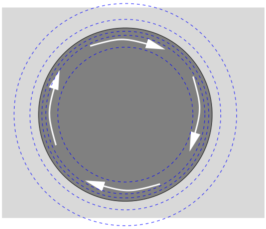
# *Guided wave interaction in photonic integrated circuits*

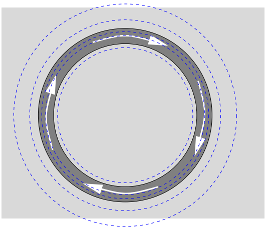


# *Guided wave interaction in photonic integrated circuits*

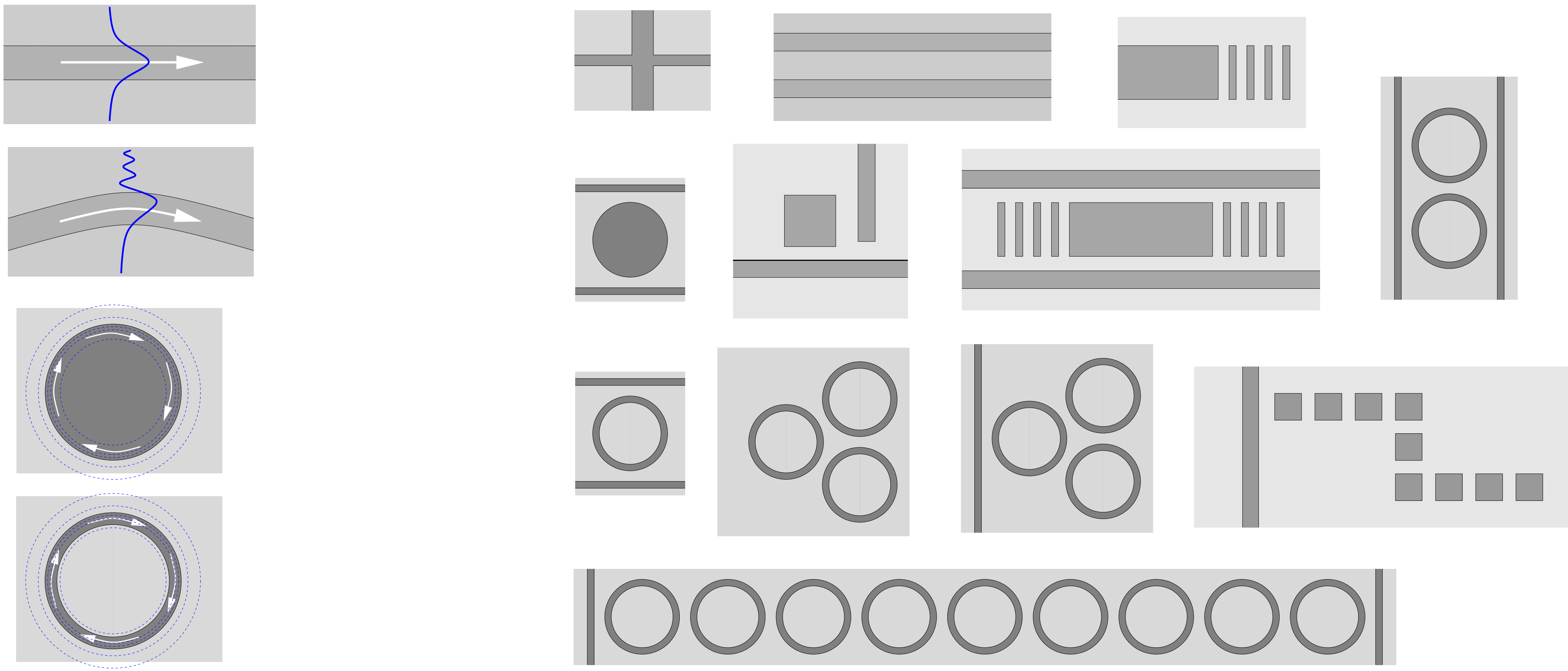




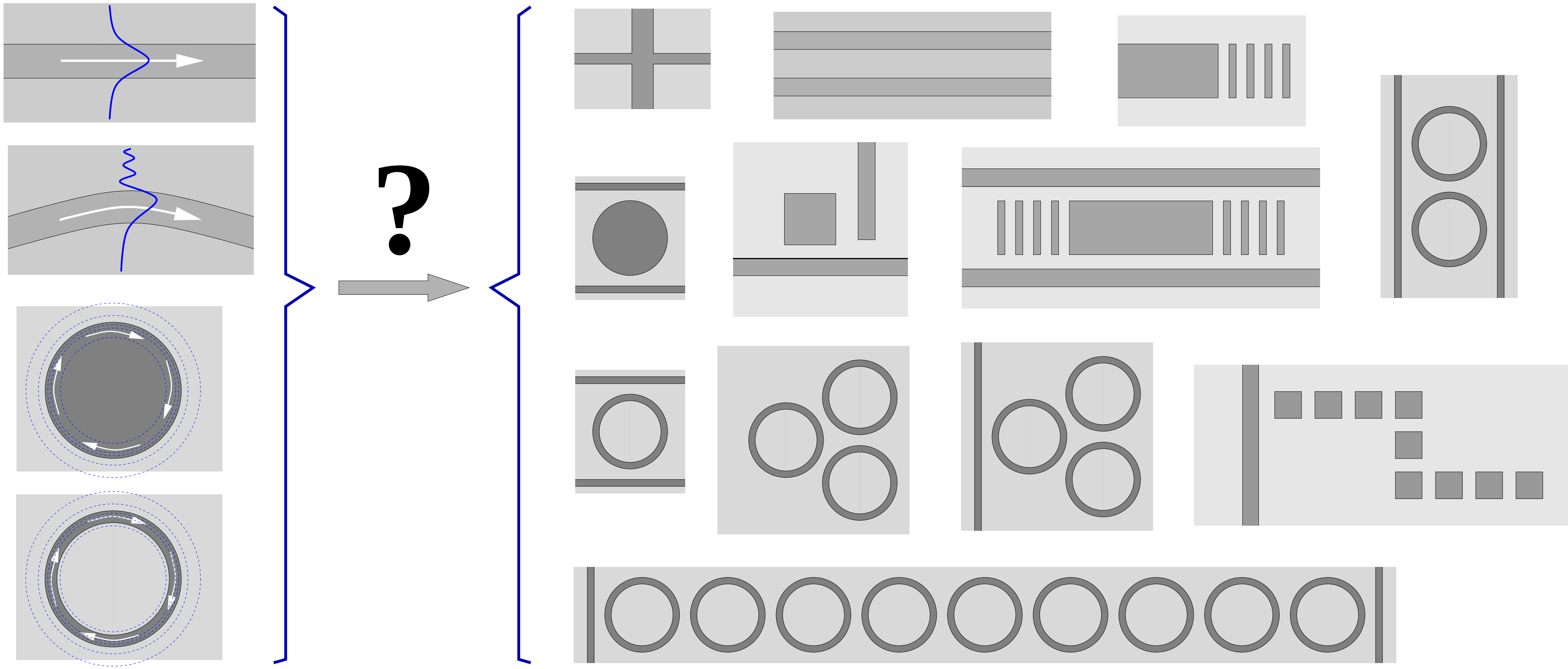




# *Guided wave interaction in photonic integrated circuits*



# *Guided wave interaction in photonic integrated circuits*



## *Outline*

- Simulations in guided wave optics
  - Macroscopic Maxwell equations
  - Straight dielectric waveguides
  - Waveguide bends
  - Whispering gallery resonances
- Hybrid analytical / numerical coupled-mode modeling
  - Field ansatz
  - Amplitude discetization, 1-D FEM
  - Solution
- Examples
  - Coupled straight waveguides
  - Channel crossing
  - Corrugated waveguides
  - Ring resonators
  - Excitation of whispering gallery resonances

## *Macroscopic Maxwell equations*

... for the optical electric and magnetic fields $\boldsymbol{E}$, $\boldsymbol{H}$ (SI)

- frequency domain, $\sim \exp(\mathrm{i}\omega t)$,
- no free currents and charges, no sources, homogeneous equations,
- typical media:
  - nonmagnetic at optical frequencies,
  - linear, isotropic, lossless (transparent) dielectrica

relative permittivity $\hat{\epsilon} = \epsilon 1, \quad \epsilon = n^2$,

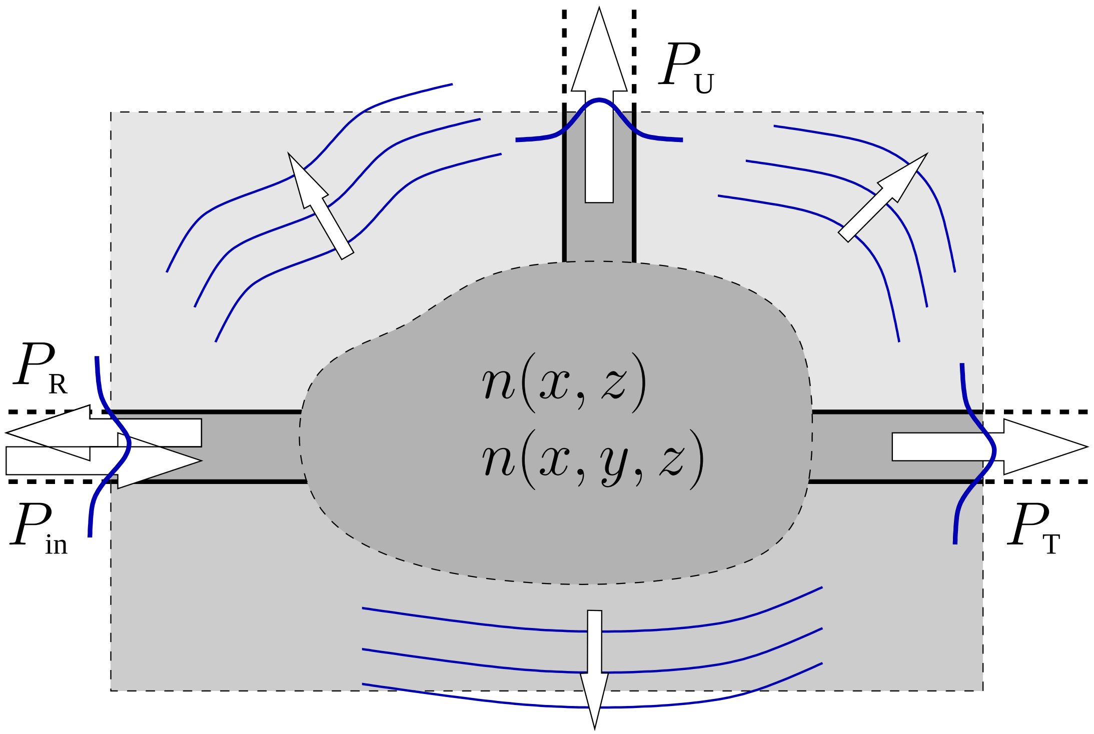
refractive index $n(x, y, z; \omega) \in \mathbb{R}$.

---

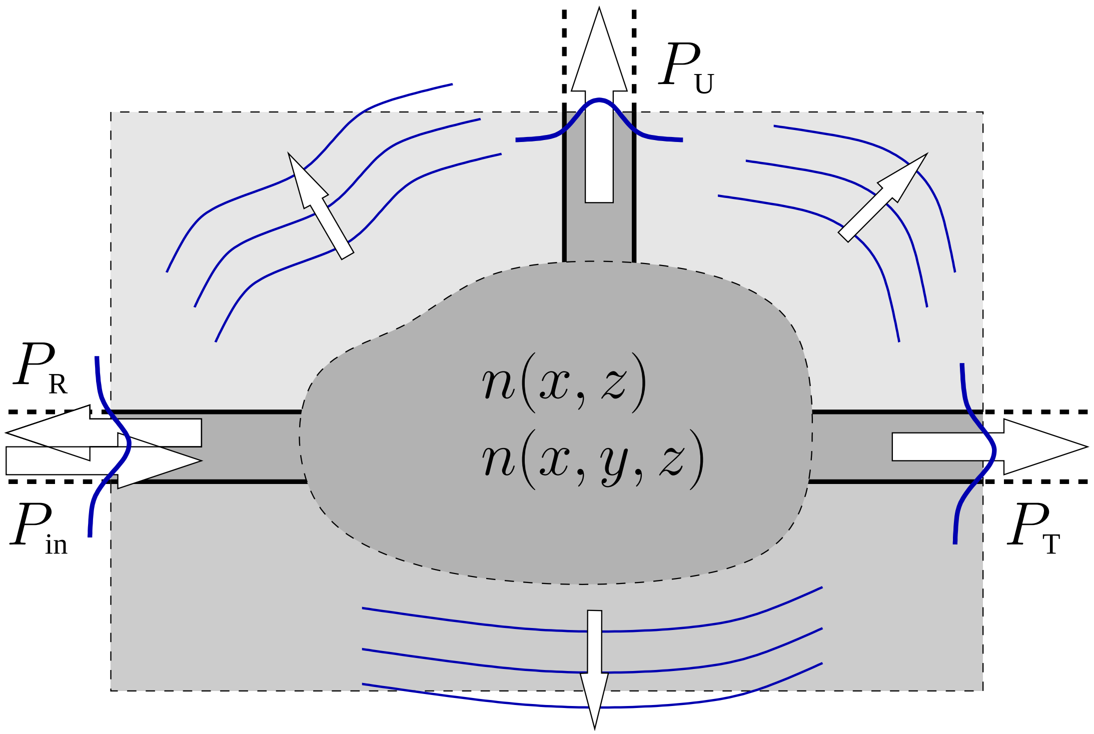
$$\mathrm{curl}\, \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H}, \quad \mathrm{curl}\, \boldsymbol{H} = \mathrm{i}\omega\epsilon_0 \hat{\epsilon} \boldsymbol{E},$$

given excitation frequency $\omega = k\mathrm{c} = 2\pi\mathrm{c}/\lambda$, scans over $\omega$ $\rightsquigarrow$ spectral data.

# *Abstract scattering problem*



## ***Abstract scattering problem***

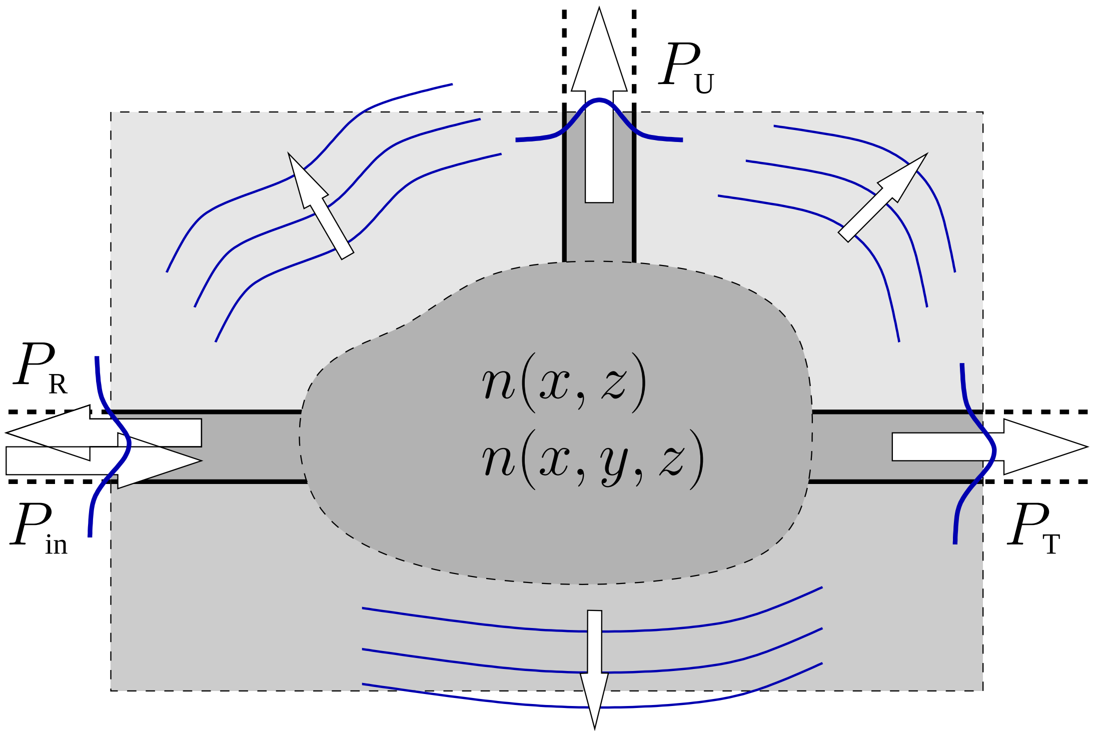



Typical parameters:

- vacuum wavelength $\lambda \in [400, 700]$ nm (visible light), $\lambda \approx 1.3\,\mu$m, $1.55\,\mu$m (optical fibers, attenuation min.),
- refractive indices $n \in [1, 3.4]$.

- Interesting domain: $(10\,\lambda — 100\,\lambda)^d$, $d = 2, 3$ (2-D, 3-D).
- Details: $\approx \lambda/10$, $\approx \lambda/100$.
- Influx and outflux: guided & nonguided waves ⟷ boundary conditions.

## *Abstract scattering problem*



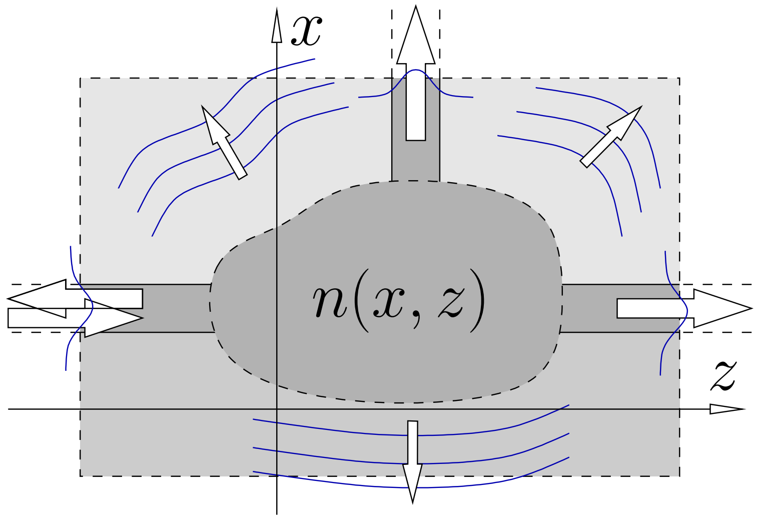

Typical parameters:

- vacuum wavelength
  $\lambda \in [400, 700]$ nm (visible light),
  $\lambda \approx 1.3\,\mu$m, $1.55\,\mu$m
  (optical fibers, attenuation min.),
- refractive indices $n \in [1, 3.4]$.

- Interesting domain: $(10\,\lambda — 100\,\lambda)^d$, $d = 2, 3$ (2-D, 3-D).
- Details: $\approx \lambda/10$, $\approx \lambda/100$.
- Influx and outflux: guided & nonguided waves $\leftrightsquigarrow$ boundary conditions.

- Emphasis: device concepts, design.

## *2-D problems*




$\partial_y \epsilon = 0,\ \epsilon(x,z) = n^2(x,z),$
$\partial_y \boldsymbol{E} = 0,\ \partial_y \boldsymbol{H} = 0;$
equations split into two subsets:

**TE**, $E_y$, $H_x$, and $H_z$, principal component $E_y$:

$$\mathrm{i}\omega\mu_0 H_x = \partial_z E_y, \quad \mathrm{i}\omega\mu_0 H_z = -\partial_x E_y, \quad \mathrm{i}\omega\epsilon_0\epsilon E_y = \partial_z H_x - \partial_x H_z,$$

or

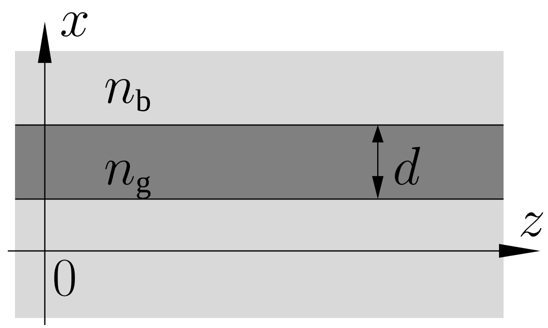
$$\partial_x^2 E_y, + \partial_z^2 E_y + k^2 \epsilon E_y = 0.$$

**TM**, $H_y$, $E_x$, and $E_z$, principal component $H_y$:

$$\mathrm{i}\omega\epsilon_0\epsilon E_x = -\partial_z H_y, \quad \mathrm{i}\omega\epsilon_0\epsilon E_z = \partial_x H_y, \quad -\mathrm{i}\omega\mu_0 H_y = \partial_z E_x - \partial_x E_z,$$

or

$$\partial_x \frac{1}{\epsilon}\partial_x H_y, + \partial_z \frac{1}{\epsilon}\partial_z H_y + k^2 H_y = 0.$$



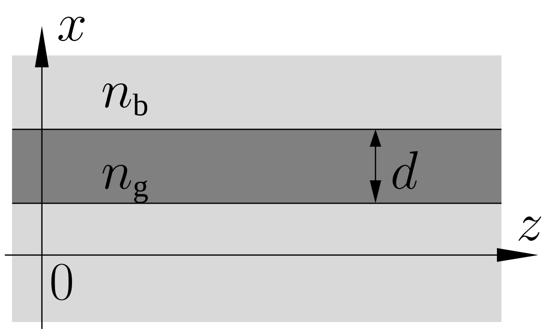

$n_{\mathrm{g}} > n_{\mathrm{b}}$

$\partial_z n = 0$, $\qquad \omega = k\mathrm{c} = 2\pi\mathrm{c}/\lambda$ given,

$E_y(x,z) = \tilde{E}_y(x)\,\mathrm{e}^{-\mathrm{i}\beta z}$,

$$\left(\partial_x^2 + k^2 n^2(x)\right)\tilde{E}_y = \beta^2 \tilde{E}_y \quad \rightsquigarrow \quad \left\{\beta, \tilde{E}_y\right\}.$$

# *Straight dielectric waveguides*




$n_g > n_b$

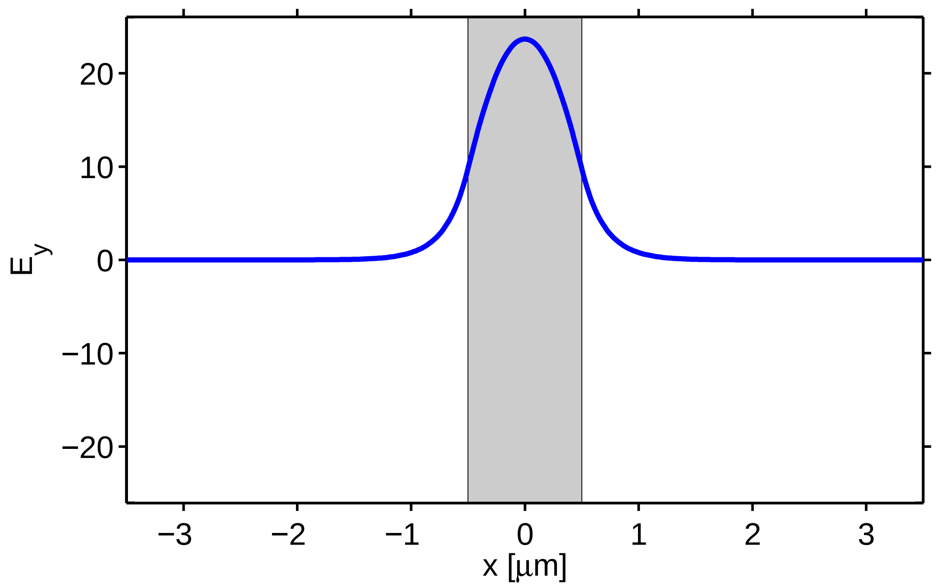
$\partial_z n = 0$, $\omega = kc = 2\pi c/\lambda$ given,

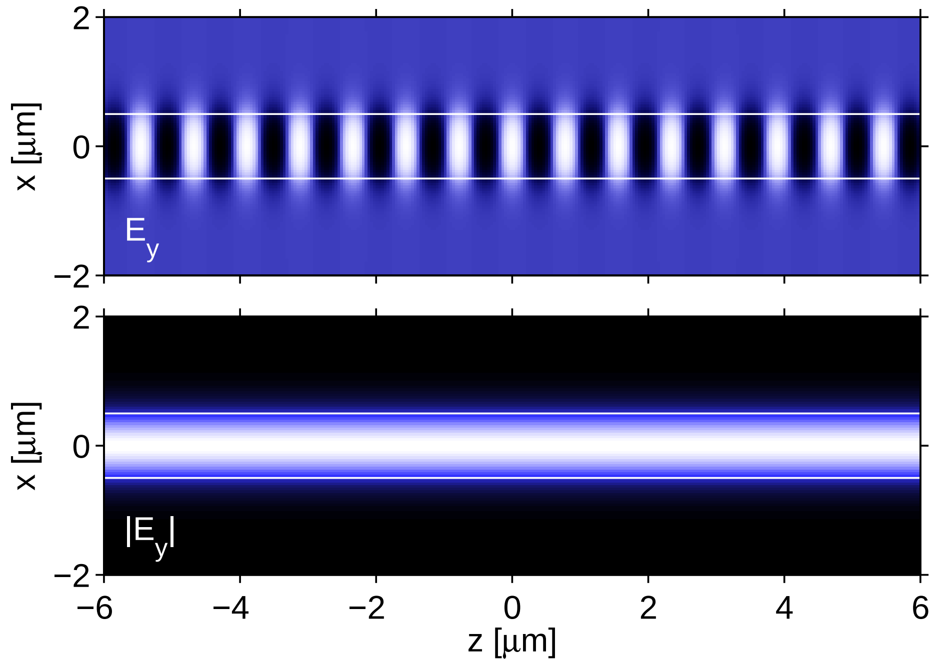
$E_y(x, z) = \tilde{E}_y(x)\, e^{-i\beta z}$,

$\left(\partial_x^2 + k^2 n^2(x)\right) \tilde{E}_y = \beta^2 \tilde{E}_y \quad \rightsquigarrow \quad \left\{\beta, \tilde{E}_y\right\}$.

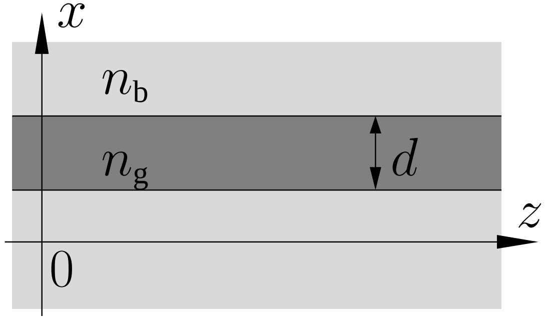
$n_b = 1.5,\ n_g = 2.0,\ d = 1.0\,\mu\text{m},\ \lambda = 1.5\,\mu\text{m}$,

$\beta_0/k = 1.924$



# *Straight dielectric waveguides*



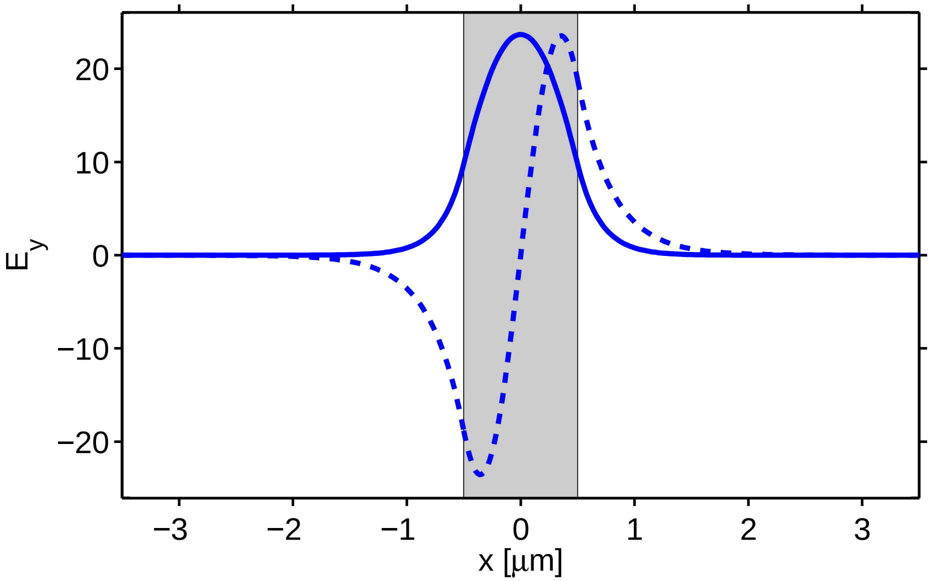

$n_\mathrm{g} > n_\mathrm{b}$

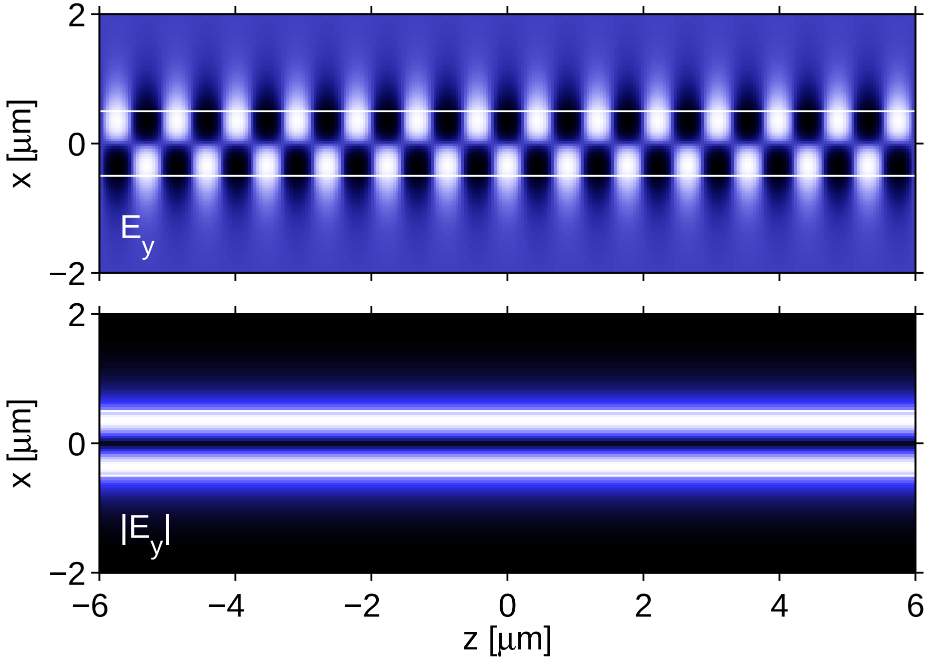
$\partial_z n = 0,$ $\qquad \omega = kc = 2\pi c/\lambda$ given,

$E_y(x,z) = \tilde{E}_y(x)\,\mathrm{e}^{-\mathrm{i}\beta z},$

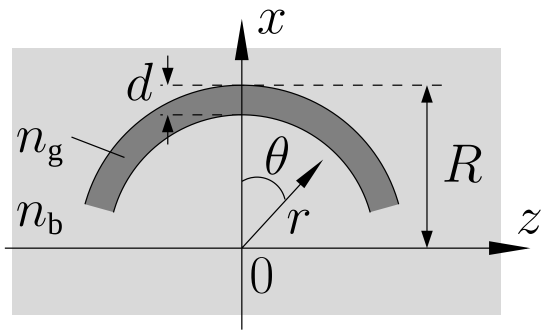
$$\left(\partial_x^2 + k^2 n^2(x)\right)\tilde{E}_y = \beta^2 \tilde{E}_y \quad \rightsquigarrow \quad \left\{\beta, \tilde{E}_y\right\}.$$

$n_\mathrm{b} = 1.5,\ n_\mathrm{g} = 2.0,\ d = 1.0\,\mu\mathrm{m},\ \lambda = 1.5\,\mu\mathrm{m},$
$\beta_0/k = 1.924,\ \beta_1/k = 1.697.$



# ***Waveguide bends***




$n_g > n_b$

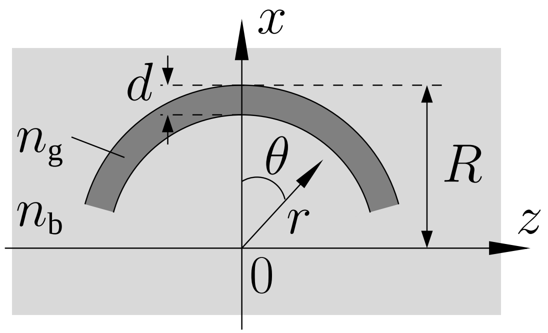
$\partial_\theta n = 0,$ $\qquad \omega = kc = 2\pi c/\lambda$ given,

$E_y(r,\theta) = \tilde{E}_y(r)\,\mathrm{e}^{-\mathrm{i}\gamma R\theta},$ $\qquad \gamma = \beta - \mathrm{i}\alpha \in \mathbb{C},$

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2}\right)\tilde{E}_y = 0$$

$$\rightsquigarrow \quad \left\{\gamma, \tilde{E}_y\right\}.$$

# *Waveguide bends*



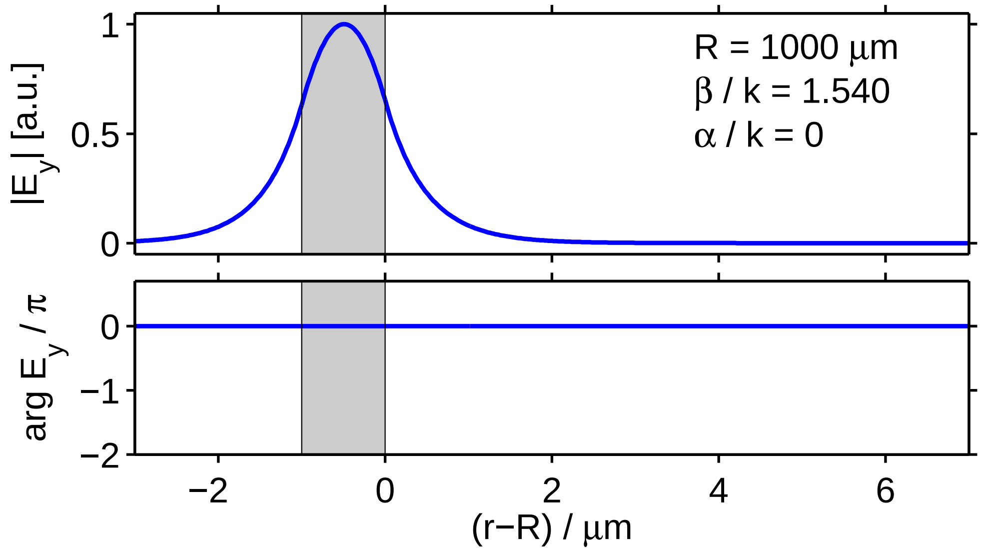

$n_g > n_b$

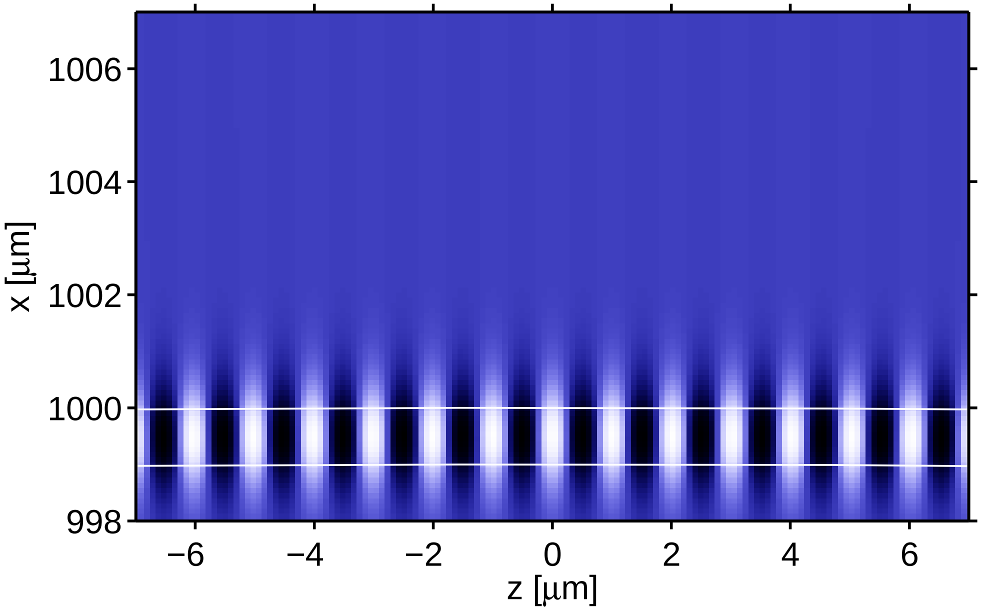
$$\partial_\theta n = 0, \qquad \omega = kc = 2\pi c/\lambda \text{ given},$$

$$E_y(r,\theta) = \tilde{E}_y(r)\, e^{-i\gamma R\theta}, \qquad \gamma = \beta - i\alpha \in \mathbb{C},$$

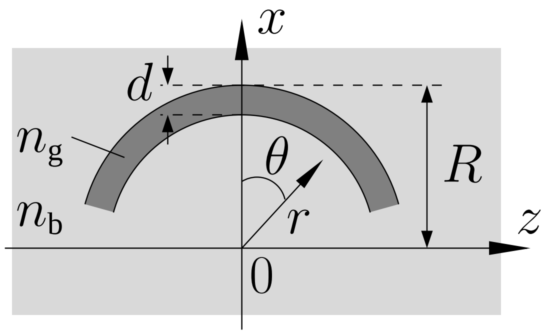
$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2}\right)\tilde{E}_y = 0 \quad \rightsquigarrow \quad \left\{\gamma, \tilde{E}_y\right\}.$$

$n_b = 1.45,\ n_g = 1.6,\ d = 1.0\,\mu\text{m},\ \lambda = 1.55\,\mu\text{m}.$



# *Waveguide bends*




$n_g > n_b$

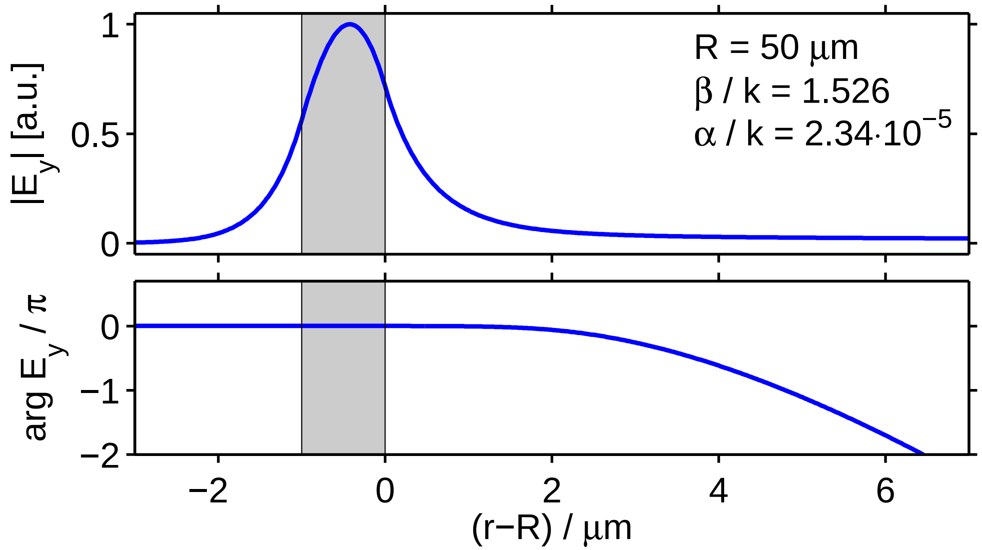
$\partial_\theta n = 0,$ $\omega = k\mathrm{c} = 2\pi\mathrm{c}/\lambda$ given,

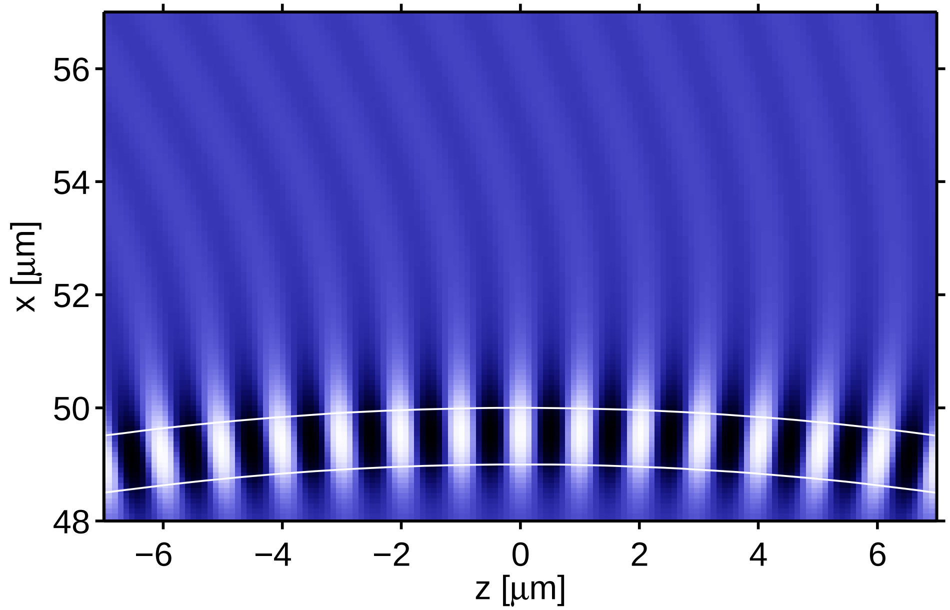
$E_y(r,\theta) = \tilde{E}_y(r)\,\mathrm{e}^{-\mathrm{i}\gamma R\theta},$ $\gamma = \beta - \mathrm{i}\alpha \in \mathbb{C},$

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\,\frac{\partial \tilde{E}_y}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2}\right)\tilde{E}_y = 0$$

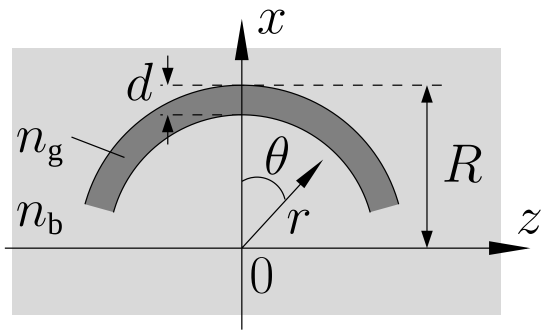
$$\rightsquigarrow \quad \{\gamma, \tilde{E}_y\}.$$

$n_b = 1.45,\ n_g = 1.6,\ d = 1.0\,\mu\mathrm{m},\ \lambda = 1.55\,\mu\mathrm{m}.$



# *Waveguide bends*



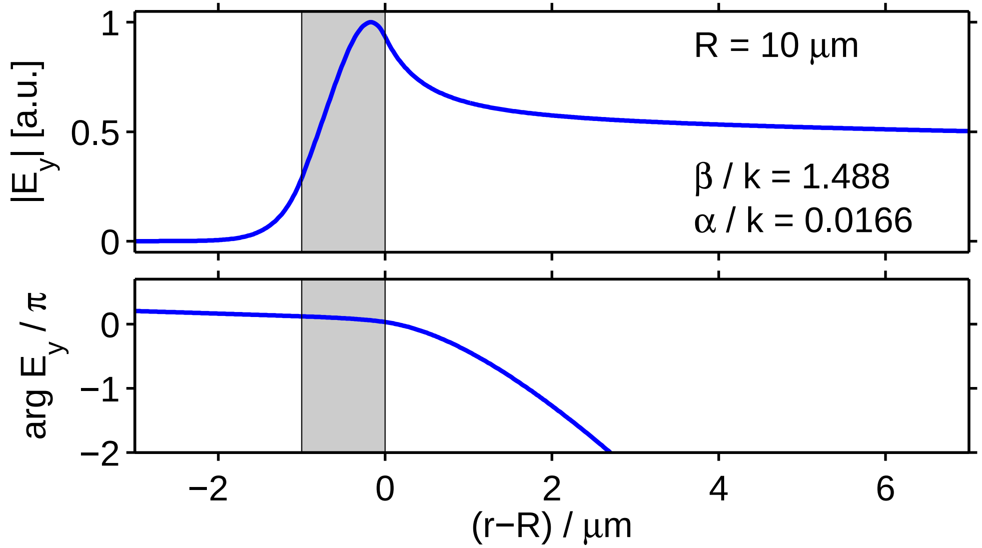

$n_\mathrm{g} > n_\mathrm{b}$

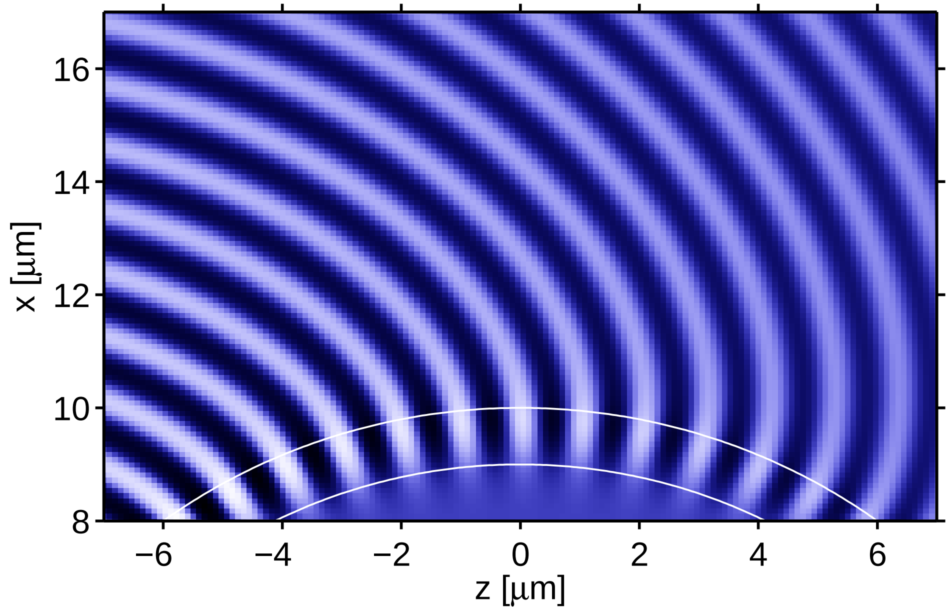
$$\partial_\theta n = 0, \qquad \omega = kc = 2\pi c/\lambda \text{ given},$$

$$E_y(r,\theta) = \tilde{E}_y(r)\,\mathrm{e}^{-\mathrm{i}\gamma R\theta}, \qquad \gamma = \beta - \mathrm{i}\alpha \in \mathbb{C},$$

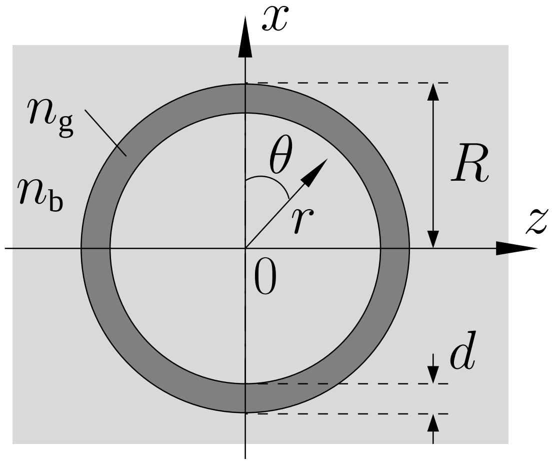
$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2}\right)\tilde{E}_y = 0 \quad \rightsquigarrow \quad \left\{\gamma, \tilde{E}_y\right\}.$$

$n_\mathrm{b} = 1.45,\ n_\mathrm{g} = 1.6,\ d = 1.0\,\mu\mathrm{m},\ \lambda = 1.55\,\mu\mathrm{m}.$



# *Whispering gallery resonances*




$n_{\mathrm{g}} > n_{\mathrm{b}}$

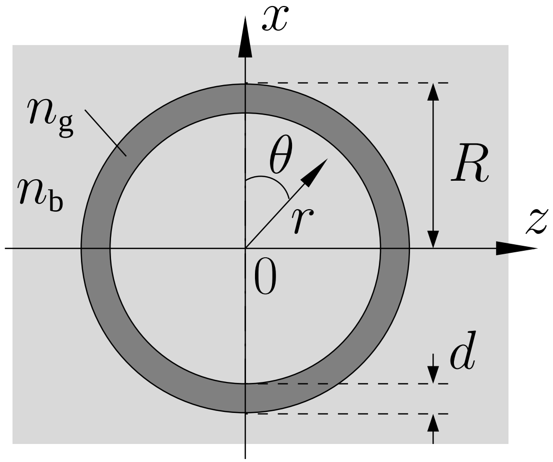
$\partial_\theta n = 0$, $\qquad \omega^{\mathrm{c}} \in \mathbb{C}$ eigenvalue,

$E_y(r,\theta) = \tilde{E}_y(r)\, \mathrm{e}^{-\mathrm{i}m\theta}$, $\qquad m \in \mathbb{Z}$,

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(\left(\frac{\omega^{\mathrm{c}}}{\mathrm{c}}\right)^2 n^2 - \frac{m^2}{r^2}\right)\tilde{E}_y = 0$$

$\rightsquigarrow \quad \left\{\omega^{\mathrm{c}}, \tilde{E}_y\right\}, \ \left\{\mathrm{WGM}(l,m)\right\}.$

## *Whispering gallery resonances*




$n_\mathrm{g} > n_\mathrm{b}$

$\partial_\theta n = 0$, $\qquad \omega^\mathrm{c} \in \mathbb{C}$ eigenvalue,

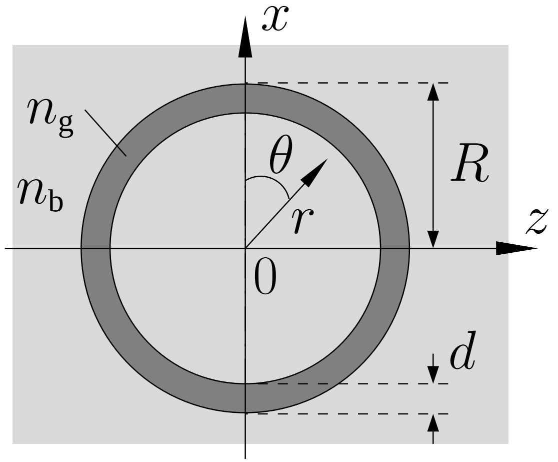
$E_y(r,\theta) = \tilde{E}_y(r)\,\mathrm{e}^{-\mathrm{i}m\theta}$, $\qquad m \in \mathbb{Z}$,

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\,\frac{\partial \tilde{E}_y}{\partial r} + \left(\left(\frac{\omega^\mathrm{c}}{\mathrm{c}}\right)^2 n^2 - \frac{m^2}{r^2}\right)\tilde{E}_y = 0$$

$$\rightsquigarrow \quad \left\{\omega^\mathrm{c}, \tilde{E}_y\right\}, \ \left\{\mathrm{WGM}(l,m)\right\}.$$

$Q = \mathrm{Re}\,\omega^\mathrm{c}/(2\mathrm{Im}\,\omega^\mathrm{c})$, $\qquad \lambda_\mathrm{r} = 2\pi\mathrm{c}/\mathrm{Re}\,\omega^\mathrm{c}$, $\qquad$ outgoing radiation, FWHM: $\Delta\lambda = \lambda_\mathrm{r}/Q$.

# *Whispering gallery resonances*



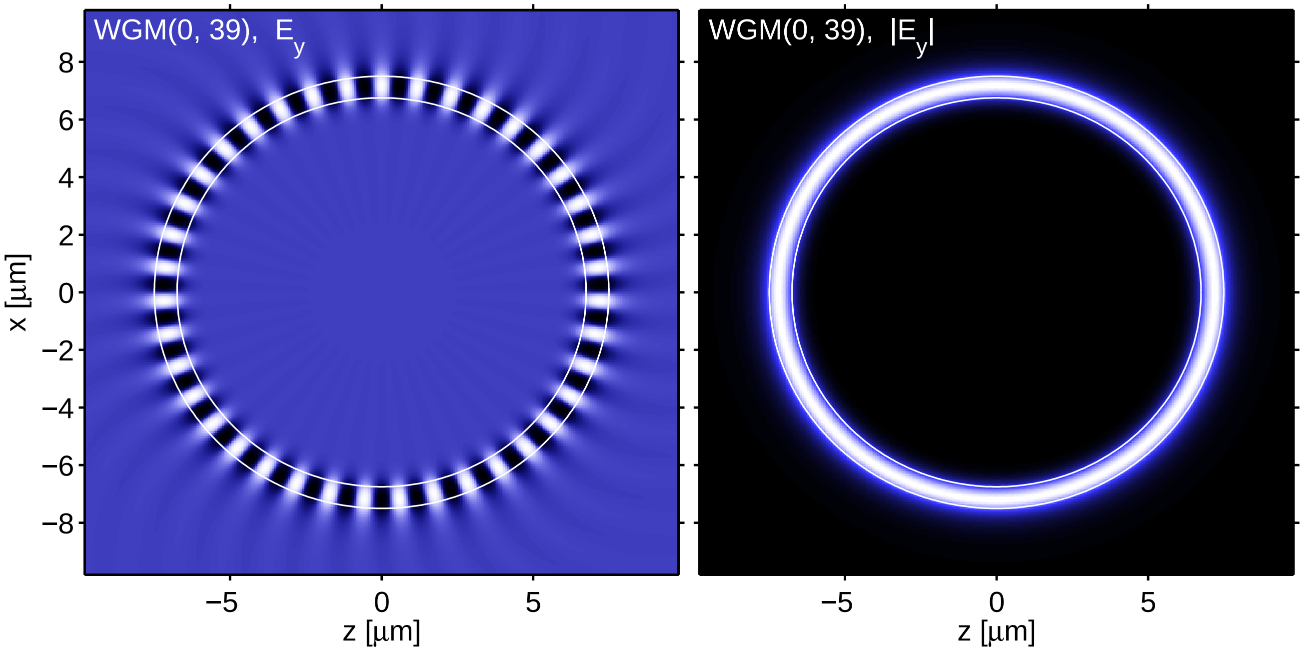

$$\partial_\theta n = 0, \qquad \omega^{\rm c} \in \mathbb{C} \;\; \text{eigenvalue},$$

$$E_y(r,\theta) = \tilde{E}_y(r)\,{\rm e}^{-{\rm i}m\theta}, \qquad m \in \mathbb{Z},$$

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(\left(\frac{\omega^{\rm c}}{\rm c}\right)^2 n^2 - \frac{m^2}{r^2}\right)\tilde{E}_y = 0$$

$n_{\rm g} > n_{\rm b}$

$$\rightsquigarrow \quad \left\{\omega^{\rm c}, \tilde{E}_y\right\}, \; \left\{\text{WGM}(l,m)\right\}.$$

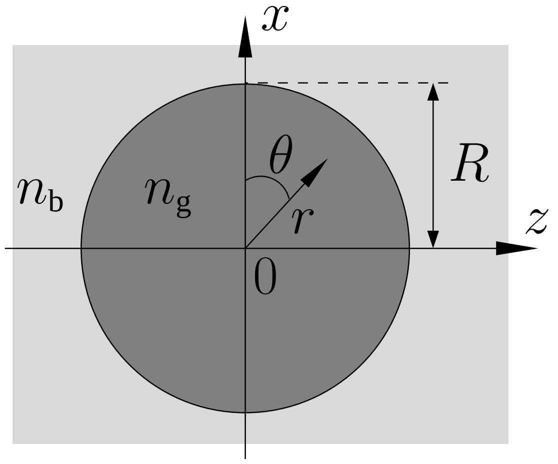



TE, $R = 7.5\,\mu$m, $d = 0.75\,\mu$m,
$n_{\rm g} = 1.5$, $n_{\rm b} = 1.0$.

WGM$(0, 39)$:

$\lambda_{\rm r} = 1.5637\,\mu$m,
$Q = 1.1 \cdot 10^5$,
$\Delta\lambda = 1.4 \cdot 10^{-5}\,\mu$m.

# *Whispering gallery resonances*



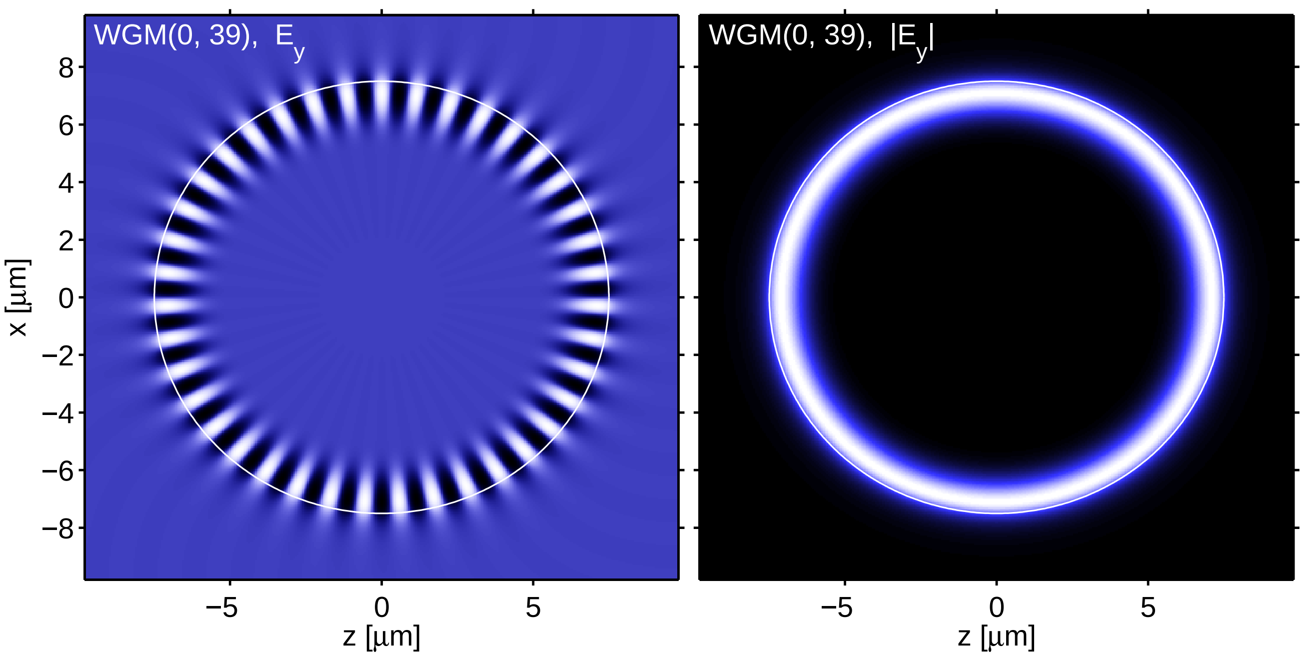

$\partial_\theta n = 0,$ $\qquad\qquad\qquad\qquad \omega^\mathrm{c} \in \mathbb{C}$ eigenvalue,

$E_y(r,\theta) = \tilde{E}_y(r)\,\mathrm{e}^{-\mathrm{i}m\theta},$ $\qquad\qquad\qquad m \in \mathbb{Z},$

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(\left(\frac{\omega^\mathrm{c}}{\mathrm{c}}\right)^2 n^2 - \frac{m^2}{r^2}\right)\tilde{E}_y = 0$$

$n_\mathrm{g} > n_\mathrm{b}$

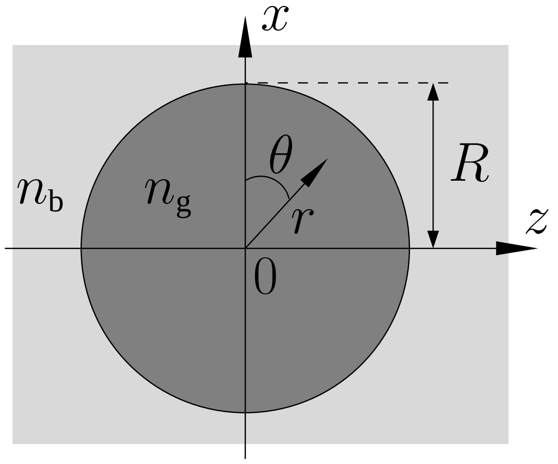
$$\rightsquigarrow \quad \left\{\omega^\mathrm{c}, \tilde{E}_y\right\}, \ \left\{\mathrm{WGM}(l,m)\right\}.$$




TE, $R = 7.5\,\mu$m,
$n_\mathrm{g} = 1.5,\ n_\mathrm{b} = 1.0.$

WGM(0, 39):

$\lambda_\mathrm{r} = 1.6025\,\mu$m,
$Q = 5.7 \cdot 10^5,$
$\Delta\lambda = 2.8 \cdot 10^{-6}\,\mu$m.



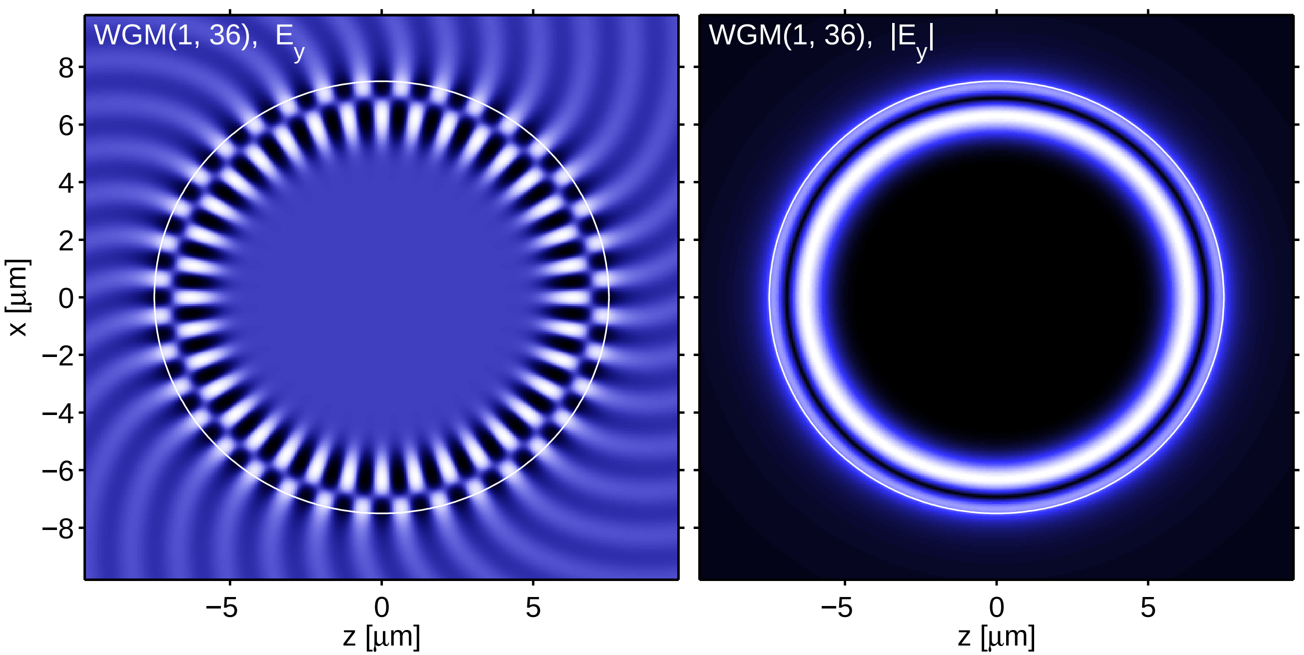

$$\partial_\theta n = 0, \qquad \omega^{\mathrm{c}} \in \mathbb{C} \ \text{ eigenvalue,}$$

$$E_y(r,\theta) = \tilde{E}_y(r)\,\mathrm{e}^{-\mathrm{i}m\theta}, \qquad m \in \mathbb{Z},$$

$n_{\mathrm{g}} > n_{\mathrm{b}}$

$$\frac{\partial^2 \tilde{E}_y}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_y}{\partial r} + \left(\left(\frac{\omega^{\mathrm{c}}}{\mathrm{c}}\right)^2 n^2 - \frac{m^2}{r^2}\right)\tilde{E}_y = 0$$

$$\rightsquigarrow \quad \left\{\omega^{\mathrm{c}}, \tilde{E}_y\right\}, \ \left\{\mathrm{WGM}(l,m)\right\}.$$

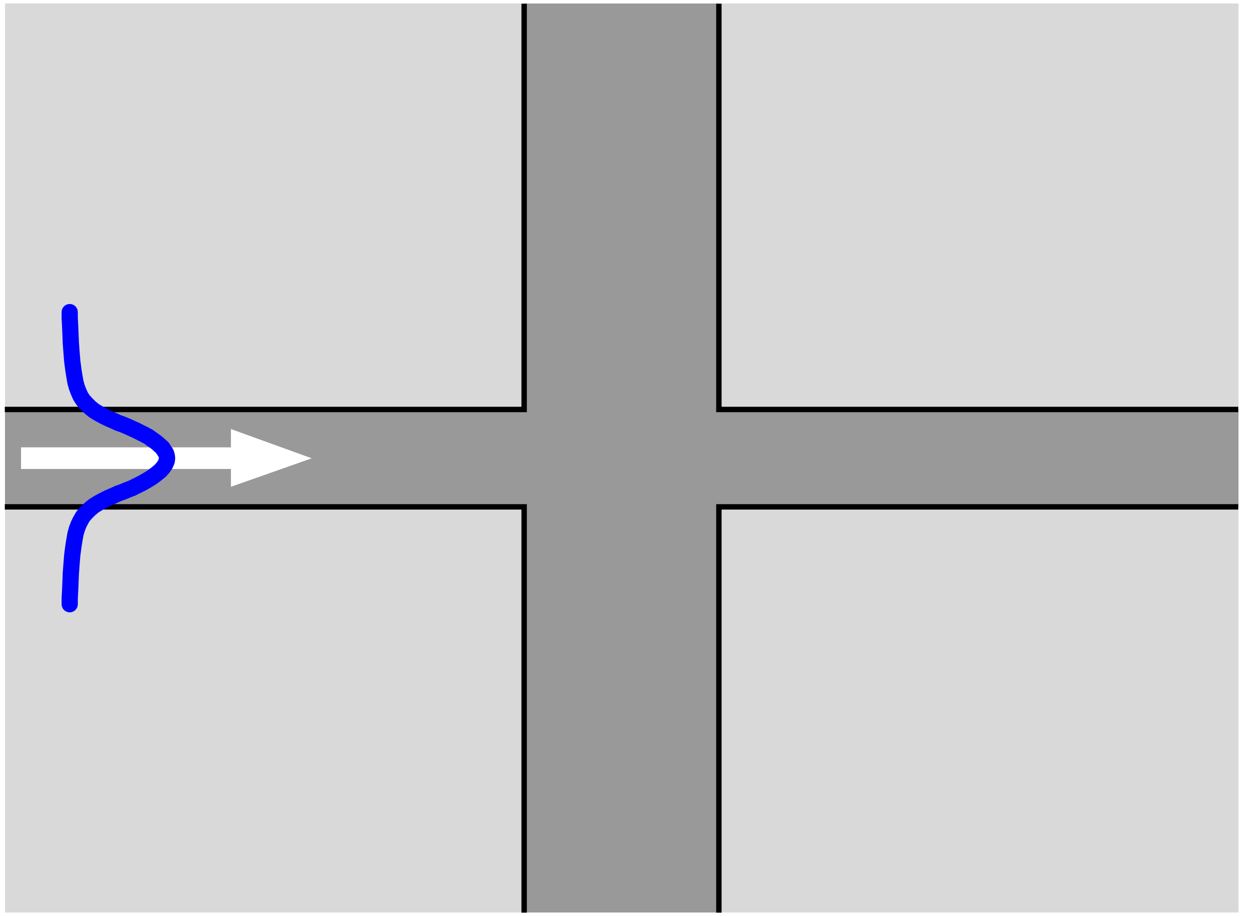



TE, $R = 7.5\,\mu$m,
$n_{\mathrm{g}} = 1.5$, $n_{\mathrm{b}} = 1.0$.

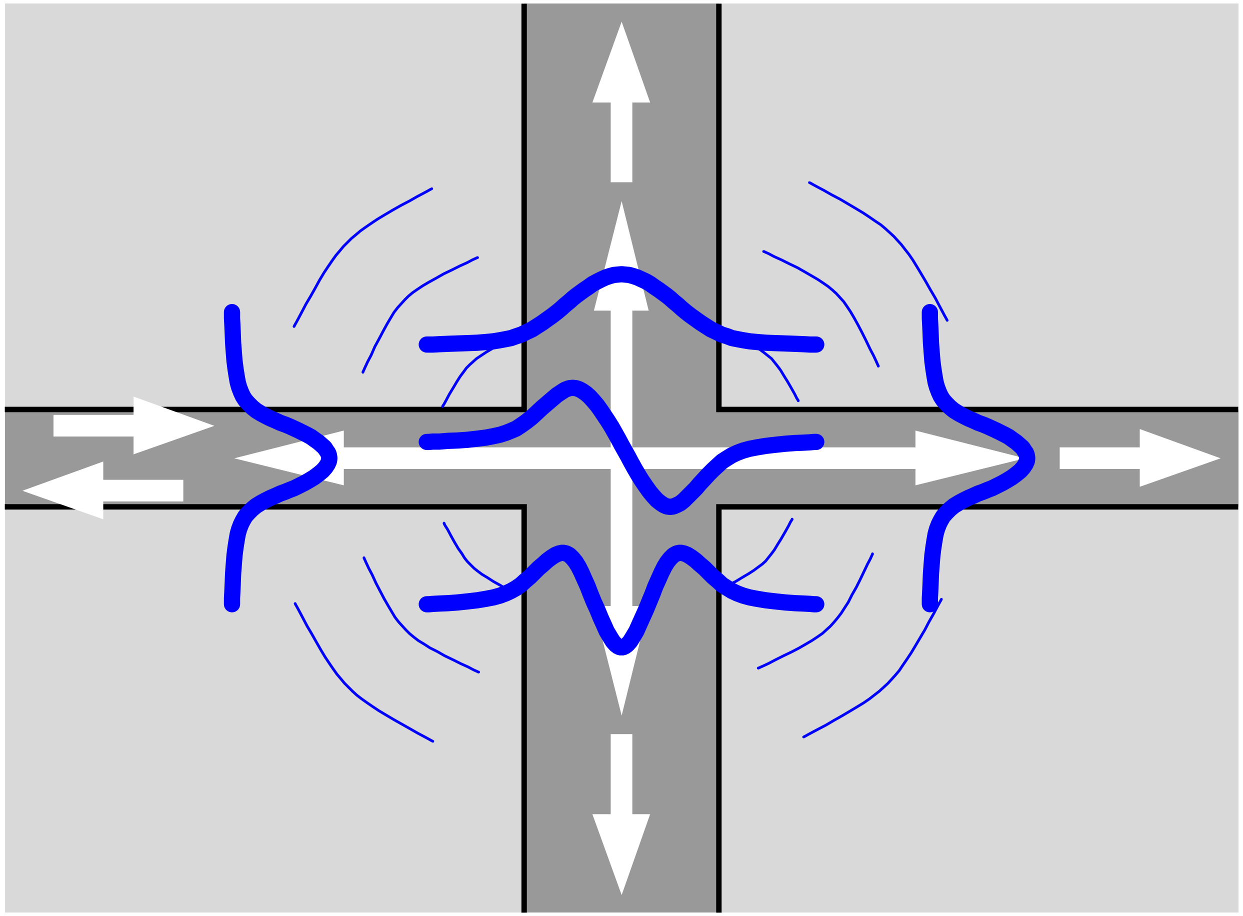
WGM(1, 36):

$\lambda_{\mathrm{r}} = 1.5367\,\mu$m,
$Q = 2.2 \cdot 10^4$,
$\Delta\lambda = 7.0 \cdot 10^{-4}\,\mu$m.

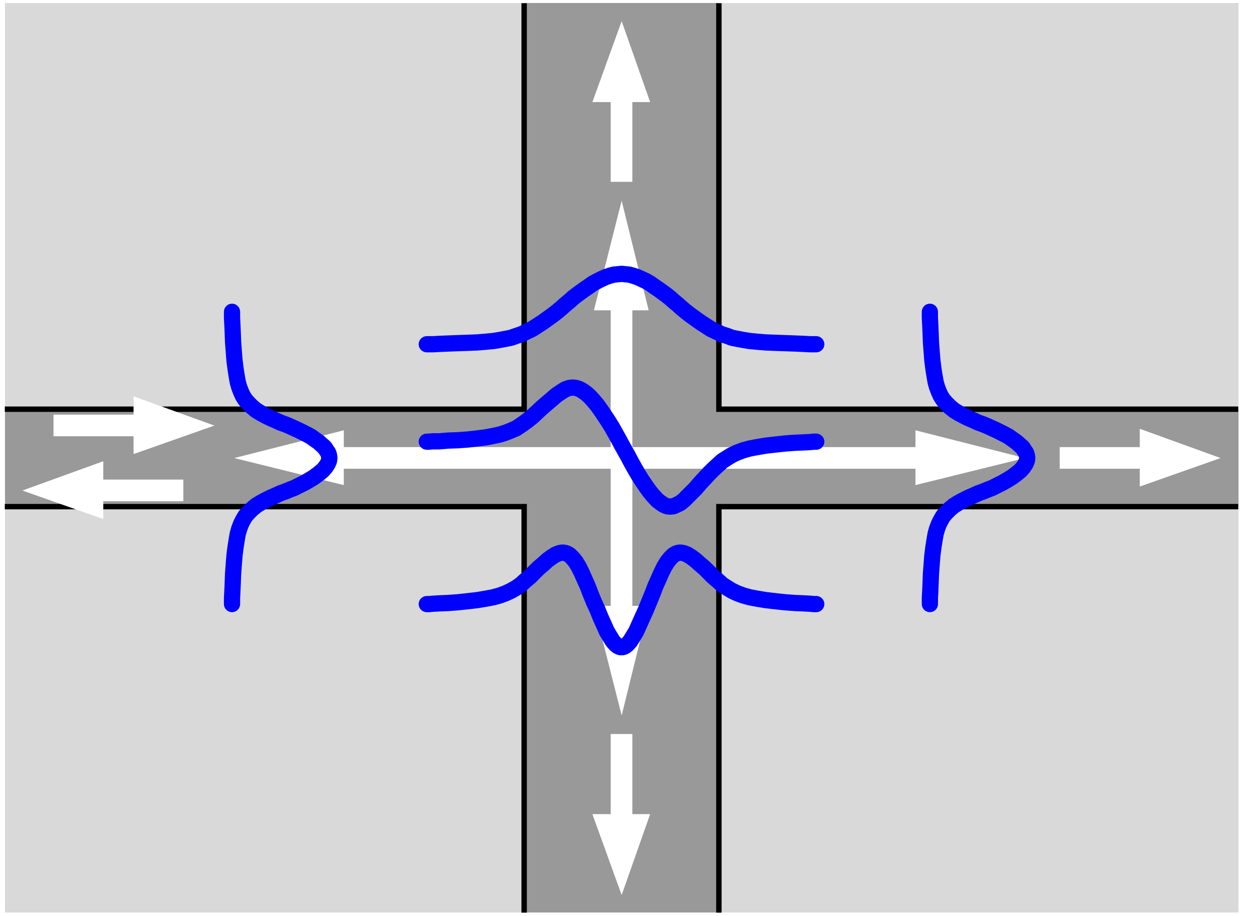
## *A waveguide crossing*



# *A waveguide crossing*

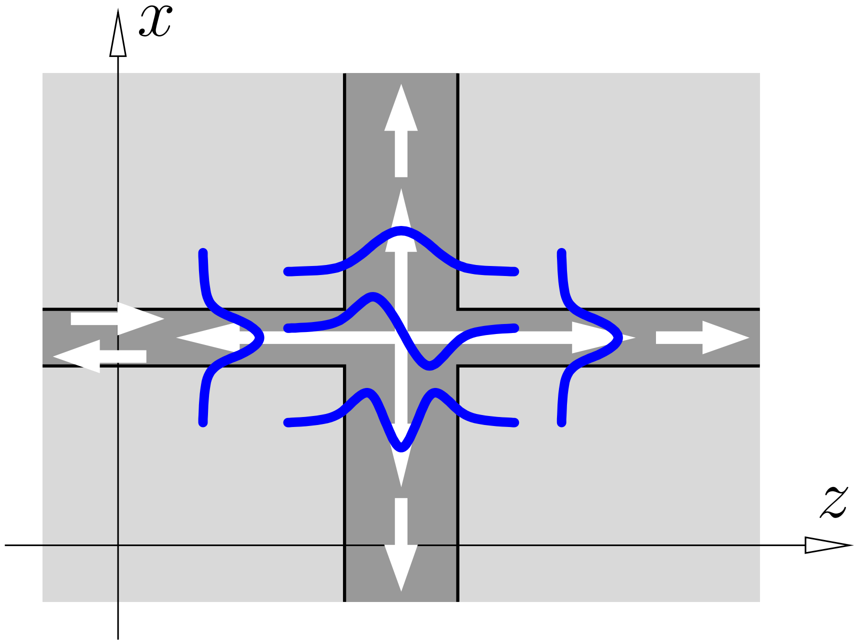


# *A waveguide crossing*



Coupled Mode Model ?

## *Field ansatz*




Basis elements (crossing):

- guided modes of the horizontal WG

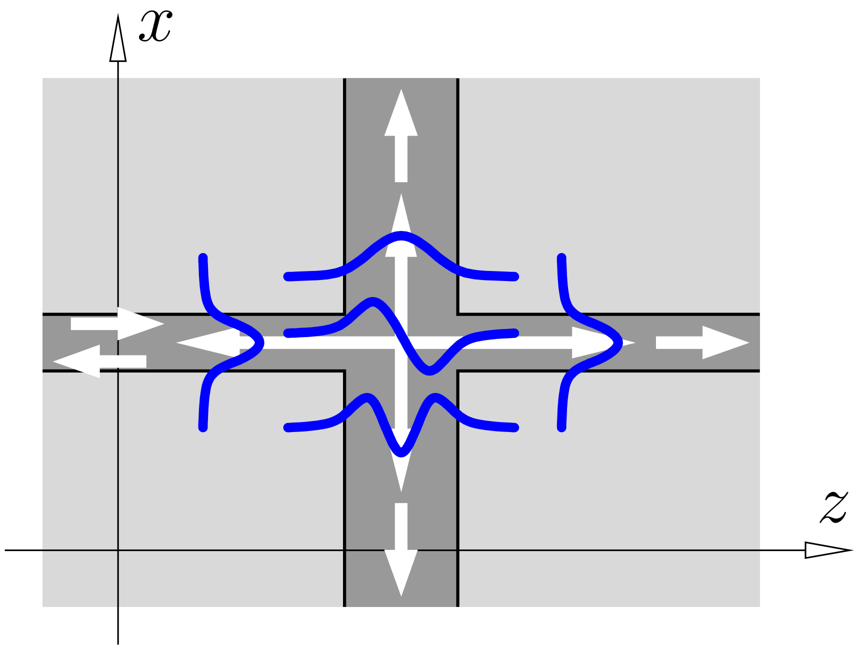
$$\boldsymbol{\psi}_m^{\mathrm{f,b}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_m^{\mathrm{f,b}}(x)\, \mathrm{e}^{\mp \mathrm{i}\beta_m^{\mathrm{f,b}} z},$$

- guided modes of the vertical WG

$$\boldsymbol{\psi}_m^{\mathrm{u,d}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_m^{\mathrm{u,d}}(z)\, \mathrm{e}^{\mp \mathrm{i}\beta_m^{\mathrm{u,d}} x}$$

- (and further terms).

## *Field ansatz*




Basis elements (crossing):

- guided modes of the horizontal WG

$$\boldsymbol{\psi}_m^{\mathrm{f,b}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_m^{\mathrm{f,b}}(x)\, \mathrm{e}^{\mp \mathrm{i}\beta_m^{\mathrm{f,b}} z},$$

- guided modes of the vertical WG

$$\boldsymbol{\psi}_m^{\mathrm{u,d}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_m^{\mathrm{u,d}}(z)\, \mathrm{e}^{\mp \mathrm{i}\beta_m^{\mathrm{u,d}} x}$$

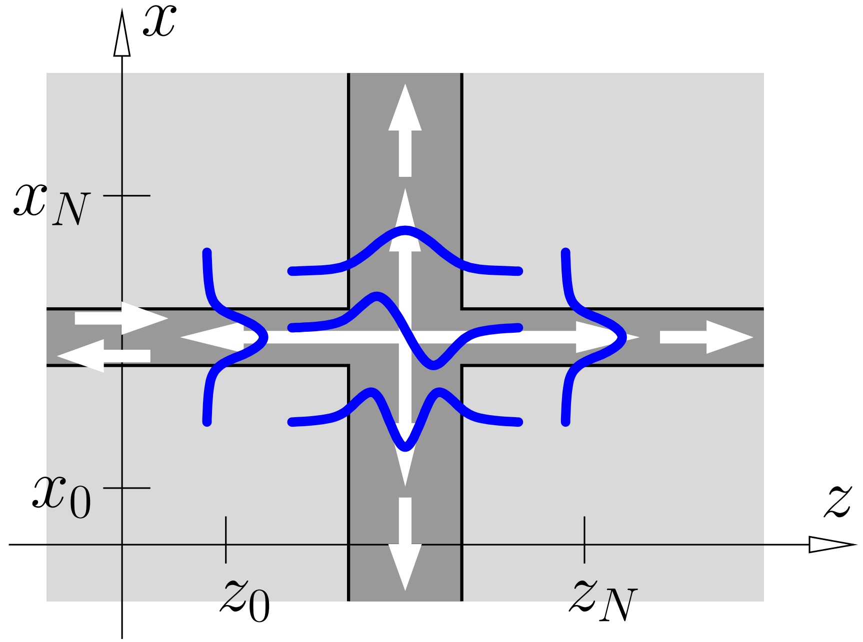
- (and further terms).

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \sum_m f_m(z)\boldsymbol{\psi}_m^{\mathrm{f}}(x,z) + \sum_m b_m(z)\boldsymbol{\psi}_m^{\mathrm{b}}(x,z) + \sum_m u_m(x)\boldsymbol{\psi}_m^{\mathrm{u}}(x,z) + \sum_m d_m(x)\boldsymbol{\psi}_m^{\mathrm{d}}(x,z)$$

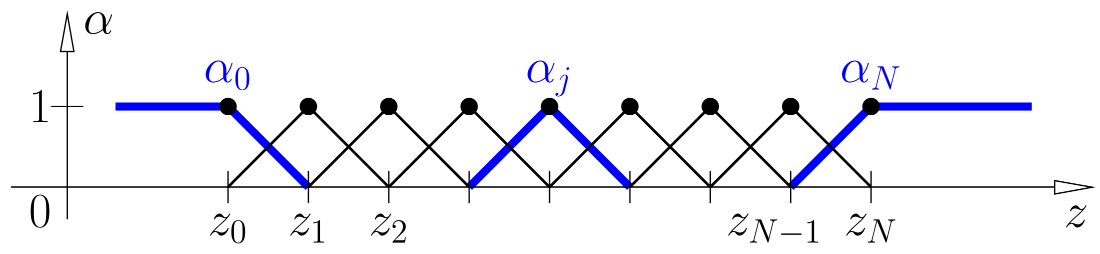
$f_m, b_m, u_m, d_m$: ?

General: a reasonable superposition of known fields with amplitudes that are functions of suitable propagation coordinate(s).

# ***Amplitude functions, discretization***




1-D linear finite elements




$$f_m(z) = \sum_{j=0}^{N} f_{m,j}\, \alpha_j(z)\,,$$

$b_m(z)$, $u_m(x)$, $d_m(x)$ analogous.

$$\hookrightarrow \qquad \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \sum_k a_k \Big(\alpha_{.}(\cdot)\boldsymbol{\psi}_{:}^{:}\Big)(x,z) =: \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}(x,z),$$

$k \in \{$waveguides, modes, elements$\}$, $\quad a_k \in \{f_{m,j}, b_{m,j}, u_{m,j}, d_{m,j}\}$, $\qquad a_k$: ?

## *Galerkin procedure*

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E} &= 0 \\ -\boldsymbol{\nabla} \times \boldsymbol{E} - \mathrm{i}\omega\mu_0\boldsymbol{H} &= 0 \end{aligned} \quad \Bigg| \quad \cdot \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix}^*, \quad \iint_{\text{comp. domain}}$$

$$\leftrightsquigarrow \quad \iint \mathcal{K}(\boldsymbol{F}, \boldsymbol{G}; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z = 0 \quad \text{for all } \boldsymbol{F},\ \boldsymbol{G},$$

where

$$\mathcal{K}(\boldsymbol{F}, \boldsymbol{G}; \boldsymbol{E}, \boldsymbol{H}) = \boldsymbol{F}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) - \boldsymbol{G}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{F}^* \cdot \boldsymbol{E} - \mathrm{i}\omega\mu_0\boldsymbol{G}^* \cdot \boldsymbol{H}\ .$$

## *Galerkin procedure, continued*

- Insert $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}$,
- select $\{\mathrm{u}\}$: indices of unknown coefficients,
  $\{\mathrm{g}\}$: given values related to prescribed influx,
- require $\iint \mathcal{K}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z = 0$ for $l \in \{\mathrm{u}\}$,
- compute $K_{lk} = \iint \mathcal{K}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_k, \boldsymbol{H}_k)\, \mathrm{d}x\, \mathrm{d}z$.

---

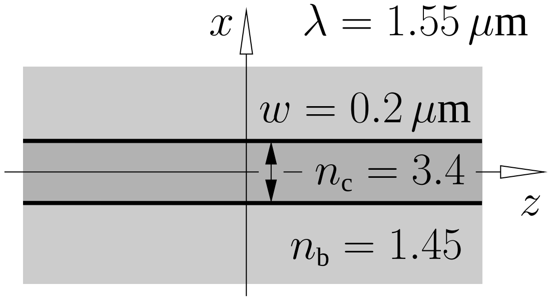
$$\sum_{k \in \{\mathrm{u,g}\}} K_{lk} a_k = 0\,, \;\; l \in \{\mathrm{u}\}\,,$$

$$\begin{pmatrix} \mathsf{K}_{\mathrm{u\,u}} & \mathsf{K}_{\mathrm{u\,g}} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_{\mathrm{u}} \\ \boldsymbol{a}_{\mathrm{g}} \end{pmatrix} = 0\,, \qquad \text{or} \qquad \mathsf{K}_{\mathrm{u\,u}} \boldsymbol{a}_{\mathrm{u}} = -\mathsf{K}_{\mathrm{u\,g}} \boldsymbol{a}_{\mathrm{g}}\,.$$

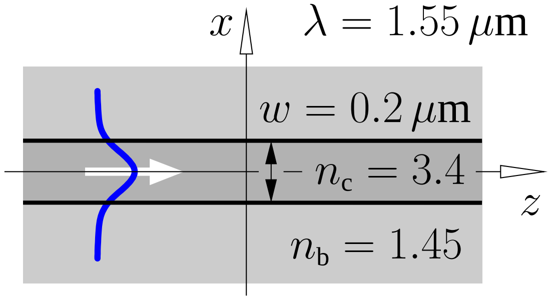
# *Further issues*

… plenty.

# *Straight waveguide*

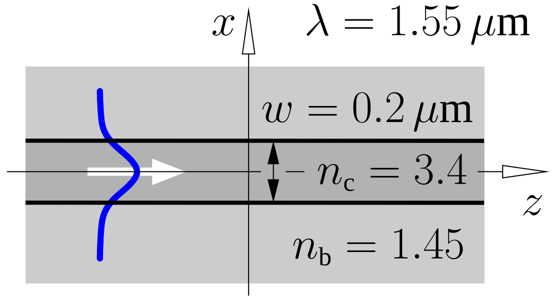


# *Straight waveguide*

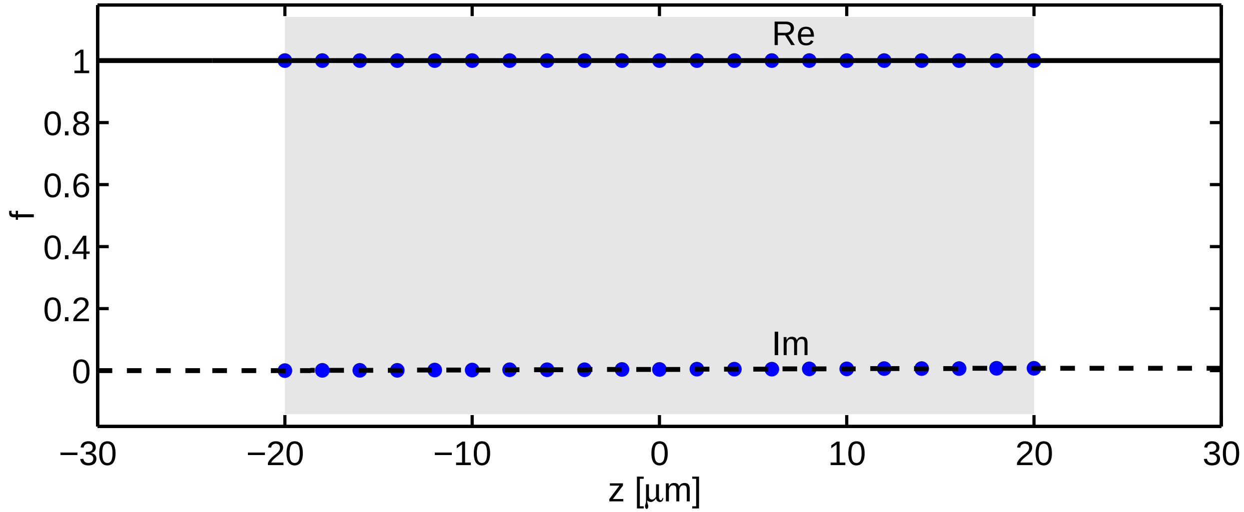


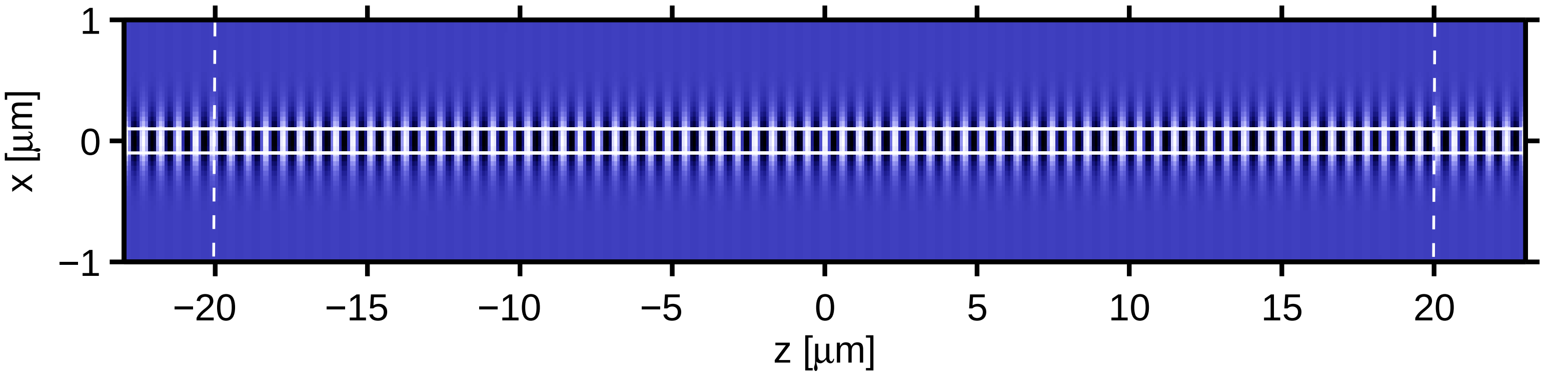

Basis element: fundamental forward propagating TE mode,
input amplitude $f_0 = 1$,
FEM discretization in $z \in [-20, 20]\,\mu$m, $\Delta z = 2\,\mu$m,
computational domain $z \in [< -20, > 20]\,\mu$m, $x \in [-3.0, 3.0]\,\mu$m.

# *Straight waveguide*

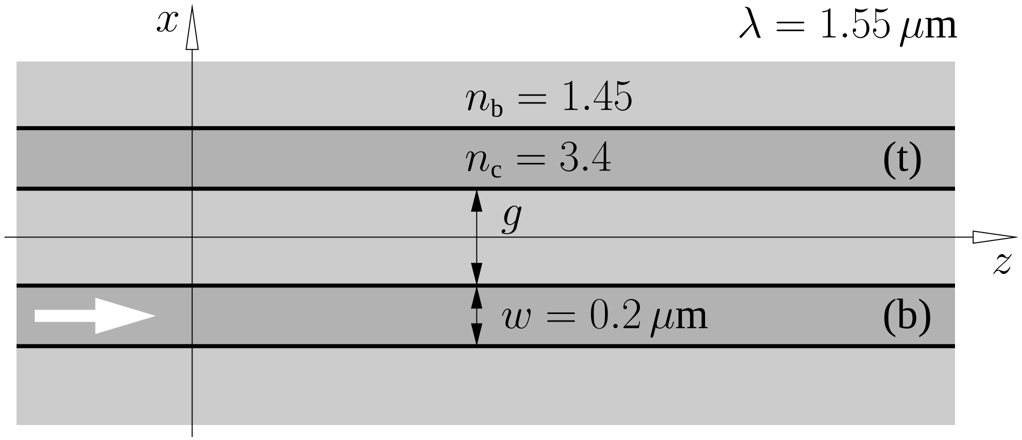



Basis element: fundamental forward propagating TE mode,
input amplitude $f_0 = 1$,
FEM discretization in $z \in [-20, 20]\,\mu$m, $\Delta z = 2\,\mu$m,
computational domain $z \in [< -20, > 20]\,\mu$m, $x \in [-3.0, 3.0]\,\mu$m.

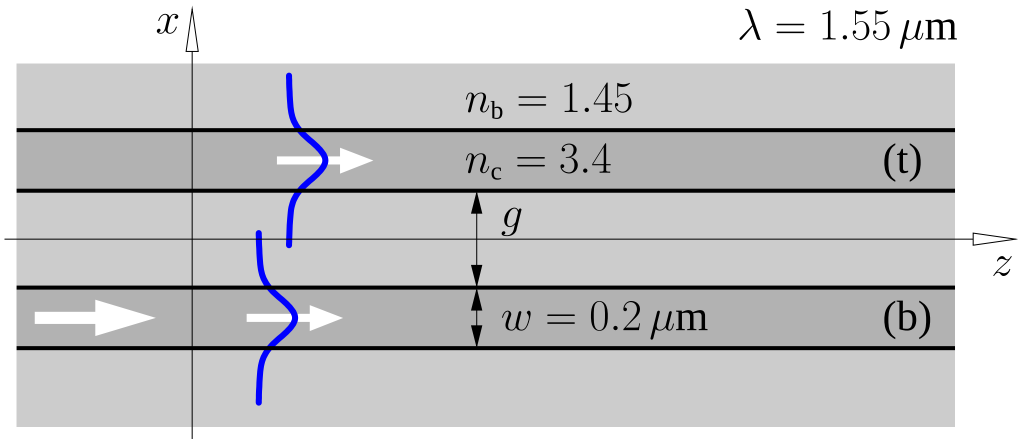


# *Two coupled parallel cores, amplitudes*



# *Two coupled parallel cores, amplitudes*

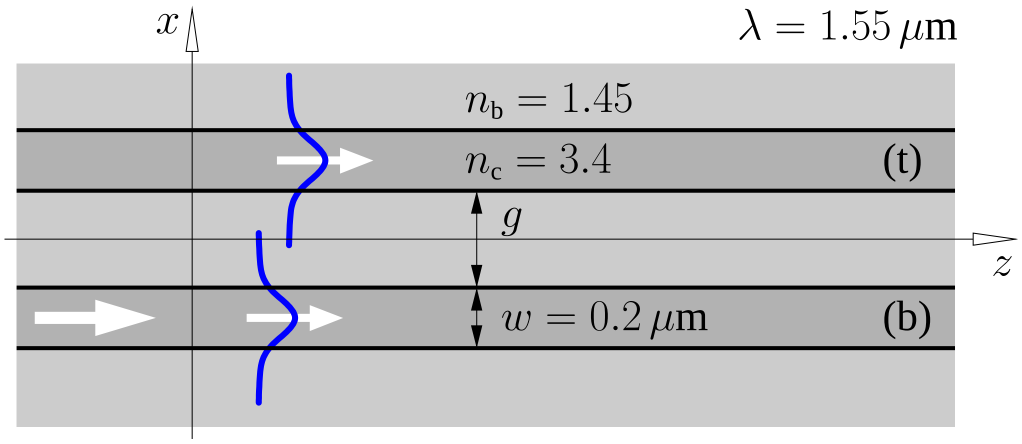



Basis elements: forward propagating fundamental TE modes of the separate cores, input amplitude $f_b = 1$,

FEM discretization:
$z \in [-20, 20]\,\mu$m, $\Delta z = 0.5\,\mu$m,

computational domain:
$z \in [-20, 20]\,\mu$m, $x \in [-3.0, 3.0]\,\mu$m.

# *Two coupled parallel cores, amplitudes*

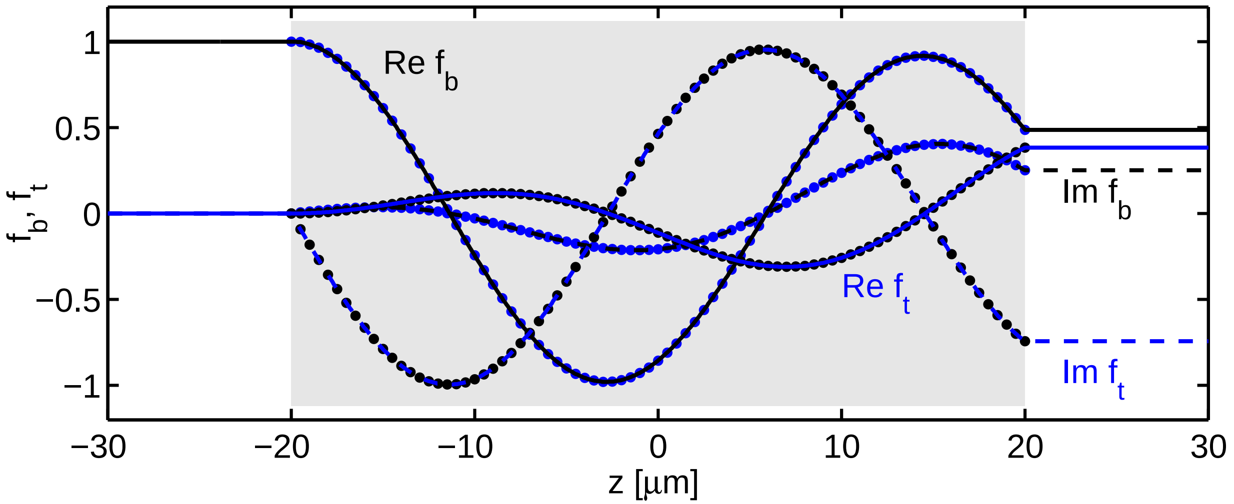


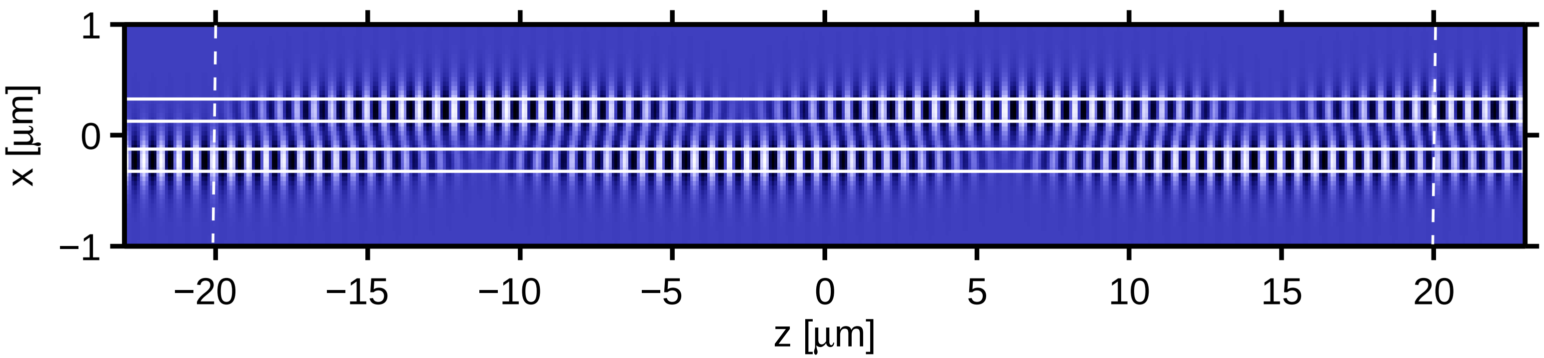

Basis elements: forward propagating fundamental TE modes of the separate cores, input amplitude $f_b = 1$,

FEM discretization:
$z \in [-20, 20]\,\mu$m, $\Delta z = 0.5\,\mu$m,

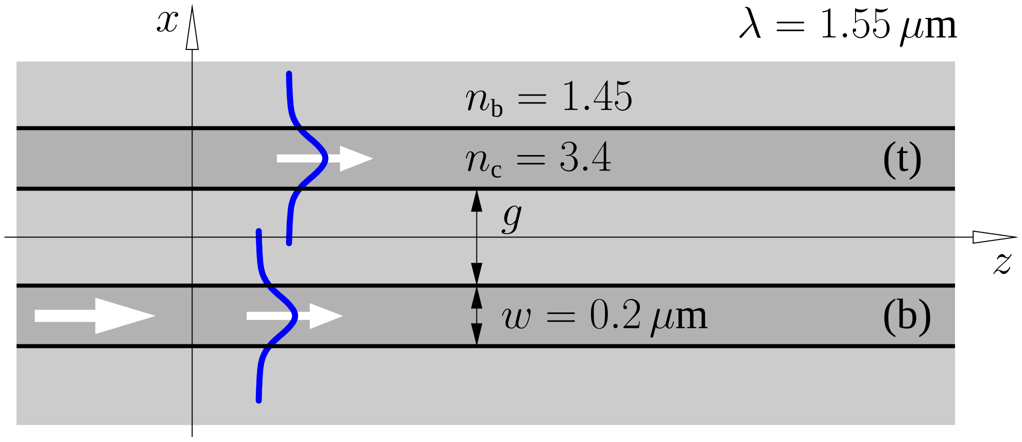
computational domain:
$z \in [-20, 20]\,\mu$m, $x \in [-3.0, 3.0]\,\mu$m.

$g = 0.25\,\mu$m:



# *Two coupled parallel cores, modal power*

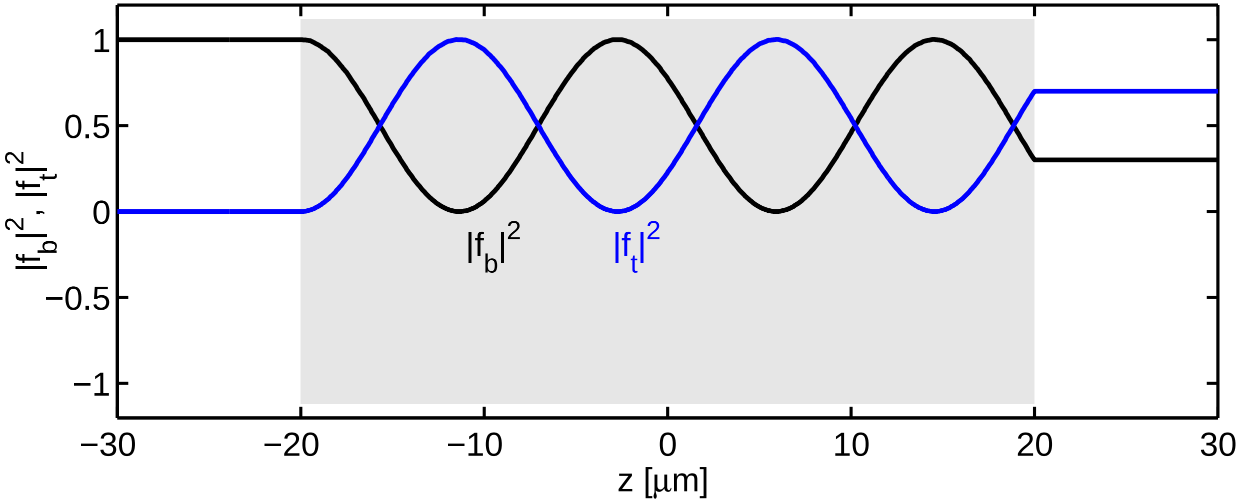


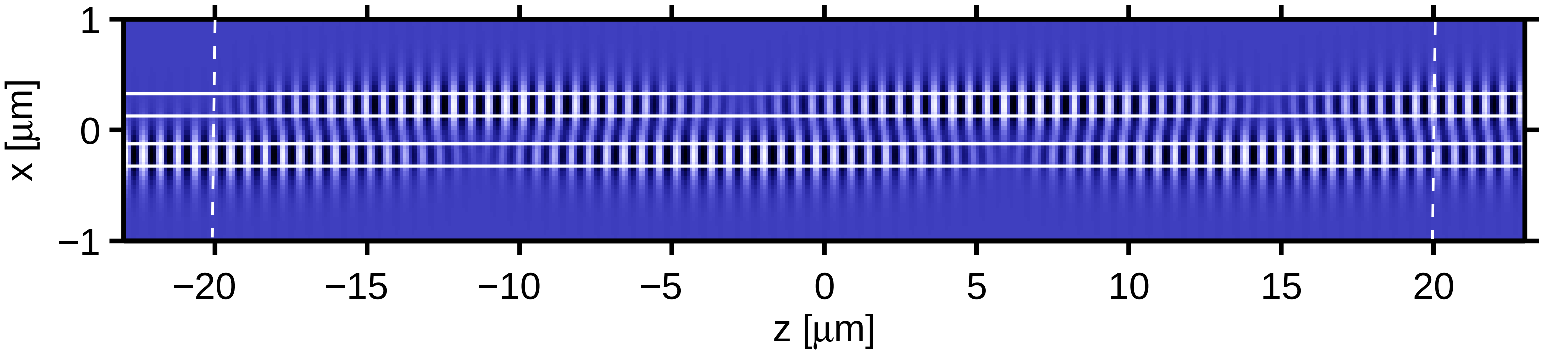

Basis elements: forward propagating fundamental TE modes of the separate cores, input amplitude $f_{\rm b} = 1$,

FEM discretization:
$z \in [-20, 20]\,\mu$m, $\Delta z = 0.5\,\mu$m,

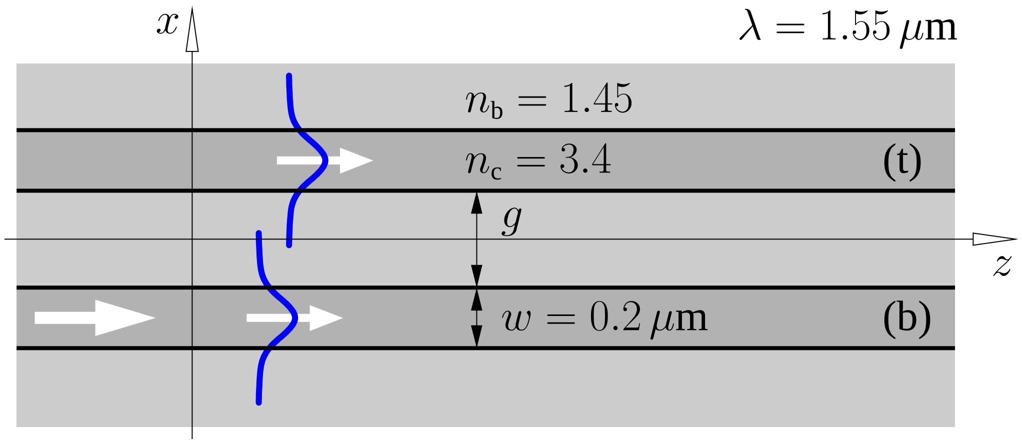
computational domain:
$z \in [-20, 20]\,\mu$m, $x \in [-3.0, 3.0]\,\mu$m.

$g = 0.25\,\mu$m:



# *Two coupled parallel cores, coupling length*

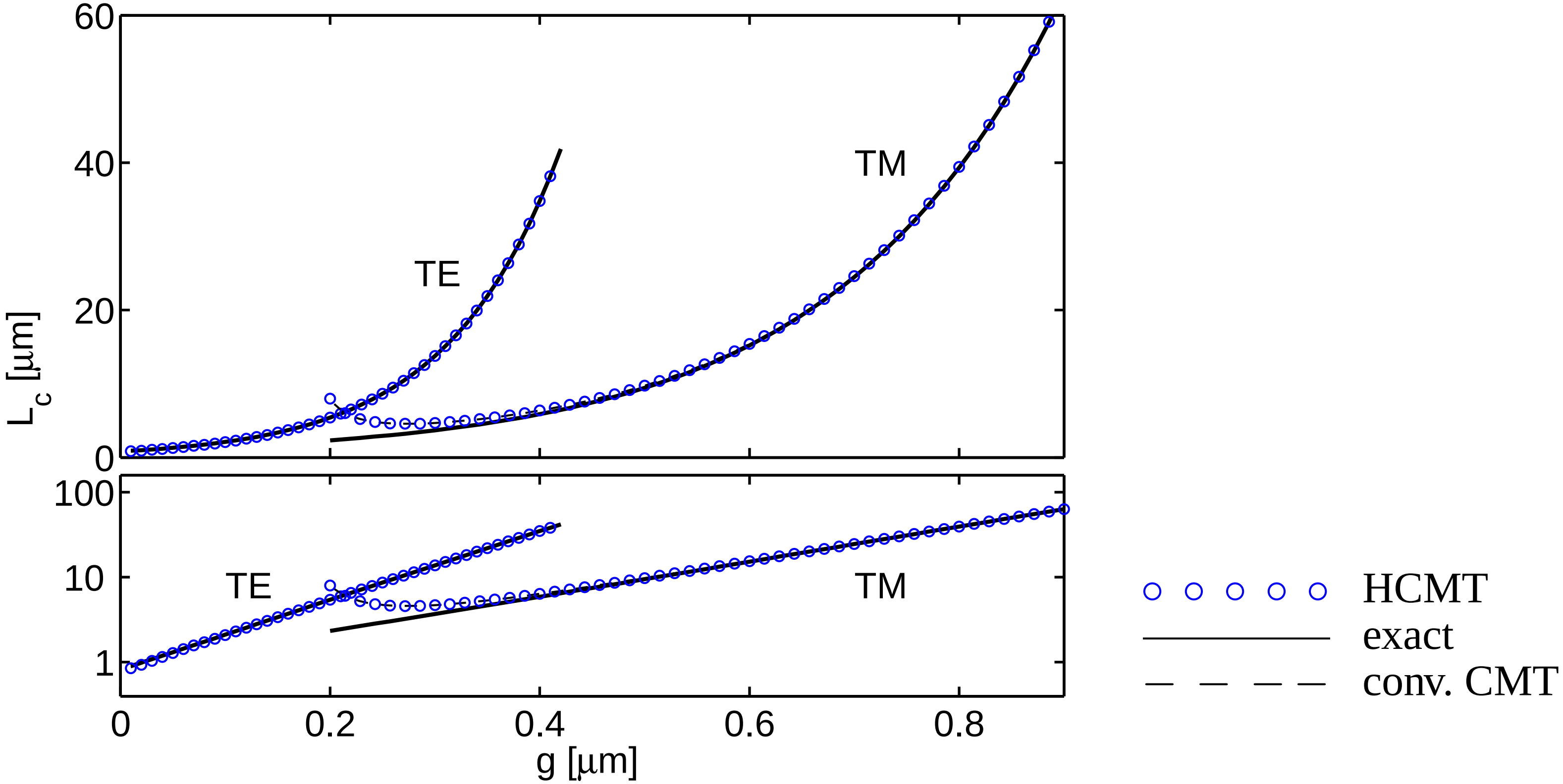



Basis elements: polarized forward propagating fundamental modes of the separate cores, input amplitude $f_b = 1$,

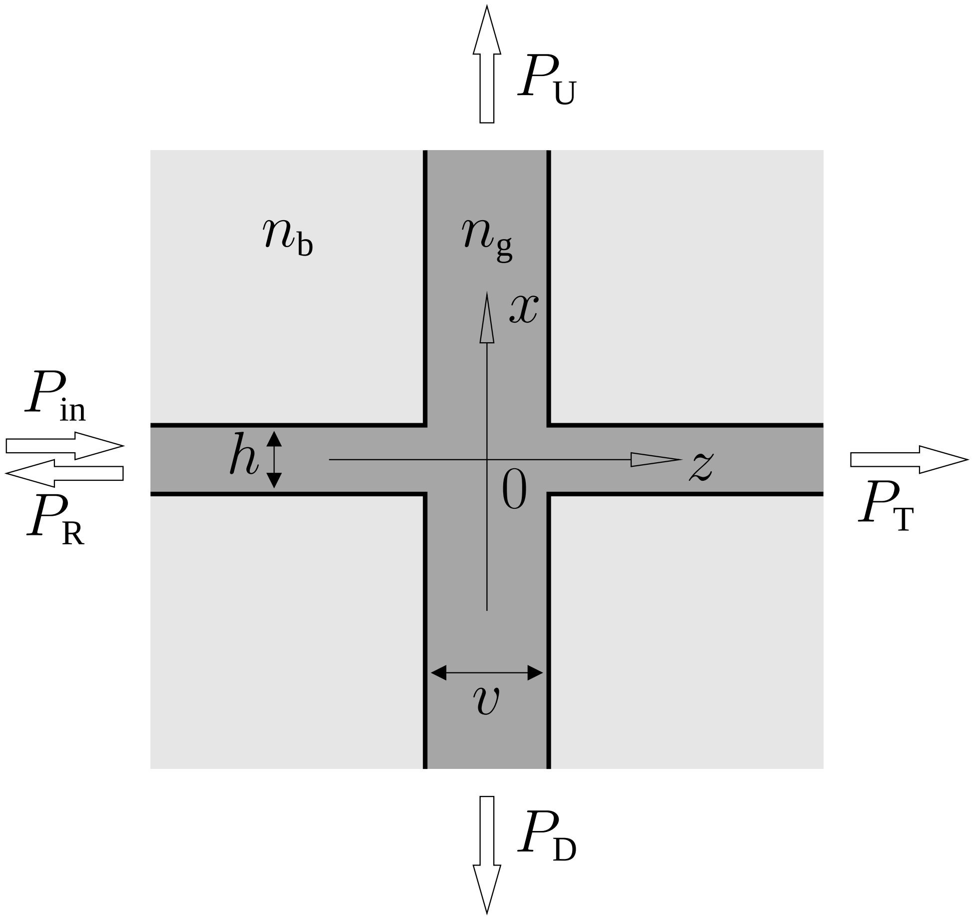
FEM discretization (TE):
$z \in [-20, 20]\,\mu$m, $\Delta z = 0.5\,\mu$m,

computational domain (TE):
$z \in [-20, 20]\,\mu$m, $x \in [-3.0, 3.0]\,\mu$m.

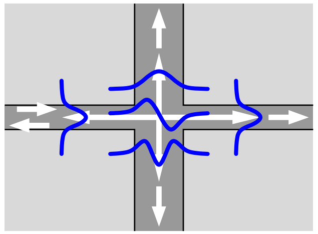
Coupling length:



## ***Waveguide crossing***




$n_g = 3.4,\ n_b = 1.45,\ \lambda = 1.55\,\mu$m,
$h = 0.2\,\mu$m, $v$ variable, TE polarization.

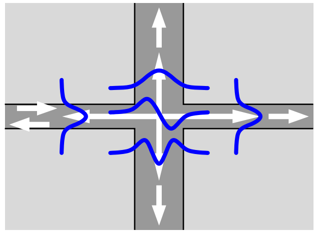


Basis elements:
guided modes of the horizontal and vertical cores
(directional variants).

FEM discretization:
$z \in [v/2 - 1.5\,\mu\text{m}, v/2 + 1.5\,\mu\text{m}]$, $\Delta x = 0.025\,\mu$m,
$x \in [w/2 - 1.5\,\mu\text{m}, w/2 + 1.5\,\mu\text{m}]$, $\Delta z = 0.025\,\mu$m.

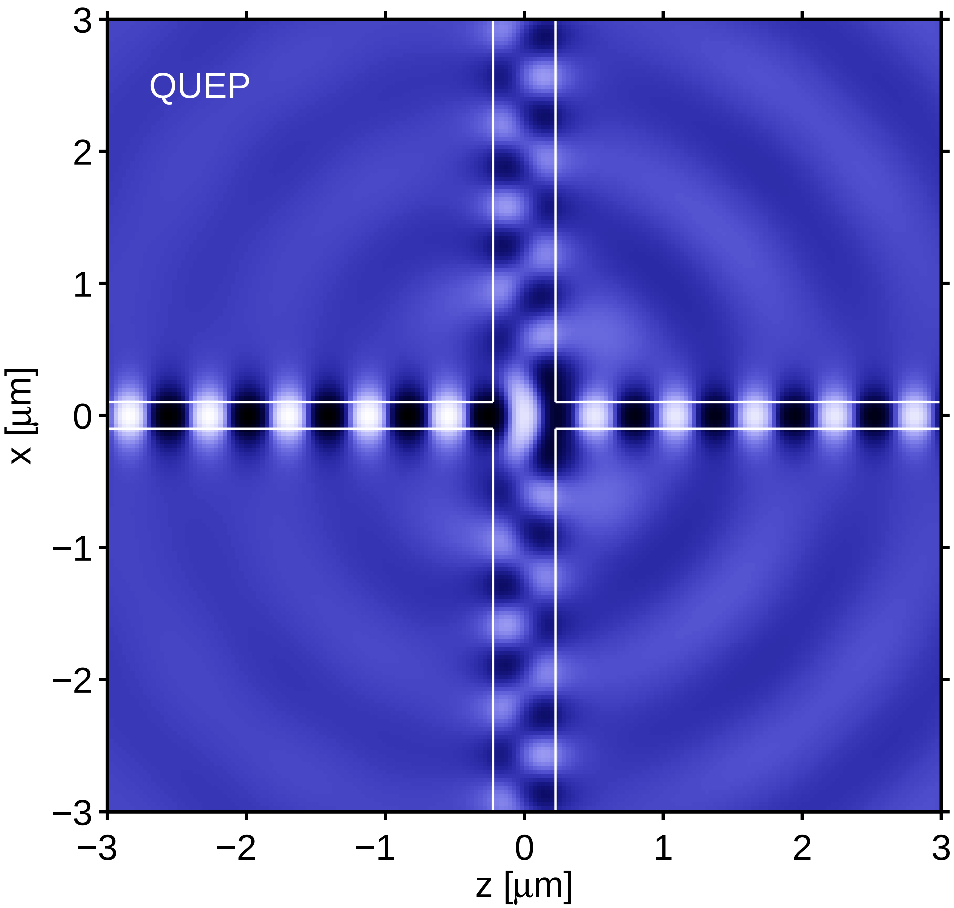
Computational window:
$z \in [-4\,\mu\text{m}, 4\,\mu\text{m}]$, $x \in [-4\,\mu\text{m}, 4\,\mu\text{m}]$.

## *Waveguide crossing, fields (I)*



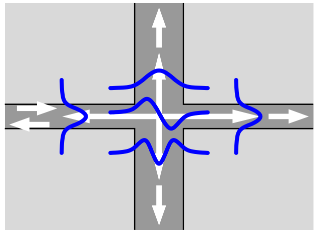
$v = 0.45\ \mu$m:

reference

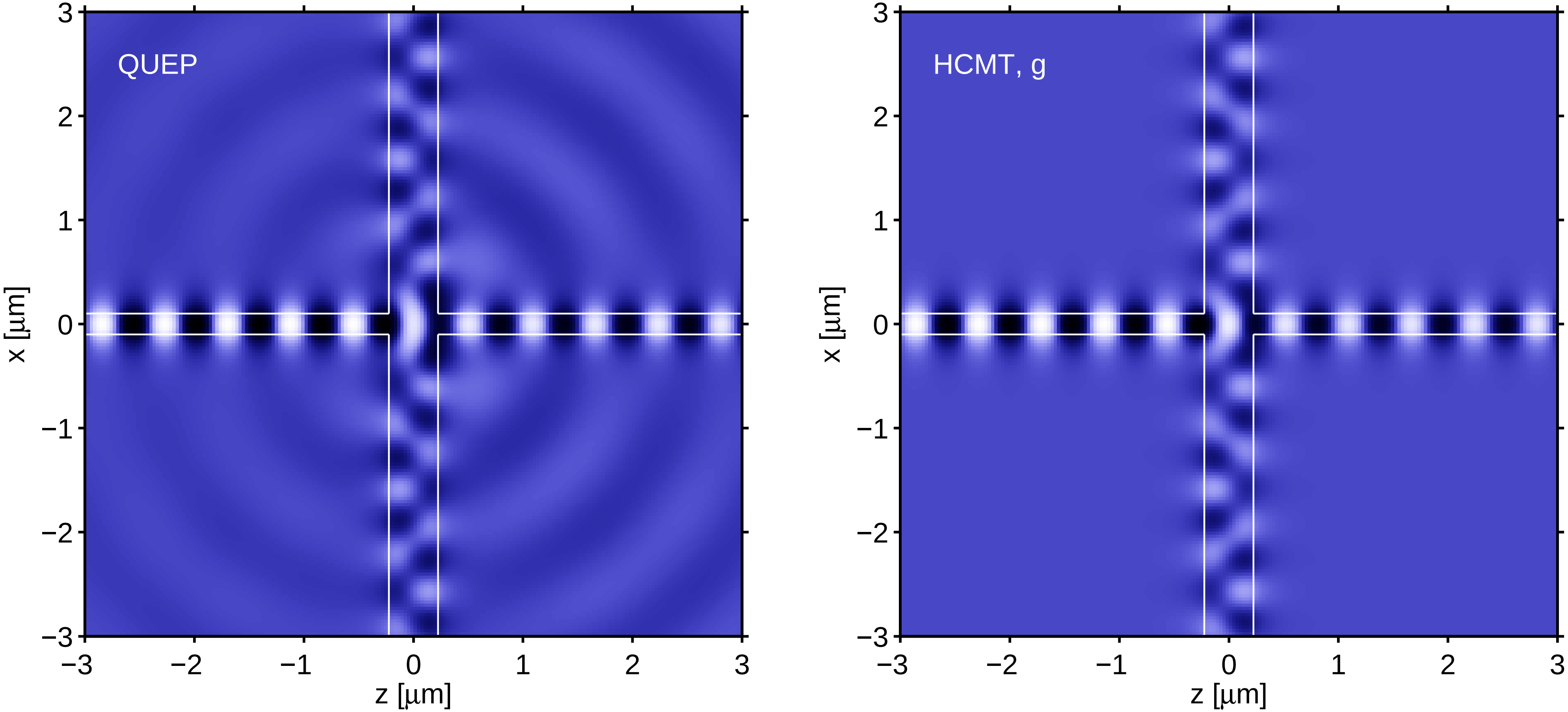


## *Waveguide crossing, fields   (I)*

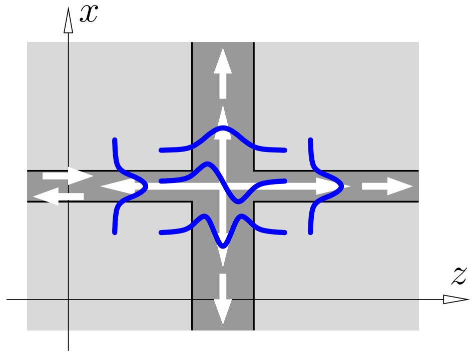
$v = 0.45\ \mu$m:



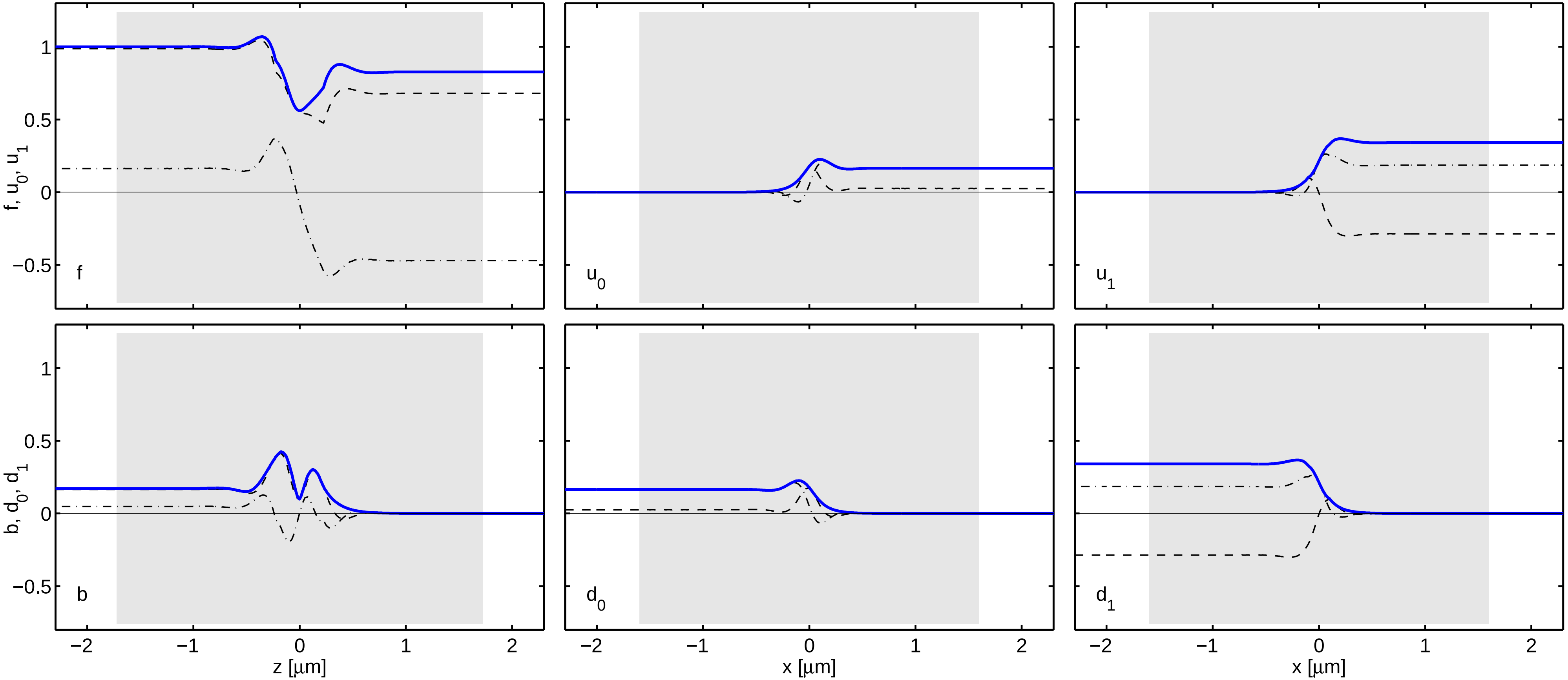
reference



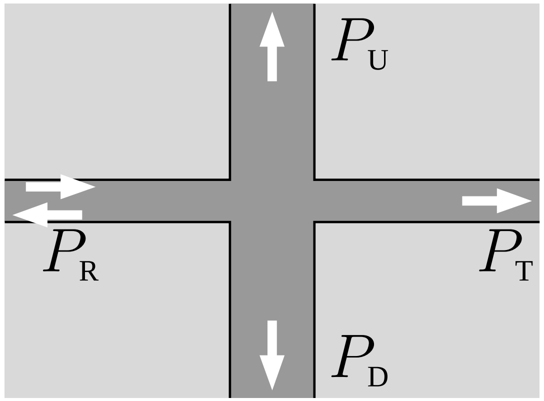
# *Waveguide crossing, amplitude functions*




$v = 0.45\,\mu$m:

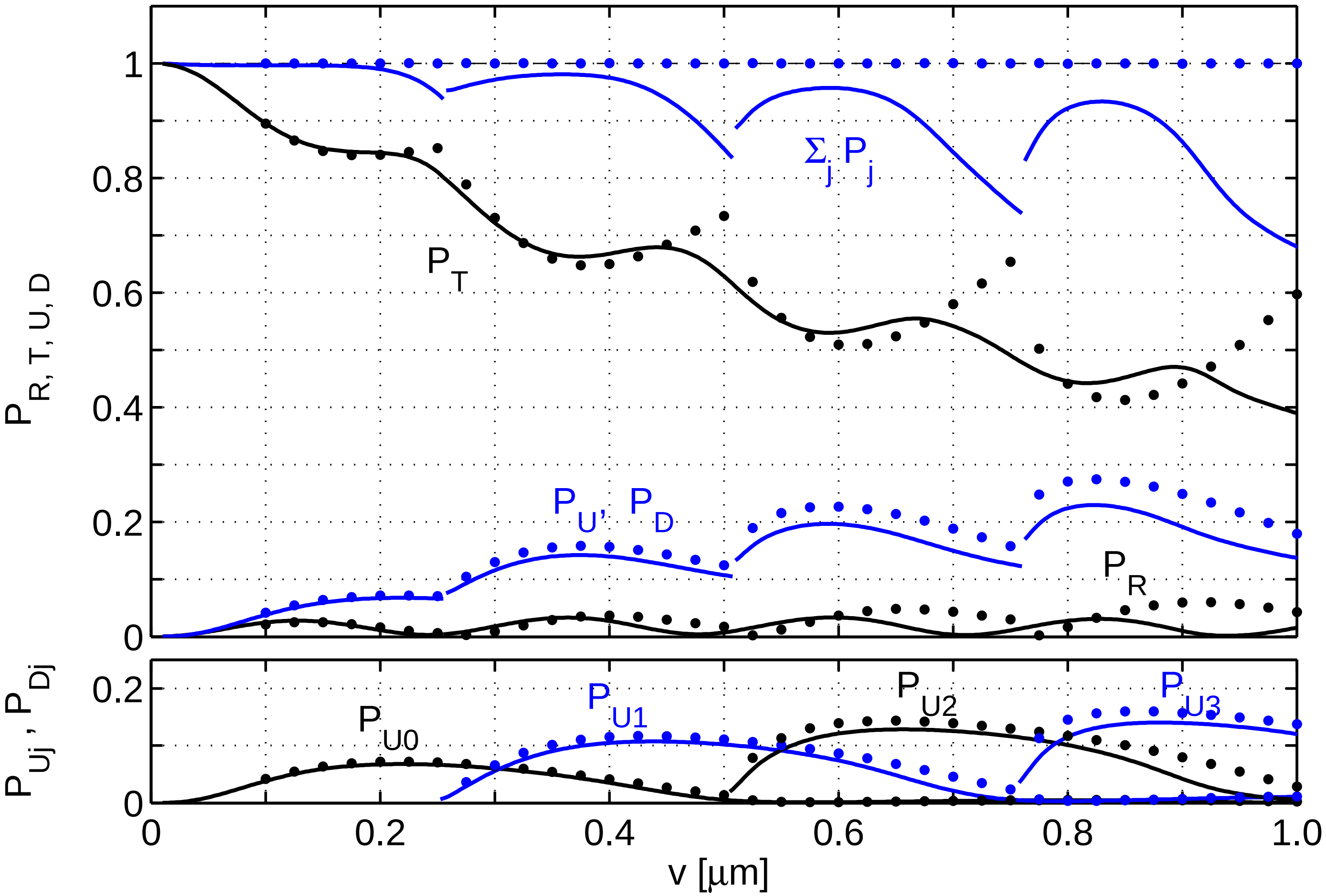


# *Waveguide crossing, power transfer (I)*



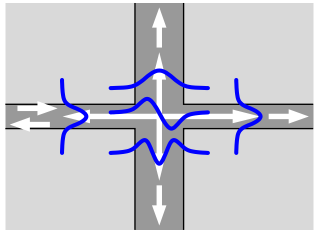

QUEP, reference

HCMT

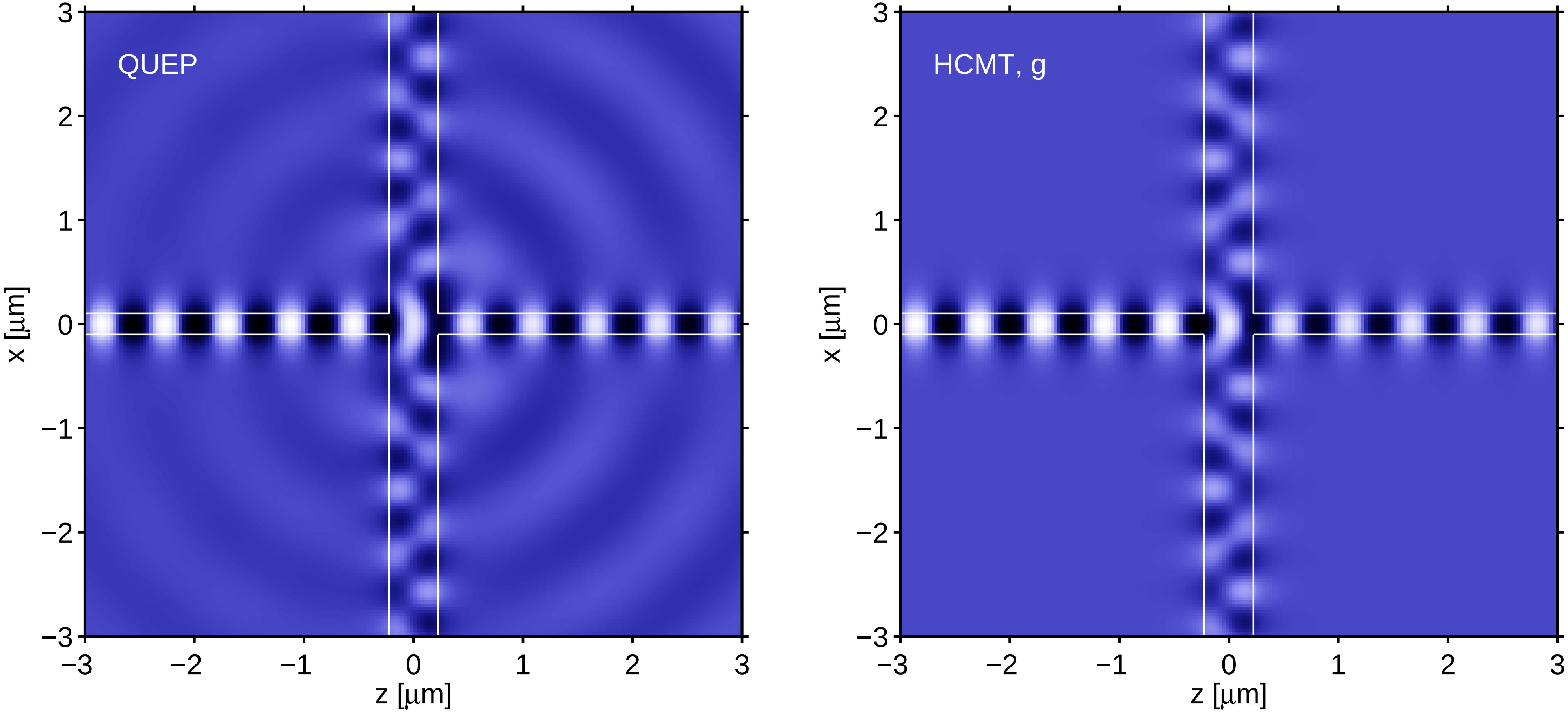


# *Waveguide crossing, fields (II)*

$v = 0.45\ \mu$m:



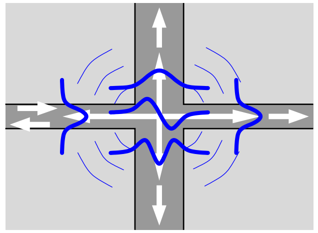
reference



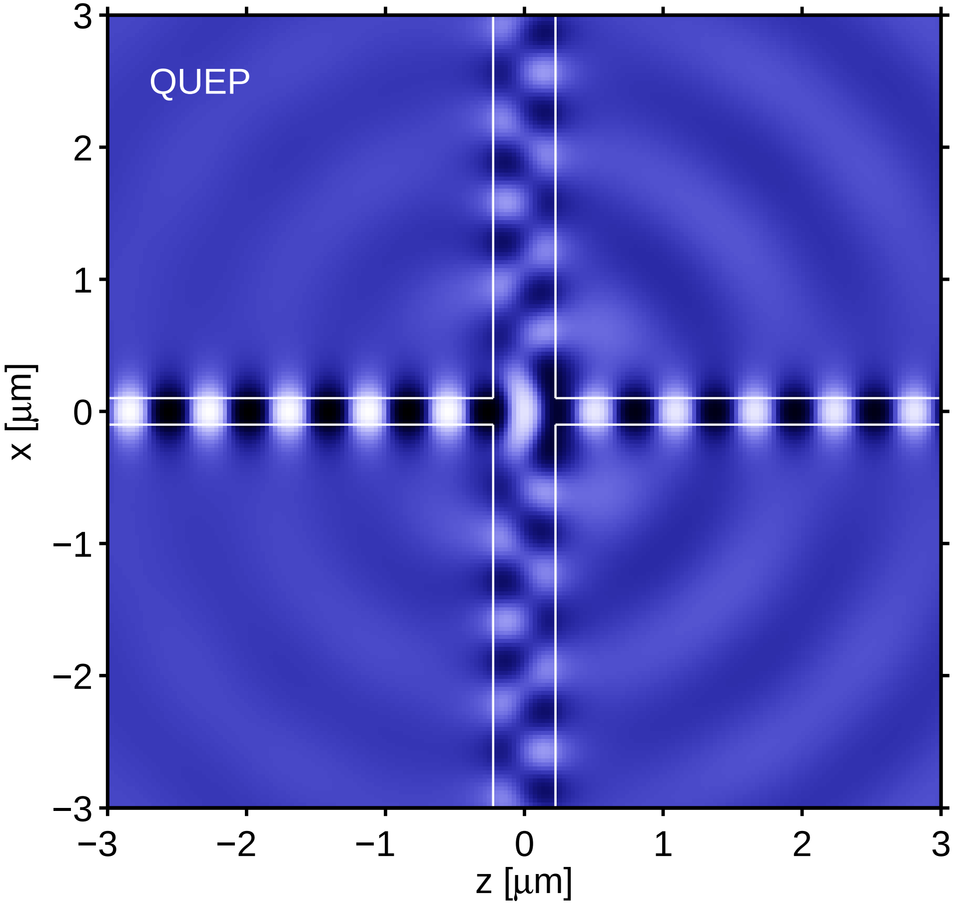
# *Waveguide crossing, fields (II)*

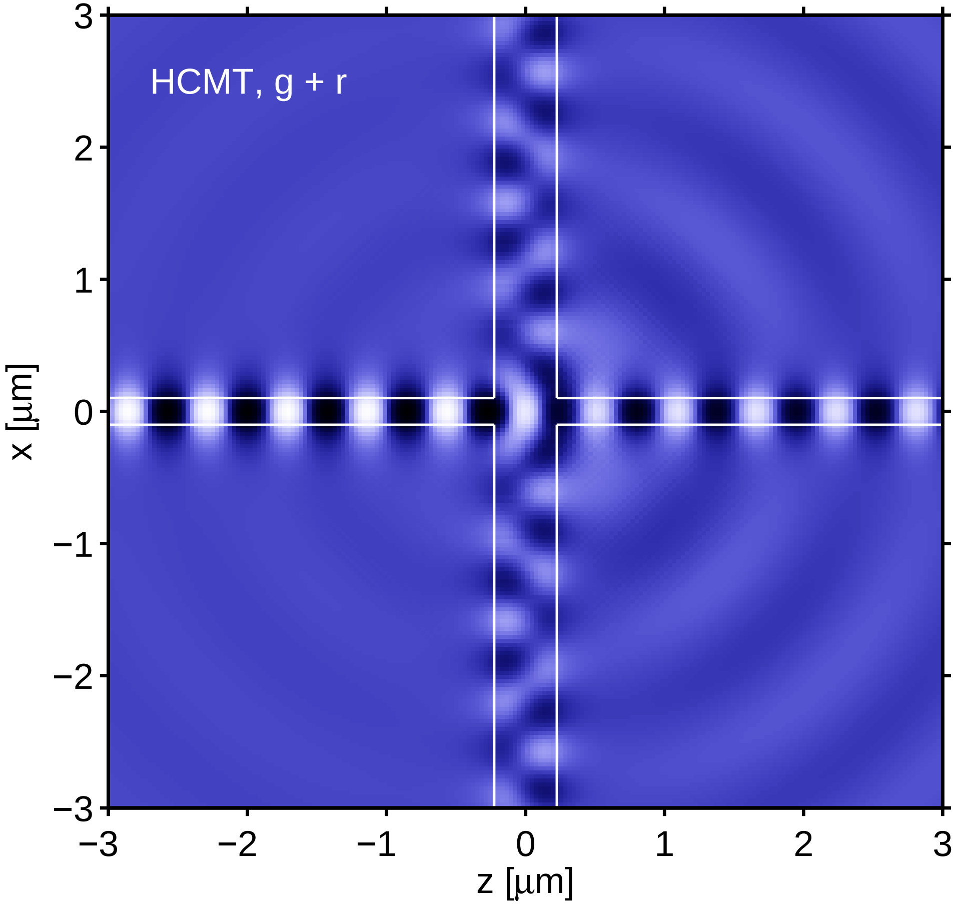
$v = 0.45\ \mu$m:

HCMT basis fields:
guided modes
$+$ 4 Gaussian beams,
outgoing along the diagonals.

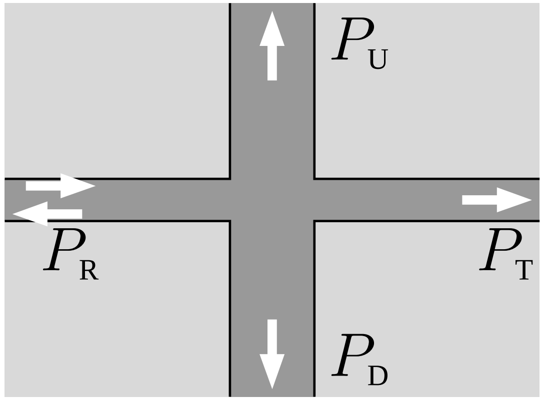


reference



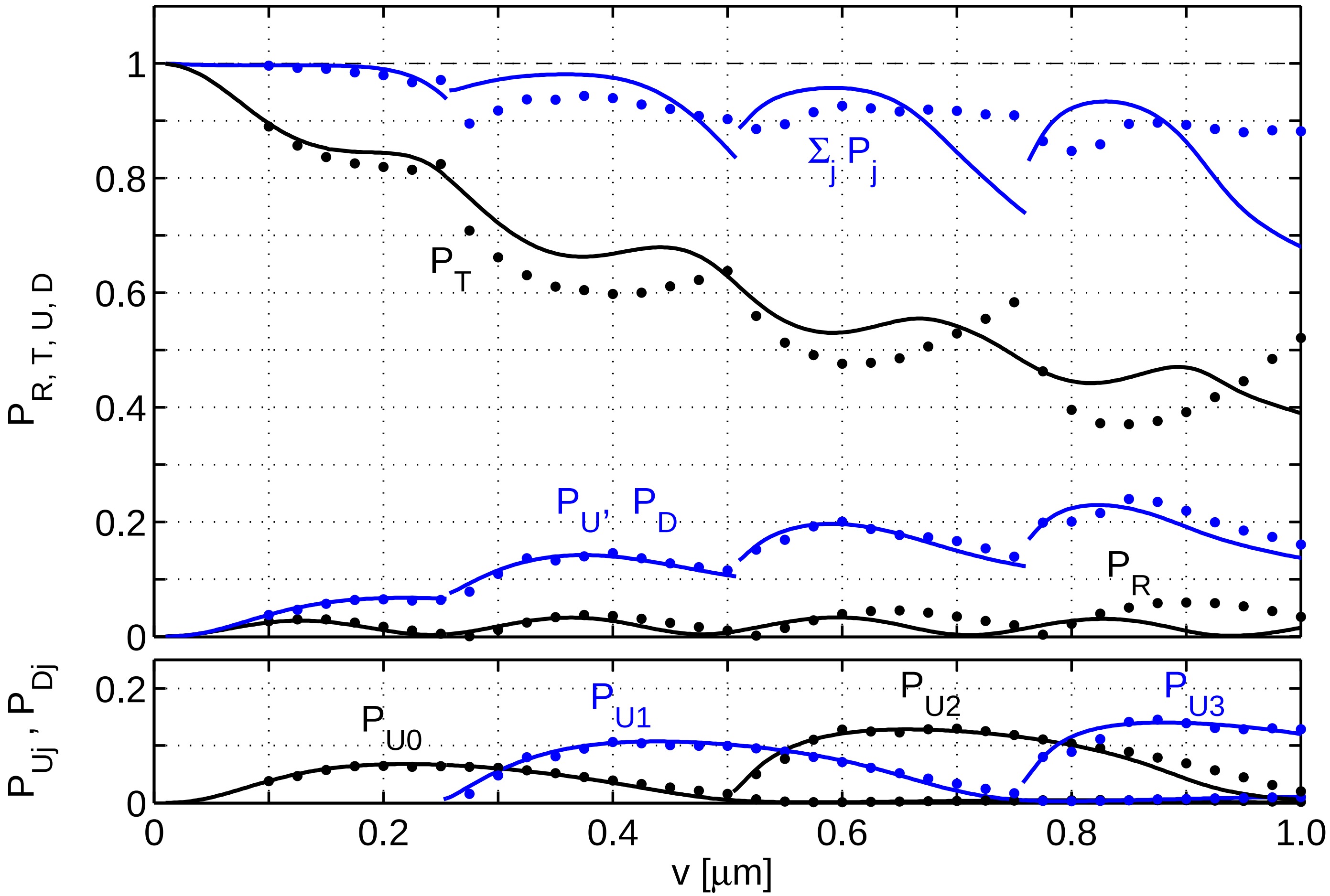



# *Waveguide crossing, power transfer (II)*

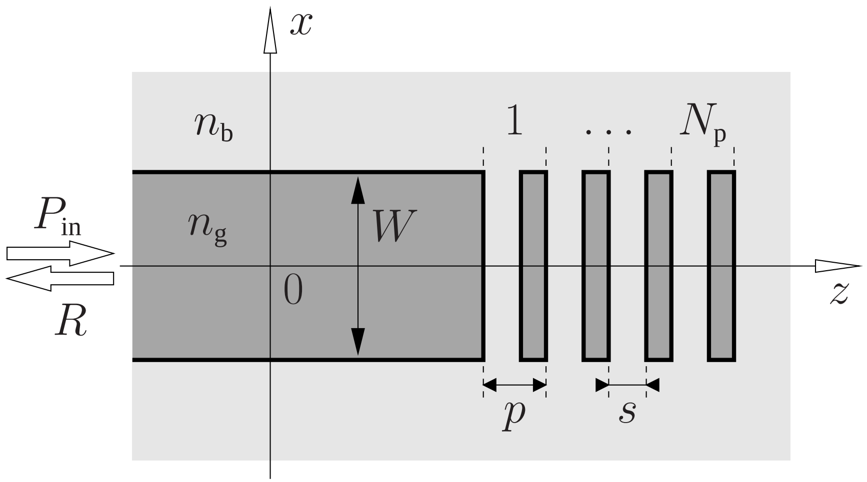



QUEP, reference

HCMT,
incl. templates
for radiated fields

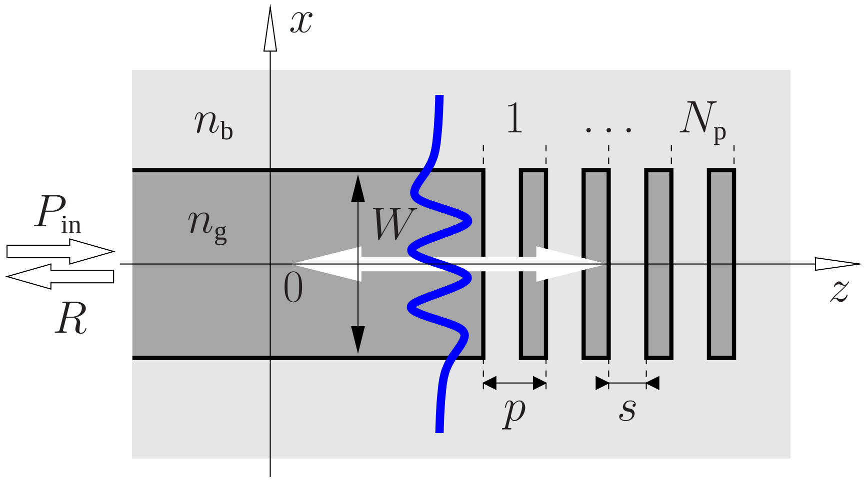


# *Waveguide Bragg reflector*



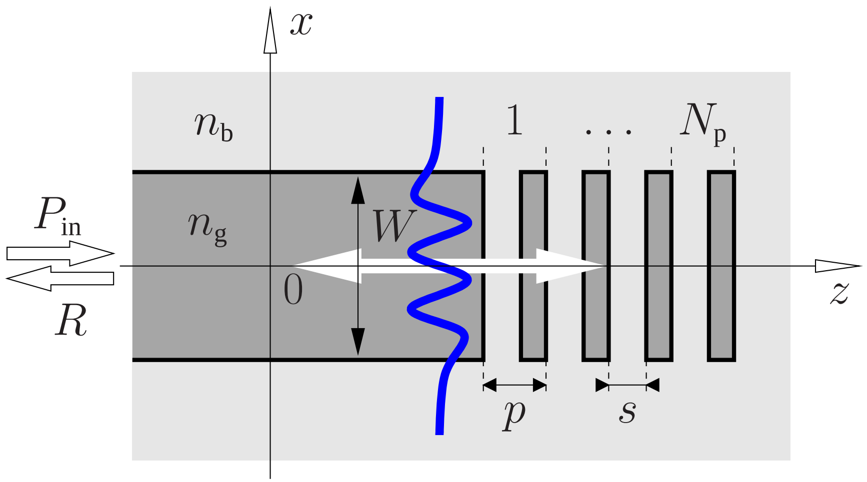

TE, $n_{\mathrm{g}} = 1.6$, $n_{\mathrm{b}} = 1.45$,
$p = 1.538\,\mu\mathrm{m}$, $s = 0.281\,\mu\mathrm{m}$,
$N_{\mathrm{p}} = 40$, $W = 9.955\,\mu\mathrm{m}$.

# *Waveguide Bragg reflector*

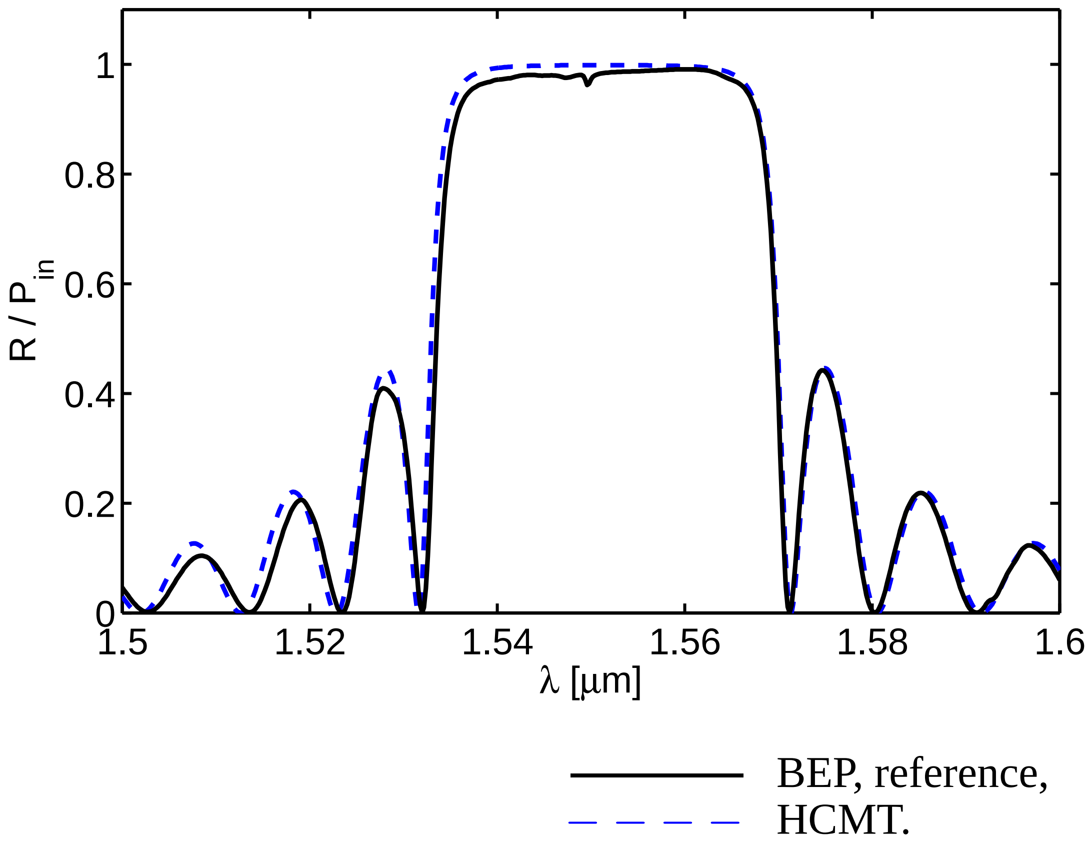



TE, $n_\mathrm{g} = 1.6$, $n_\mathrm{b} = 1.45$,
$p = 1.538\,\mu$m, $s = 0.281\,\mu$m,
$N_\mathrm{p} = 40$, $W = 9.955\,\mu$m.

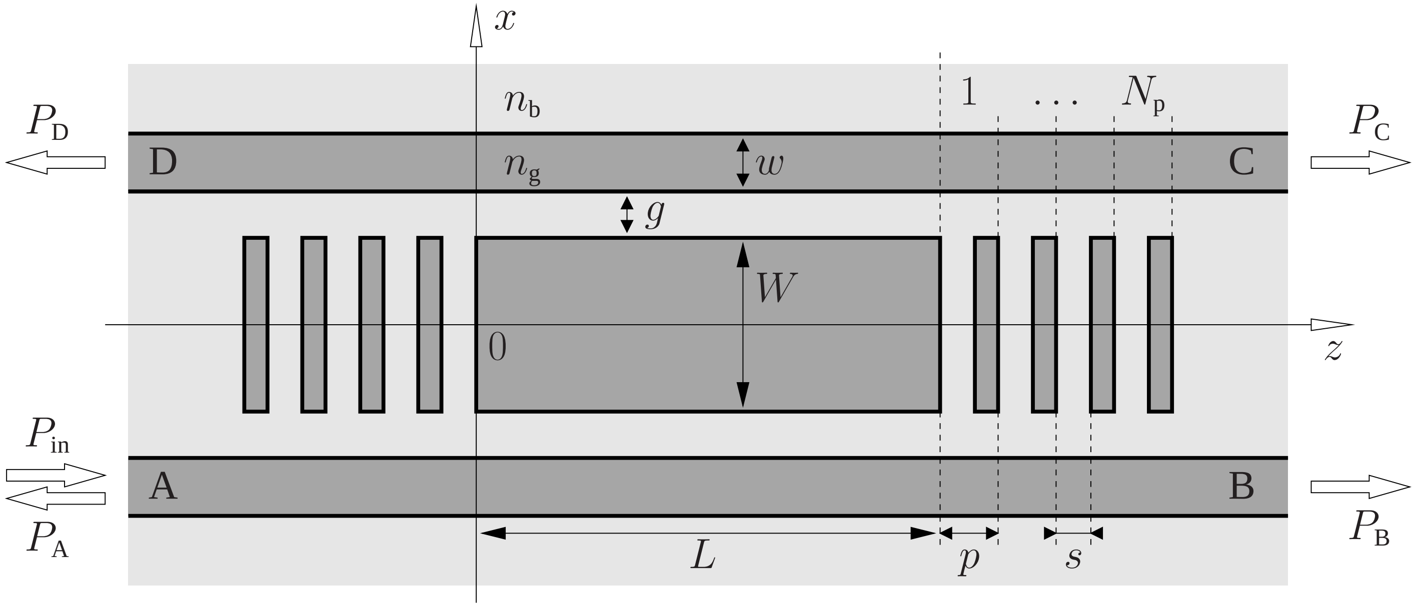
## *Waveguide Bragg reflector*




TE, $n_g = 1.6$, $n_b = 1.45$,
$p = 1.538\,\mu$m, $s = 0.281\,\mu$m,
$N_p = 40$, $W = 9.955\,\mu$m.

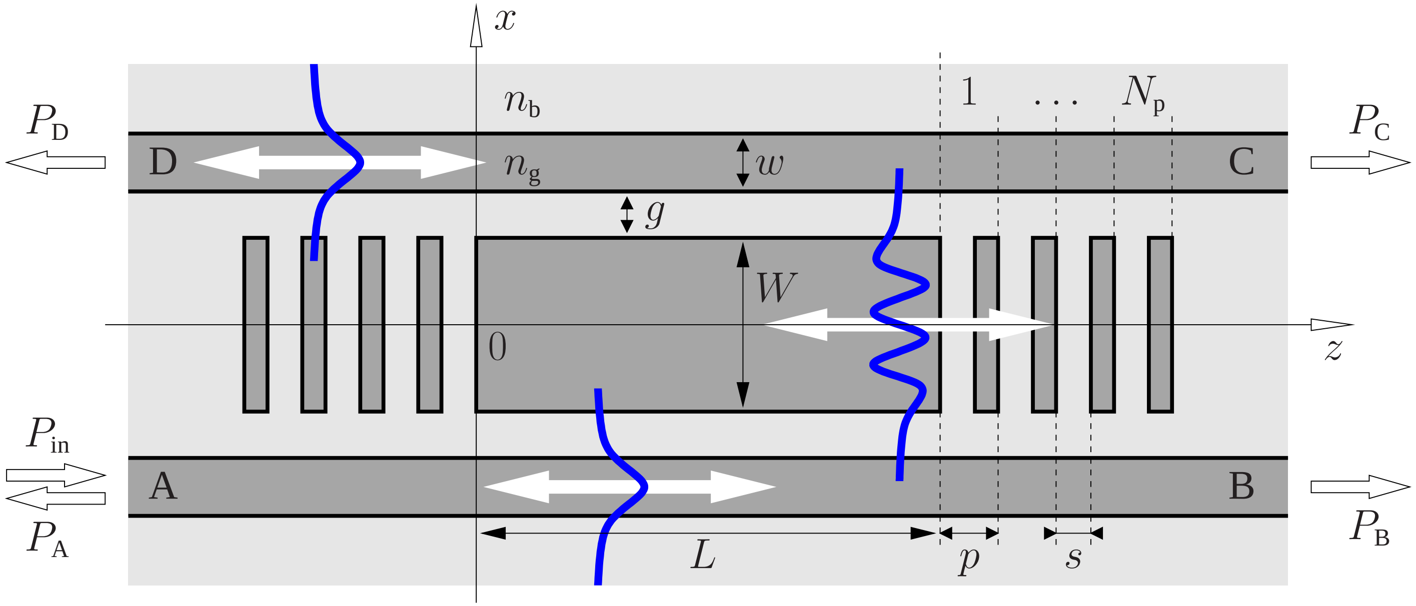


# *Grating-assisted rectangular resonator*



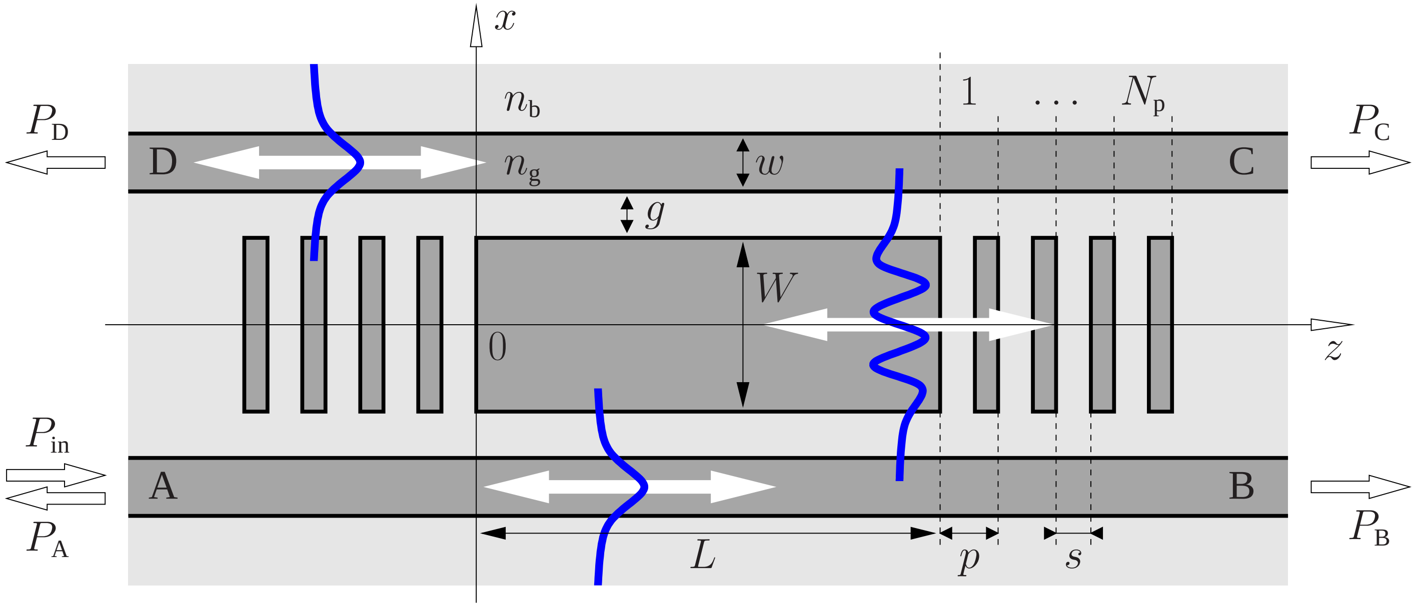

$n_g = 1.60$, $n_b = 1.45$,
$w = 1.0\,\mu$m, $g = 1.6\,\mu$m,
$W = 9.955\,\mu$m,
$L = 79.985\,\mu$m,
$p = 1.538\,\mu$m,
$s = 0.281\,\mu$m,
$N_p = 40$.

# *Grating-assisted rectangular resonator*

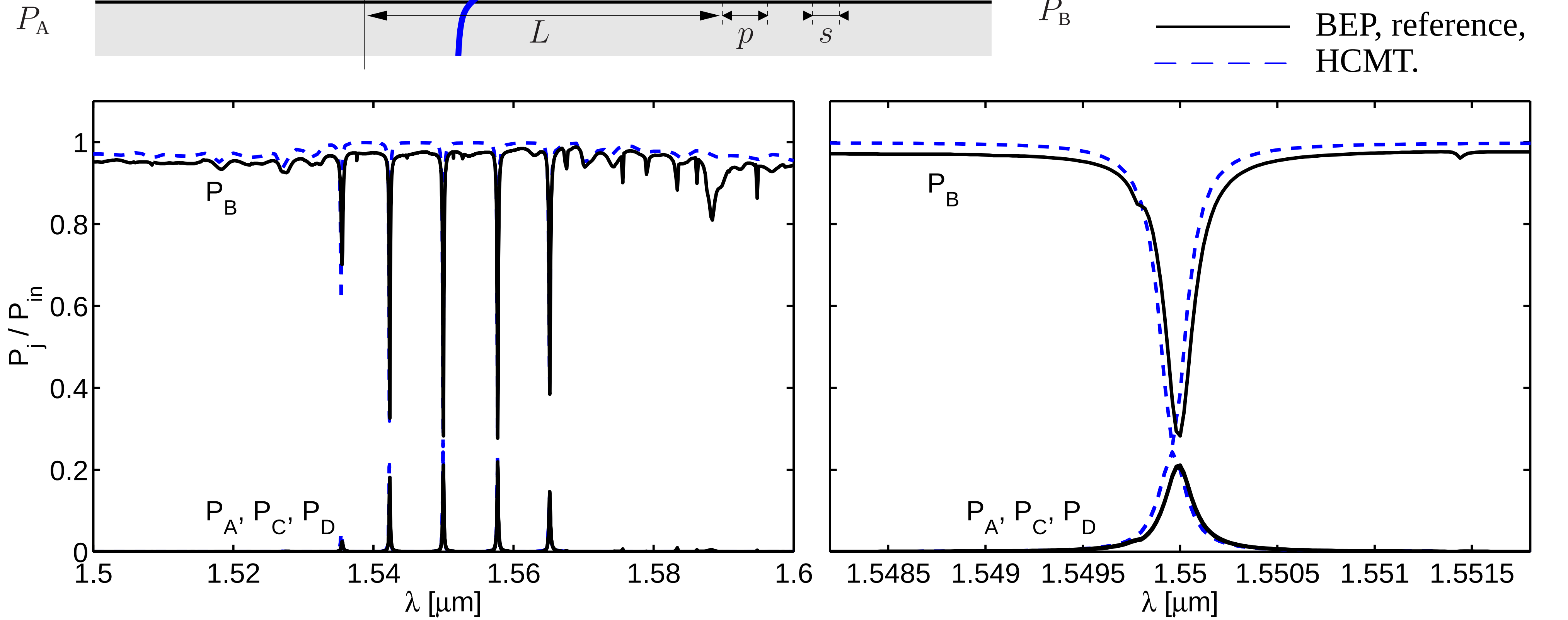



$n_g = 1.60$, $n_b = 1.45$,
$w = 1.0\,\mu$m, $g = 1.6\,\mu$m,
$W = 9.955\,\mu$m,
$L = 79.985\,\mu$m,
$p = 1.538\,\mu$m,
$s = 0.281\,\mu$m,
$N_p = 40$.

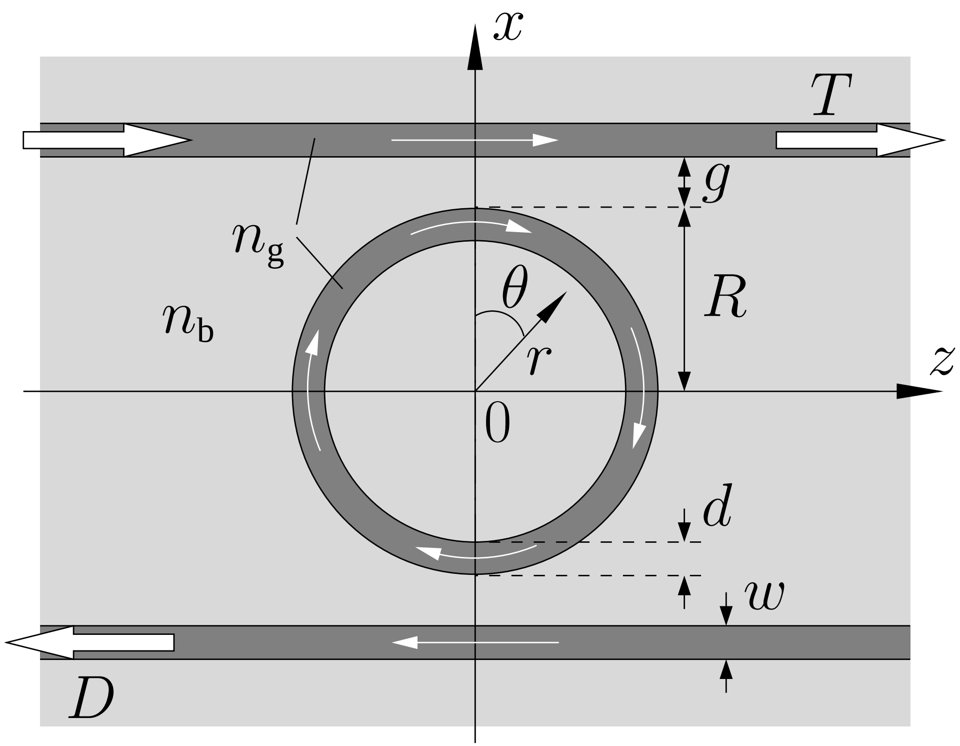
# ***Grating-assisted rectangular resonator***




$n_g = 1.60$, $n_b = 1.45$,
$w = 1.0\,\mu$m, $g = 1.6\,\mu$m,
$W = 9.955\,\mu$m,
$L = 79.985\,\mu$m,
$p = 1.538\,\mu$m,
$s = 0.281\,\mu$m,
$N_p = 40$.

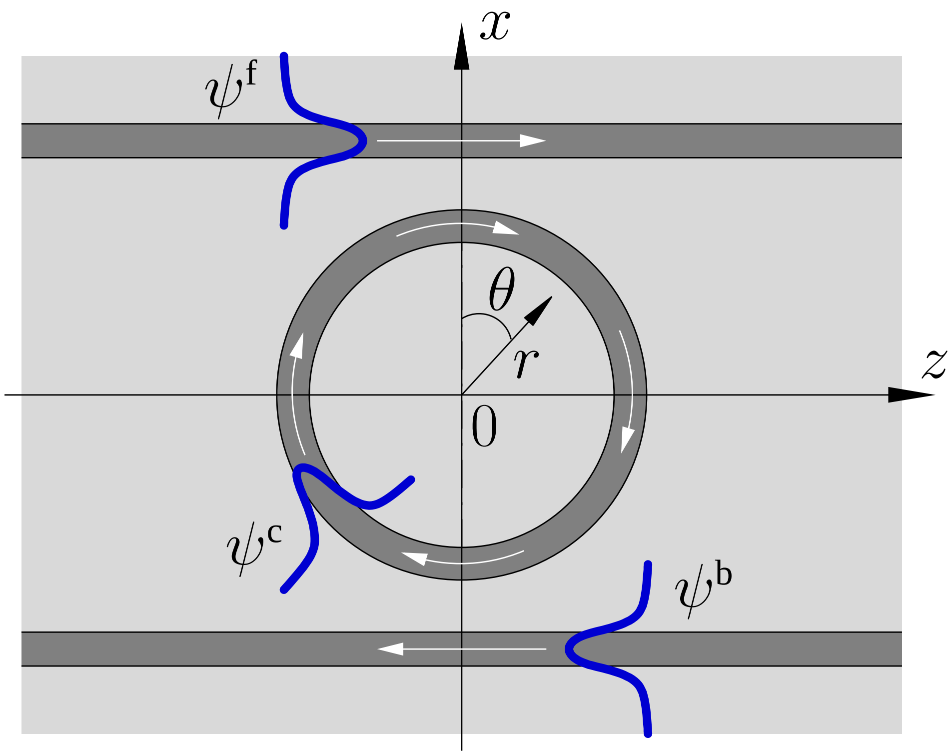


# *Ringresonator*




TE, $R = 7.5\,\mu$m, $w = 0.6\,\mu$m, $d = 0.75\,\mu$m, $g = 0.3\,\mu$m, $n_\mathrm{g} = 1.5$, $n_\mathrm{b} = 1.0$, $\lambda \approx 1.55\,\mu$m.

# *Ringresonator, field template*




Basis elements:

- bus WGs:

$$\boldsymbol{\psi}^{\rm f,b}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}^{\rm f,b}(x)\, {\rm e}^{\mp {\rm i}\beta z},$$

- cavity:

$$\boldsymbol{\psi}^{\rm c}(r,\theta) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}^{\rm c}(r)\, {\rm e}^{\mp {\rm i}\gamma R\theta},$$

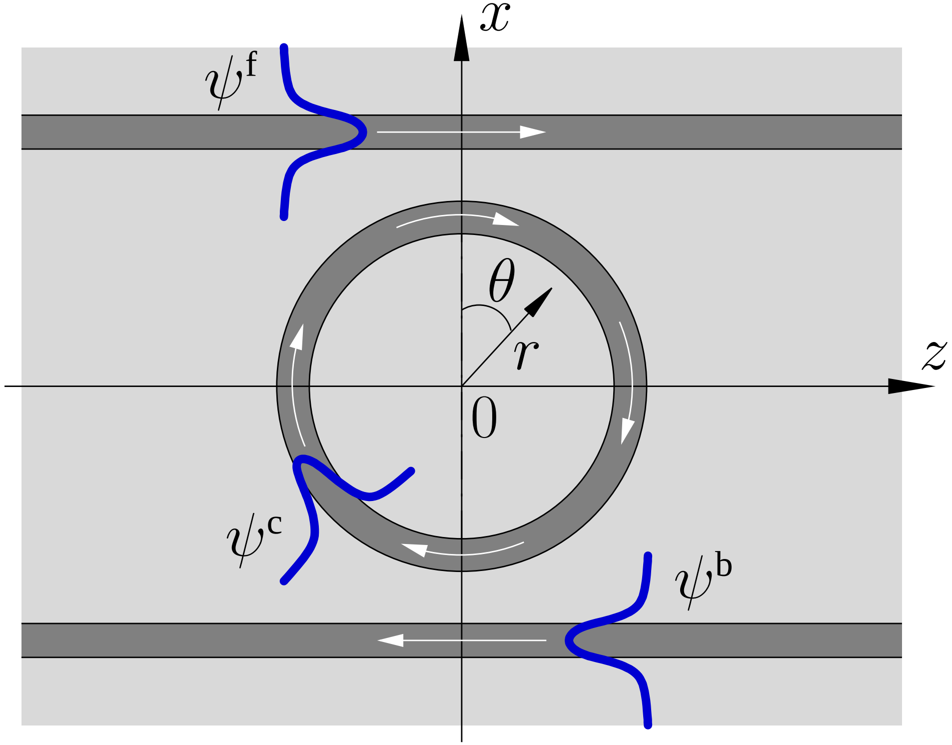
$$\gamma R \;\to\; \text{floor}(\text{Re}\gamma R + 1/2),$$

- further terms: bidirectional propagation, higher order modes, other channels, etc. .

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = f(z)\,\boldsymbol{\psi}^{\rm f}(x,z) + b(z)\,\boldsymbol{\psi}^{\rm b}(x,z) + c(\theta)\,\boldsymbol{\psi}^{\rm c}(r,\theta),$$

$$r = r(x,z),\ \theta = \theta(x,z).$$

$f$, $b$, $c$: ?




1-D FEM discretization:

$f(z) \to \{f_j\}$,
$b(z) \to \{b_j\}$,

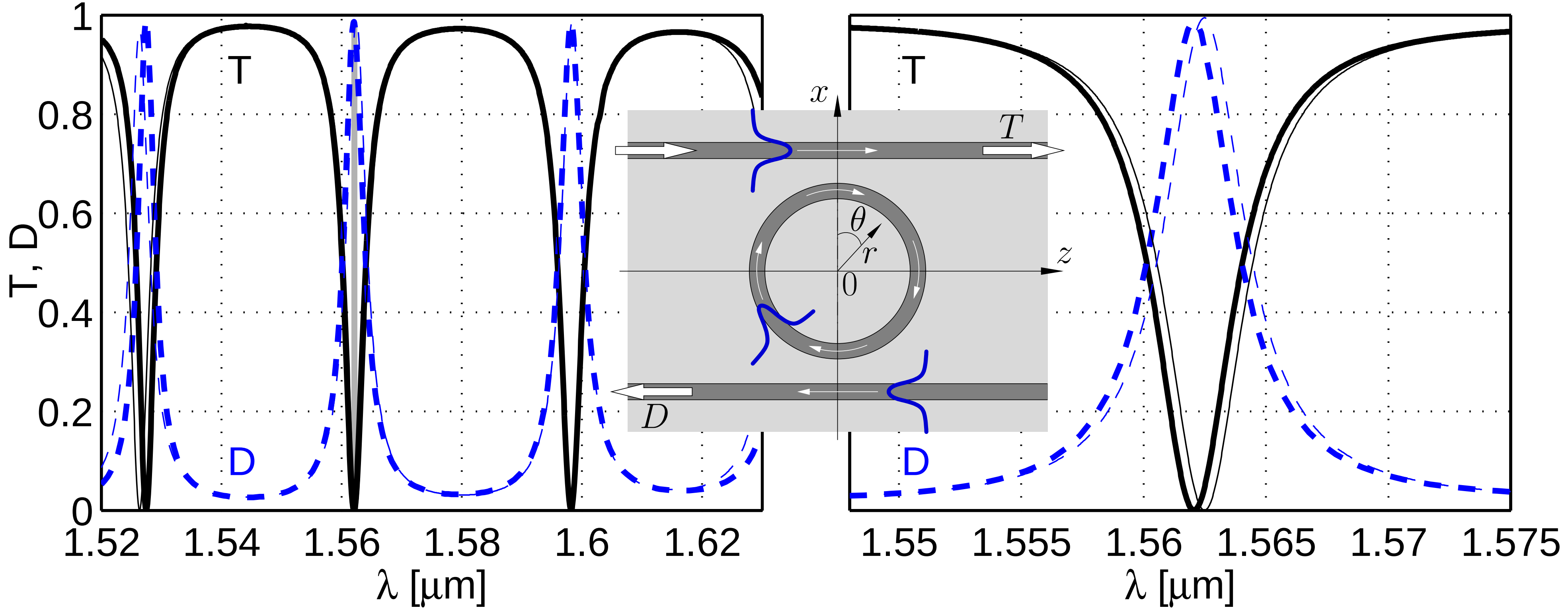
$c(\theta) \to \{c_j\}$, identify nodes 0 and $N_\theta$,
$r \to r(x,z),\ \theta \to \theta(x,z)$.

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \sum_k a_k \Big(\alpha_\cdot(\cdot)\boldsymbol{\psi}^{\cdot}_{\cdot}\Big)(x,z) =: \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}(x,z),$$

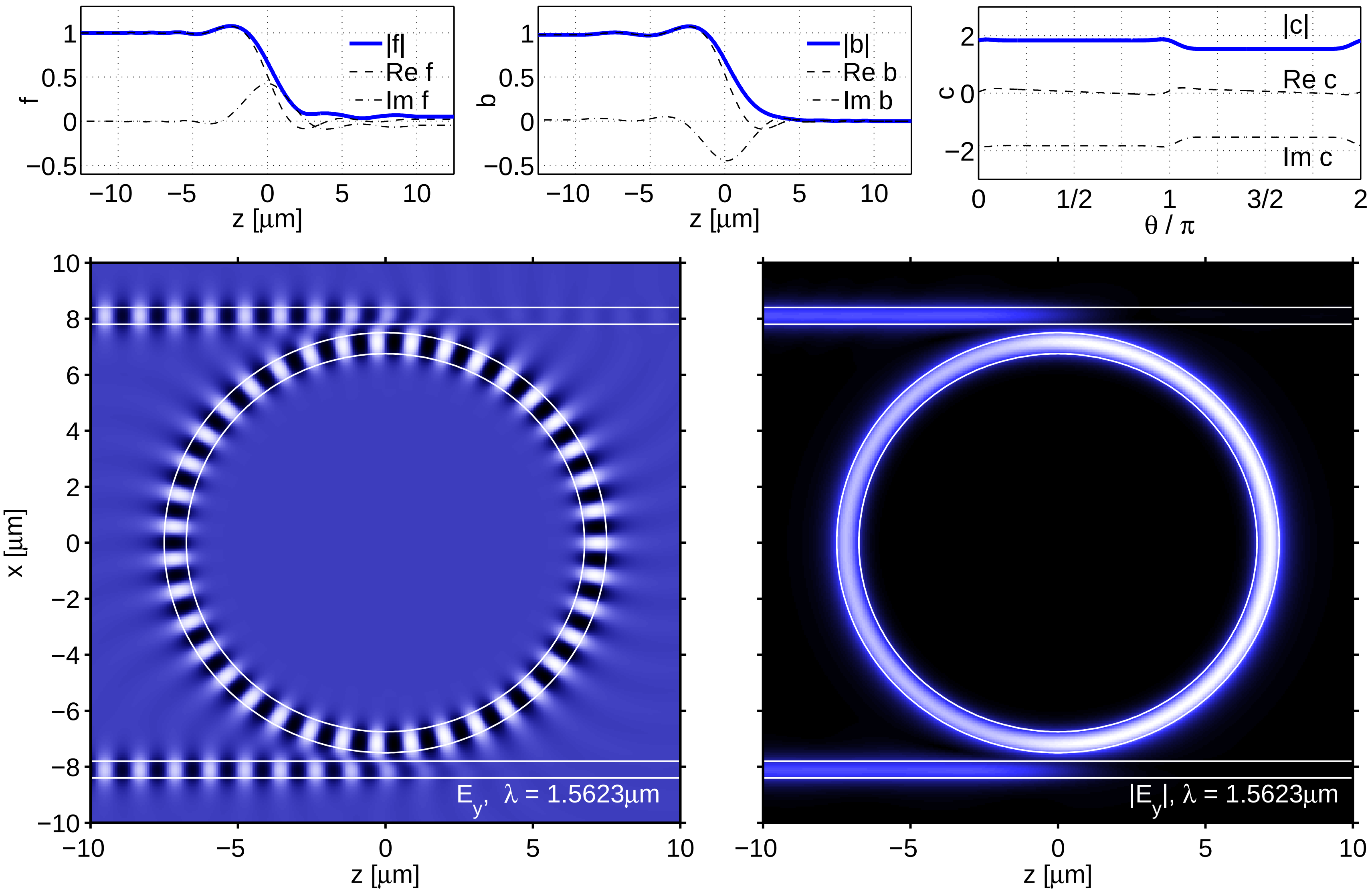
$k \in \{$channels, modes, elements$\}$, $\quad a_k \in \{f_j, b_j, c_j\}$.

HCMT solution as before.

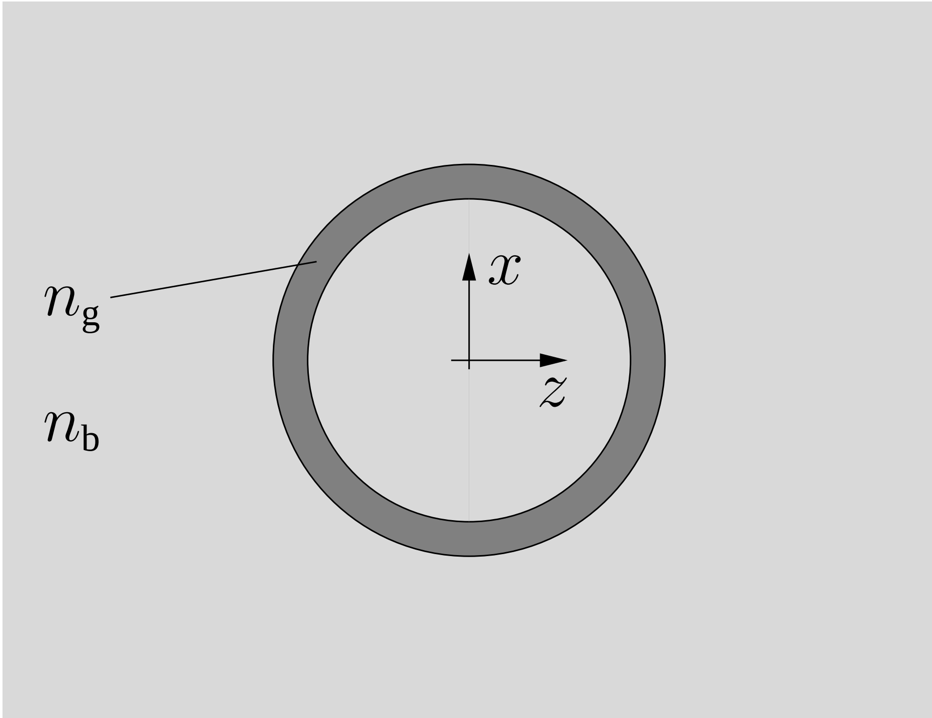
# *Single ring filter, spectral response*



# Single ring filter, resonance

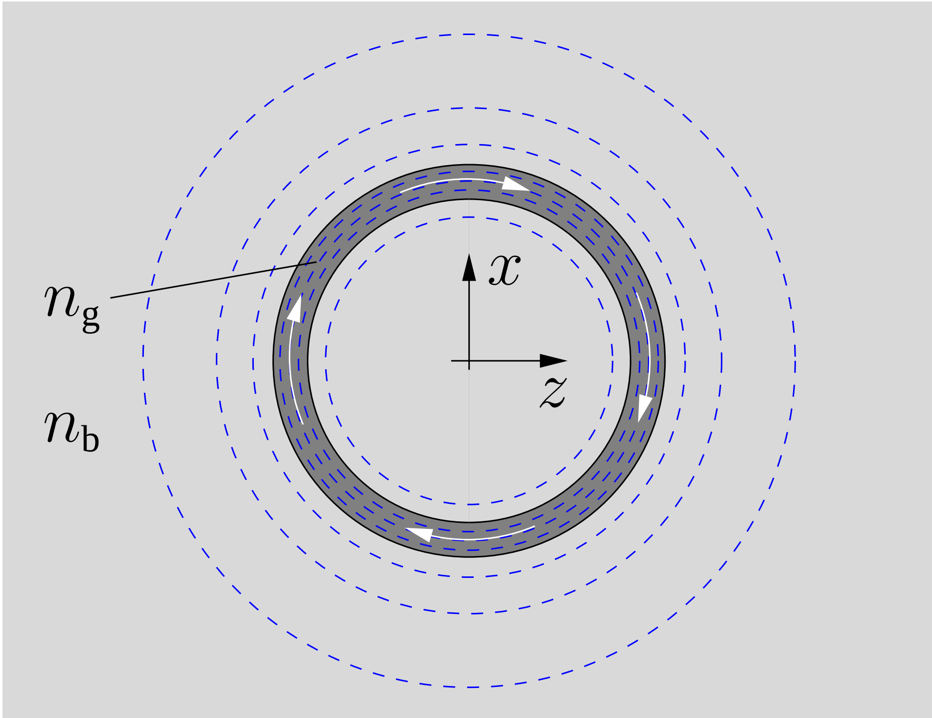


# *Excitation of whispering gallery resonances*



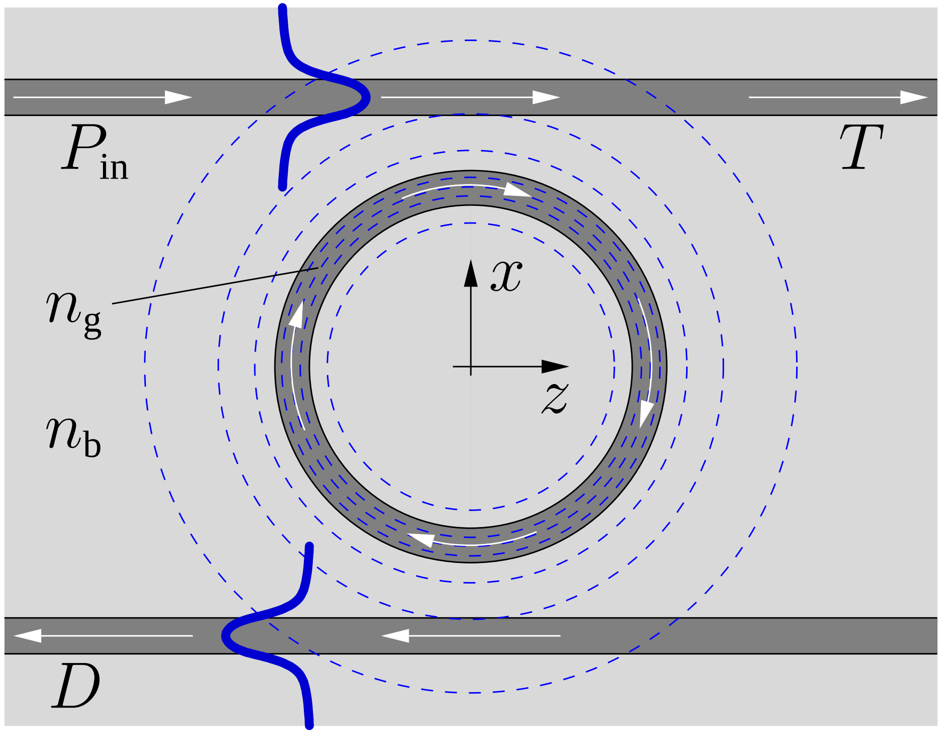

$n_{\mathrm{g}} > n_{\mathrm{b}}$

# *Excitation of whispering gallery resonances*



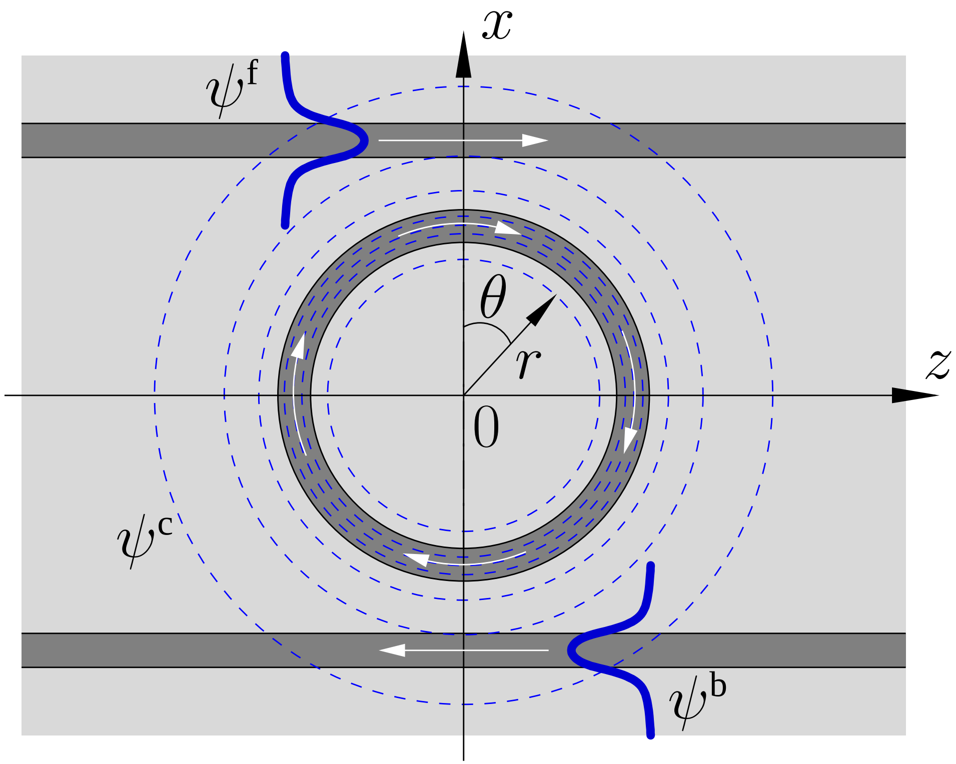

$$n_{\rm g} > n_{\rm b} \; , \qquad \left\{ \omega_j^{\rm c}, \; \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_j^{\rm c} (x, z) \right\}$$

# *Excitation of whispering gallery resonances*




$n_{\text{g}} > n_{\text{b}}$ ,  $\left\{ \omega_j^{\text{c}},\ \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_j^{\text{c}}(x,z) \right\}$,  $P_{\text{in}}(\omega)$ given: $T(\omega)$, $D(\omega)$ = ?

# *Ringresonator, field template*

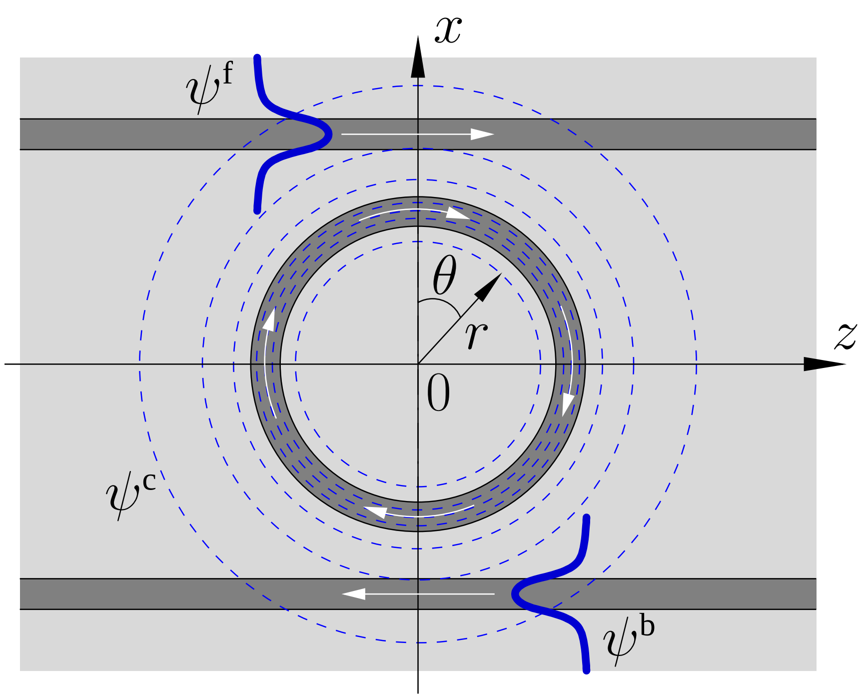



- Frequency $\omega$ given, $\sim \exp(\mathrm{i}\omega t)$.
- Bus channels:
$$\boldsymbol{\psi}^{\mathrm{f,b}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}^{\mathrm{f,b}}(x)\, \mathrm{e}^{\mp \mathrm{i}\beta z}.$$
- Cavity, WGMs:
$$\boldsymbol{\psi}^{\mathrm{c}}_j(r,\theta) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}^{\mathrm{c}}_j(r)\, \mathrm{e}^{-\mathrm{i}m_j\theta},$$
$m_j \in \mathbb{Z}$.
- Further terms:
bidirectional propagation, higher order modes, other channels, etc. .

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = f(z)\,\boldsymbol{\psi}^{\mathrm{f}}(x,z) + b(z)\,\boldsymbol{\psi}^{\mathrm{b}}(x,z) + \sum_j c_j\,\boldsymbol{\psi}^{\mathrm{c}}_j(r,\theta),$$

$$r = r(x,z),\ \theta = \theta(x,z). \qquad f,\, b,\, c_j : \ ?$$

# *Ringresonator, HCMT procedure*




Channels: 1-D FEM discretization,

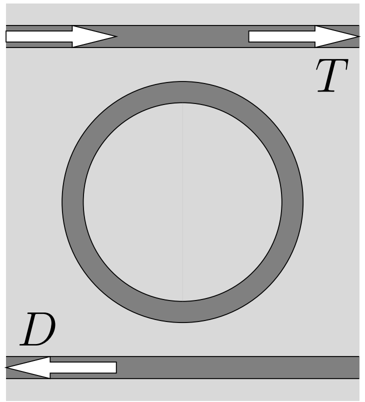
$f(z) \to \{f_j\}$,
$b(z) \to \{b_j\}$.

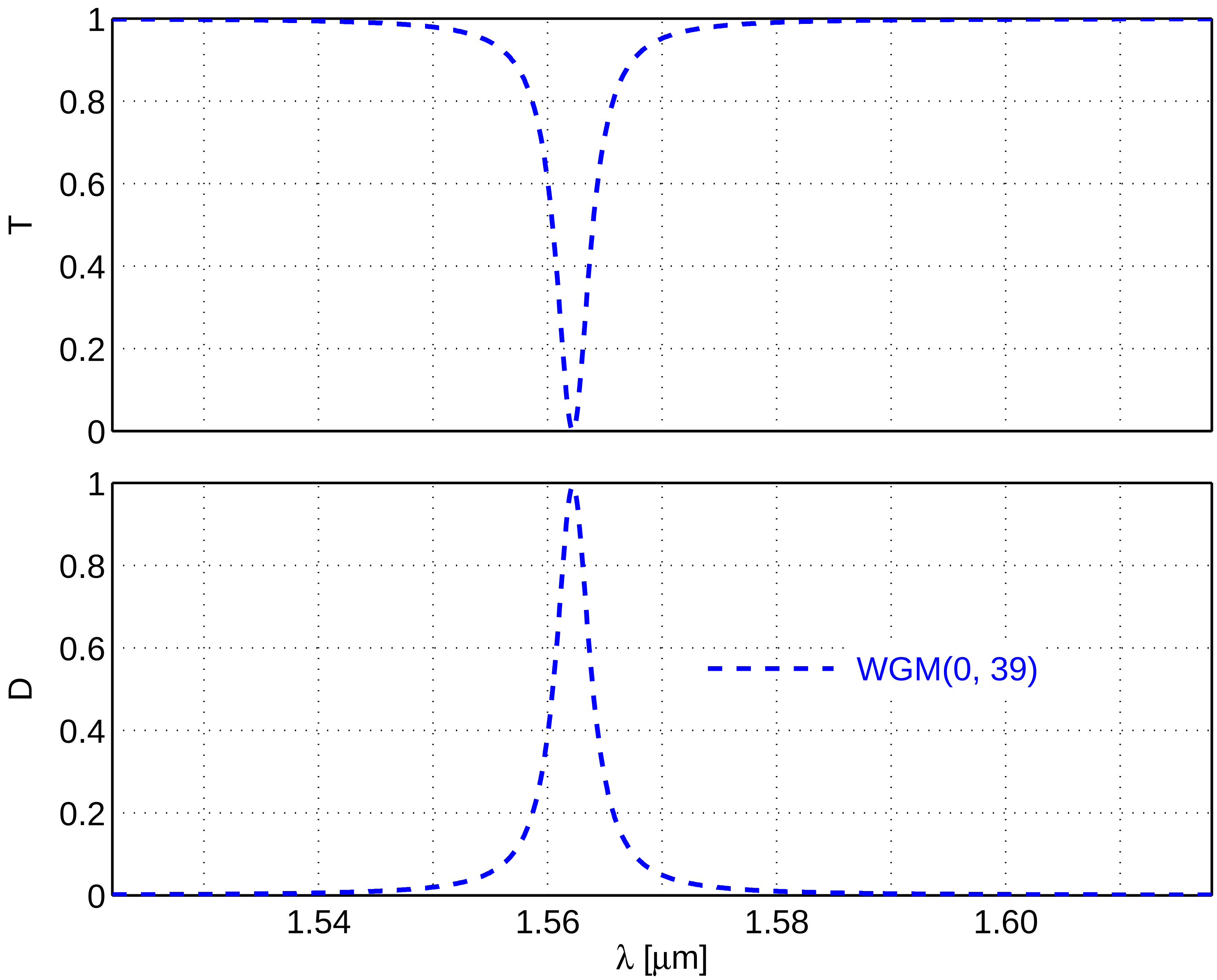
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \sum_j f_j\, (\alpha_j \boldsymbol{\psi}_j^{\mathrm{f}})(x,z) + \sum_j b_j\, (\alpha_j \boldsymbol{\psi}_j^{\mathrm{b}})(x,z) + \sum_j c_j\, \boldsymbol{\psi}_j'^{\mathrm{c}}(x,z)$$

$$=: \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}(x,z), \qquad a_k \in \{f_j, b_j, c_j\}.$$

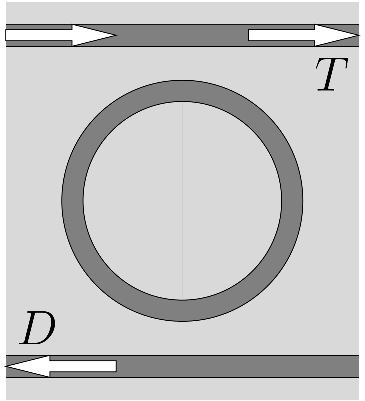
HCMT solution as before.

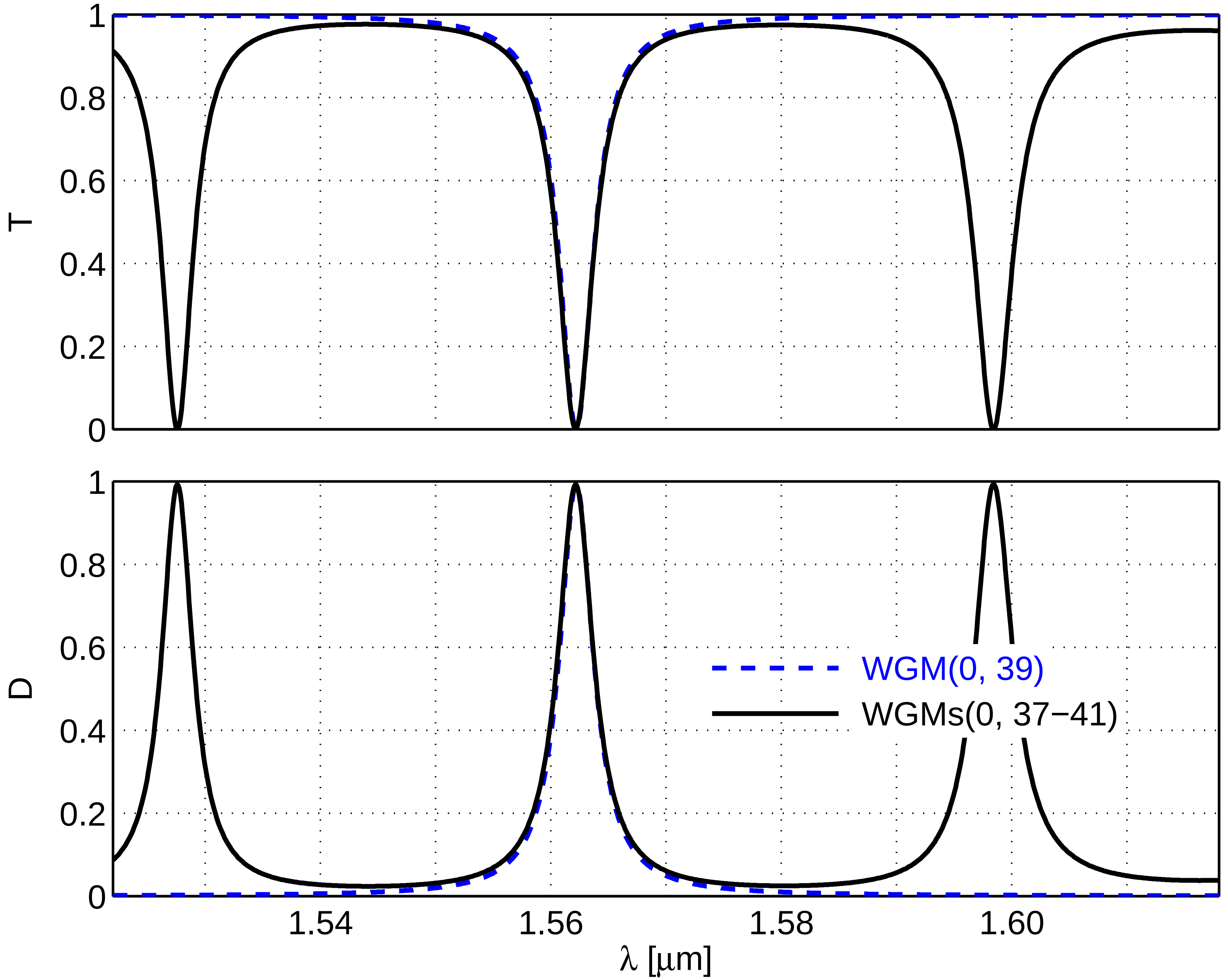
# *Single ring filter, spectral response*



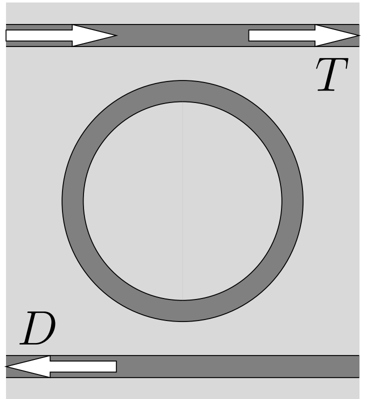



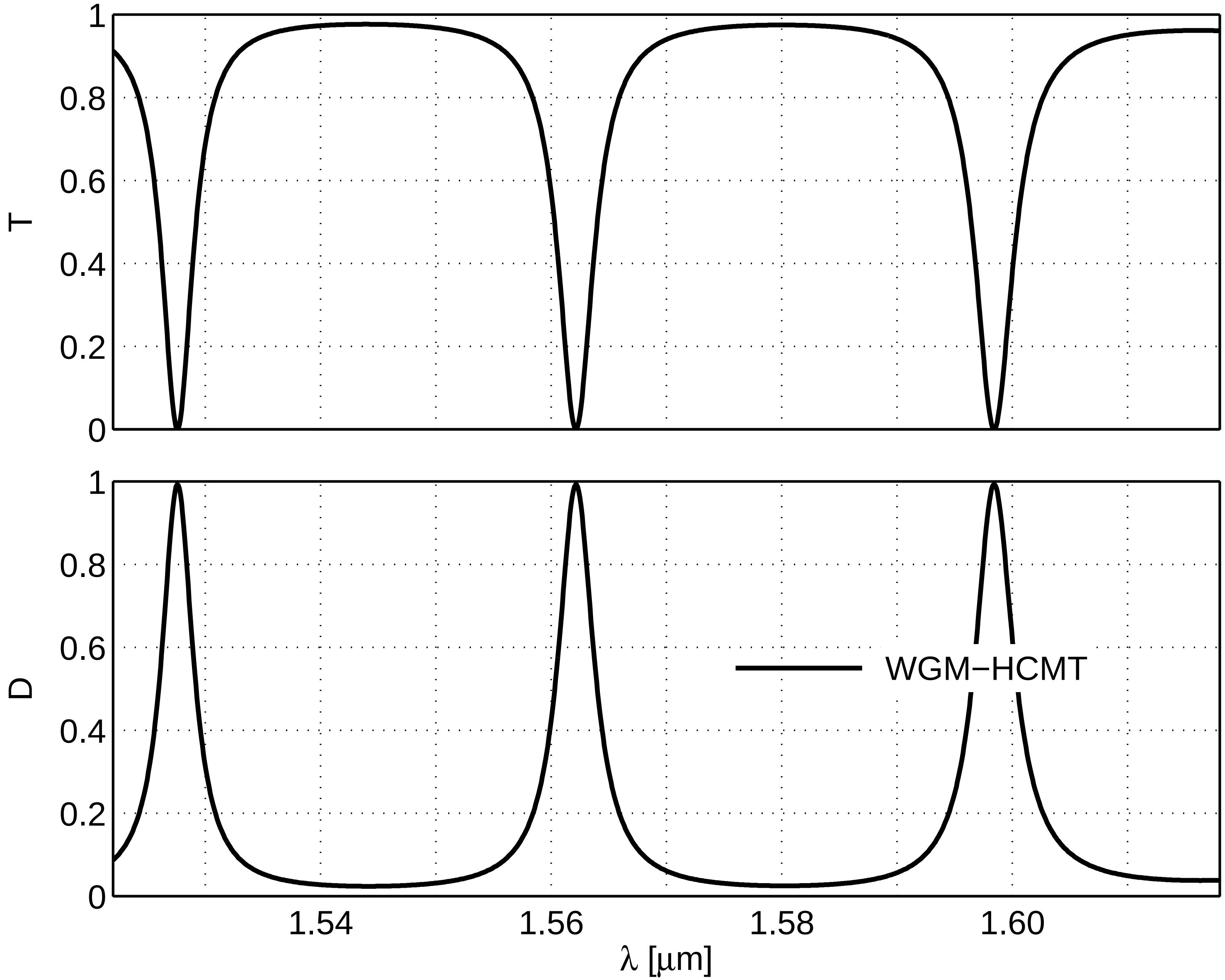
# *Single ring filter, spectral response*



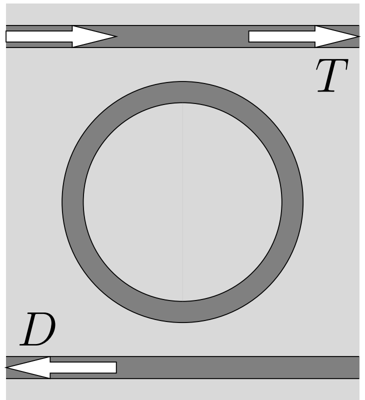



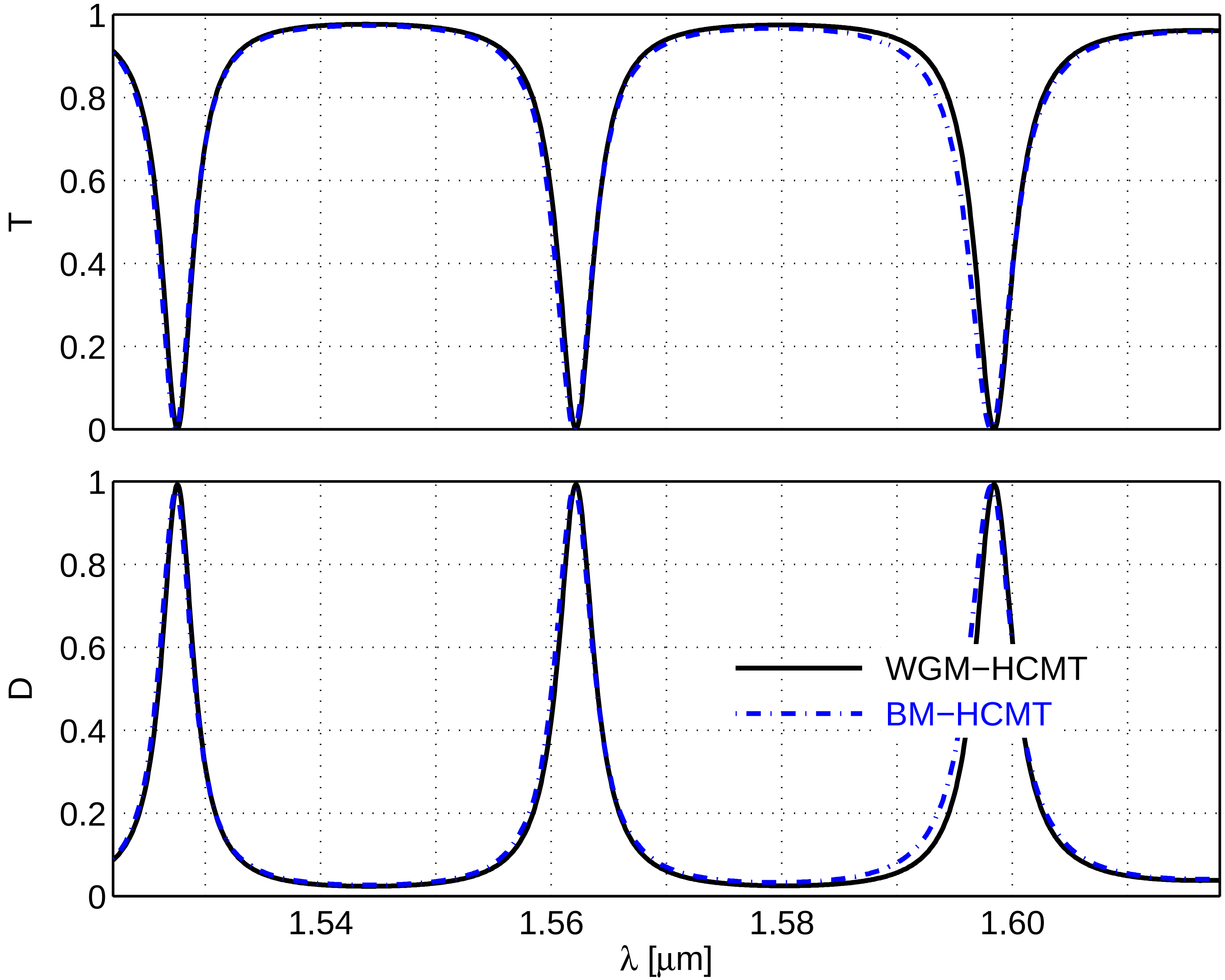
## *Single ring filter, benchmark*



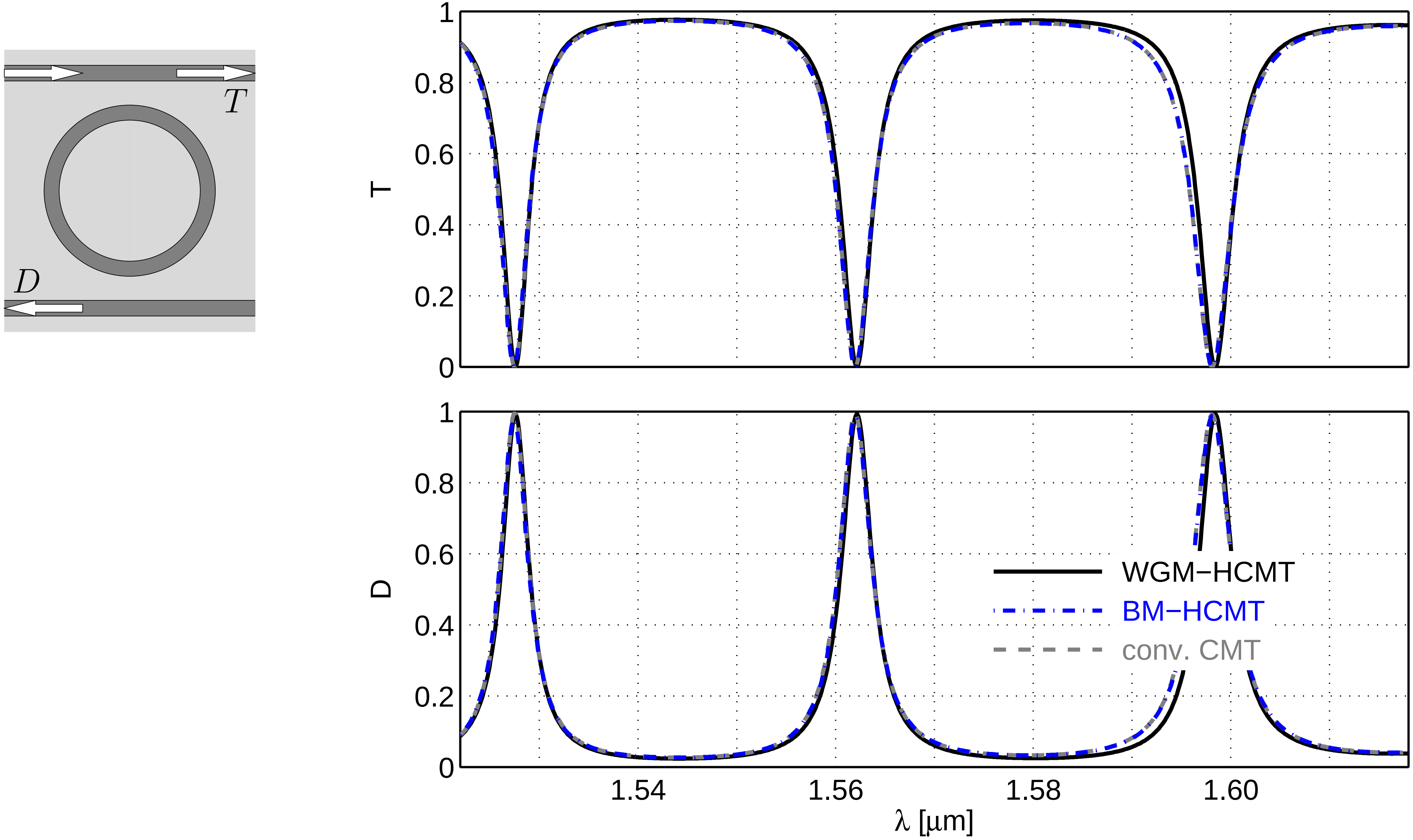



# *Single ring filter, benchmark*

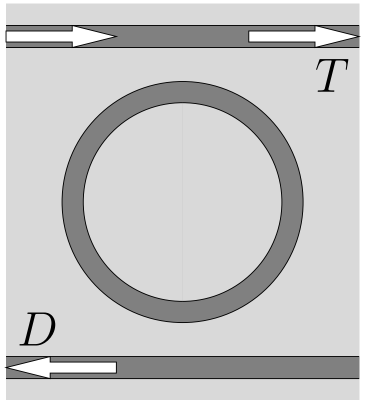


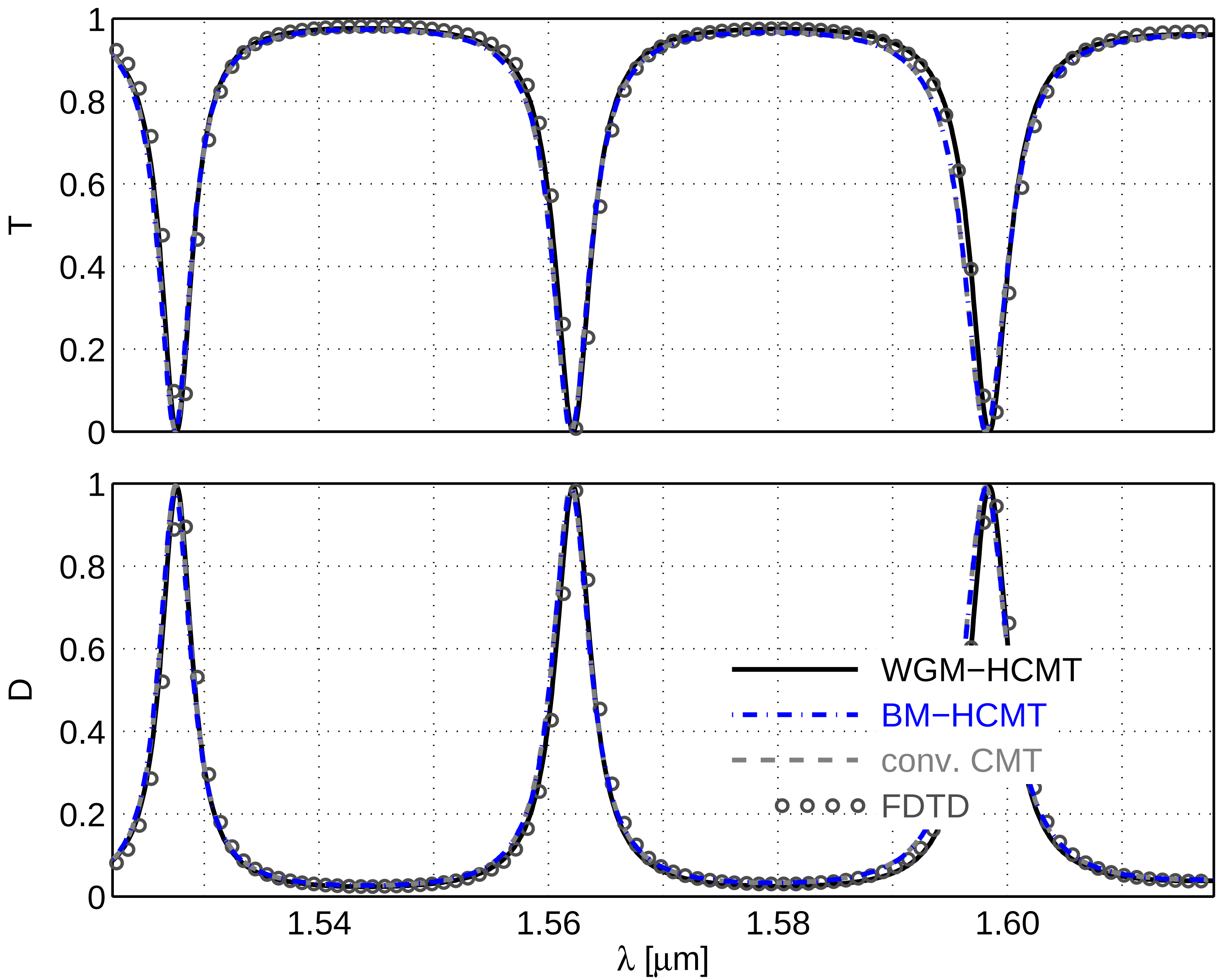



# *Single ring filter, benchmark*

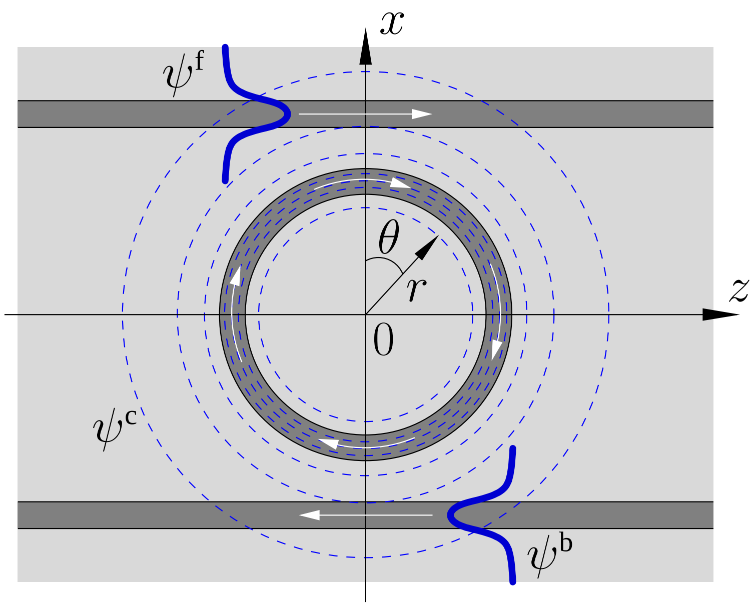


# *Single ring filter, benchmark*



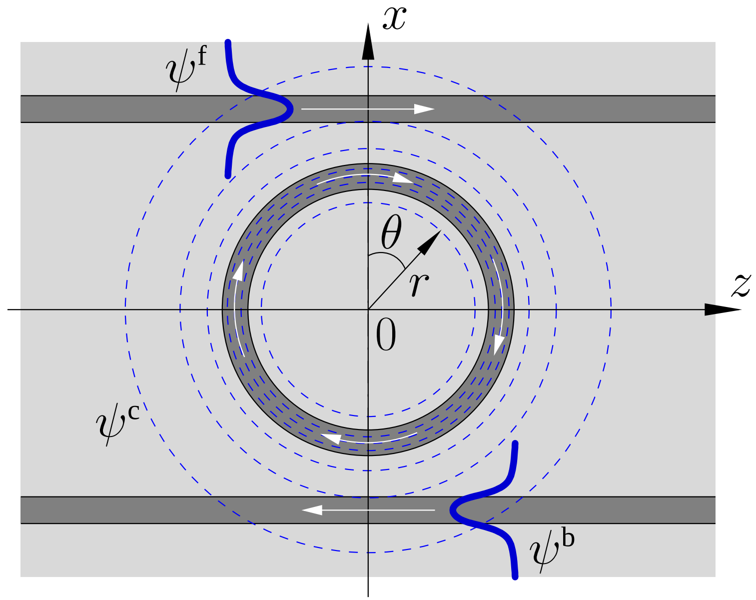



## *Single ring filter, WGM amplitudes*

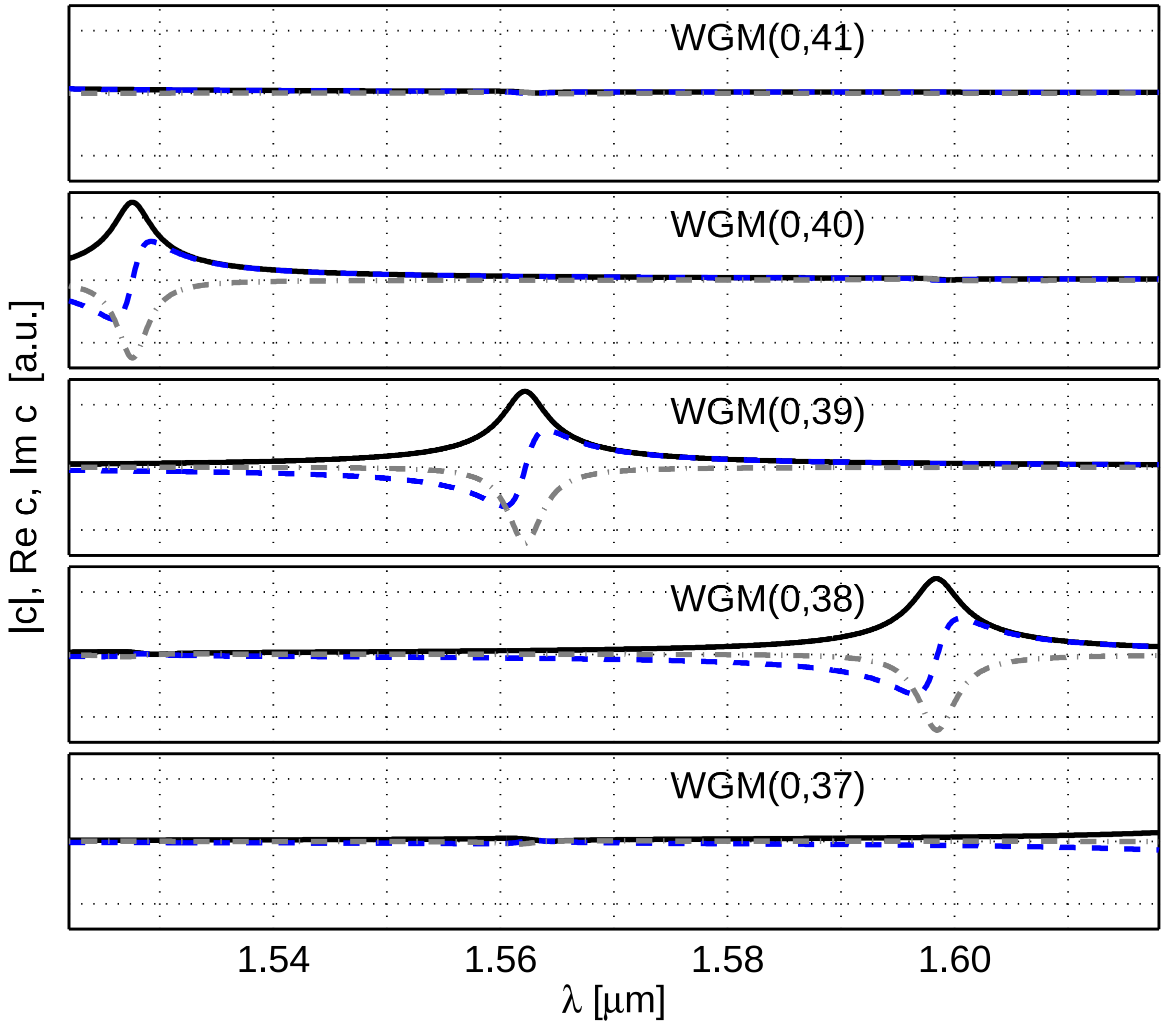



$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = f(z)\,\boldsymbol{\psi}^{\rm f}(x,z) + b(z)\,\boldsymbol{\psi}^{\rm b}(x,z) + \sum_j c_j\,\boldsymbol{\psi}'^{\rm c}_j(x,z)$$

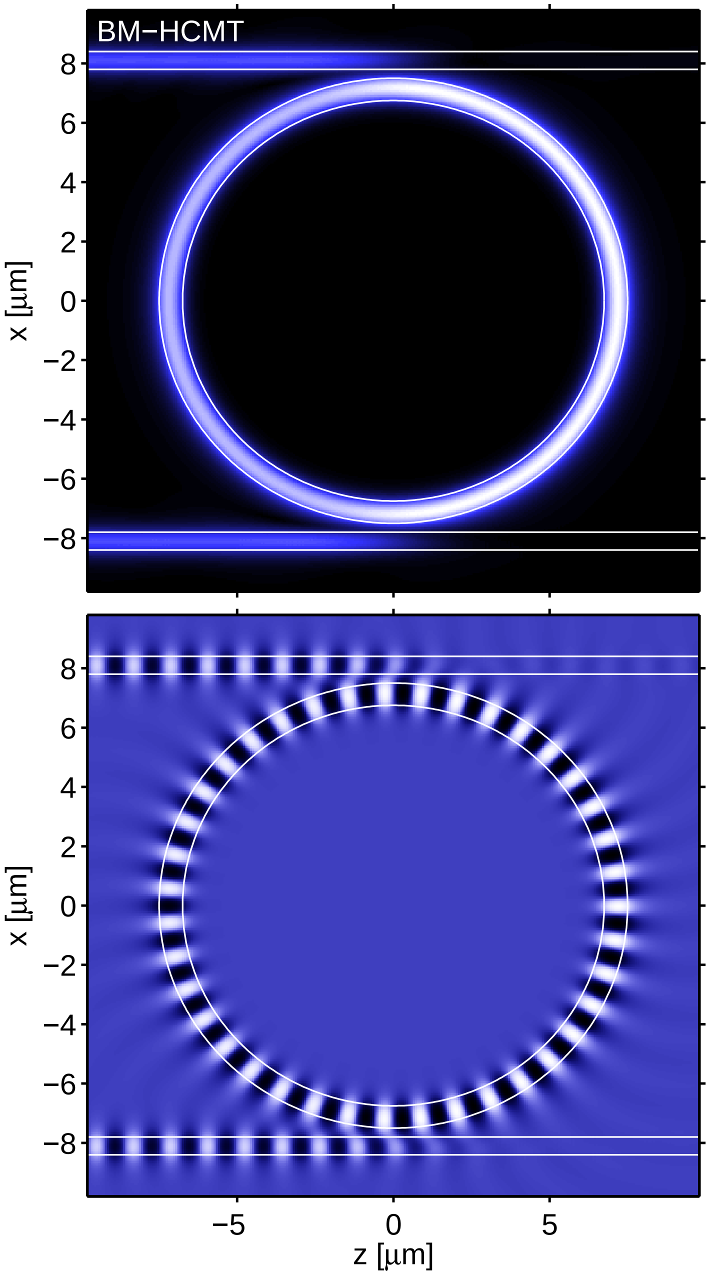
# *Single ring filter, WGM amplitudes*



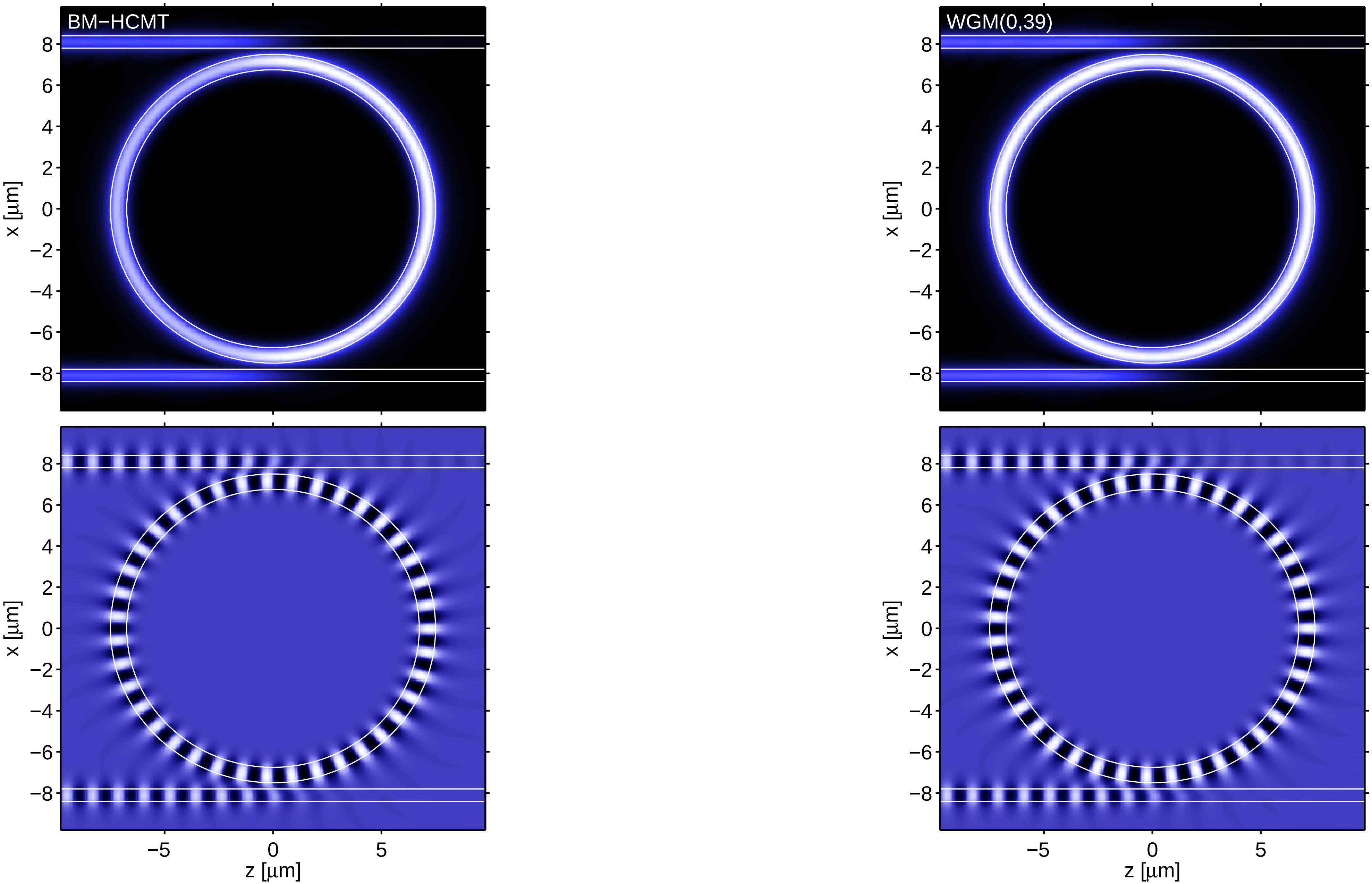

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = f(z)\,\boldsymbol{\psi}^{\mathrm{f}}(x,z) + b(z)\,\boldsymbol{\psi}^{\mathrm{b}}(x,z) + \sum_j c_j\,\boldsymbol{\psi}_j'^{\mathrm{c}}(x,z)$$



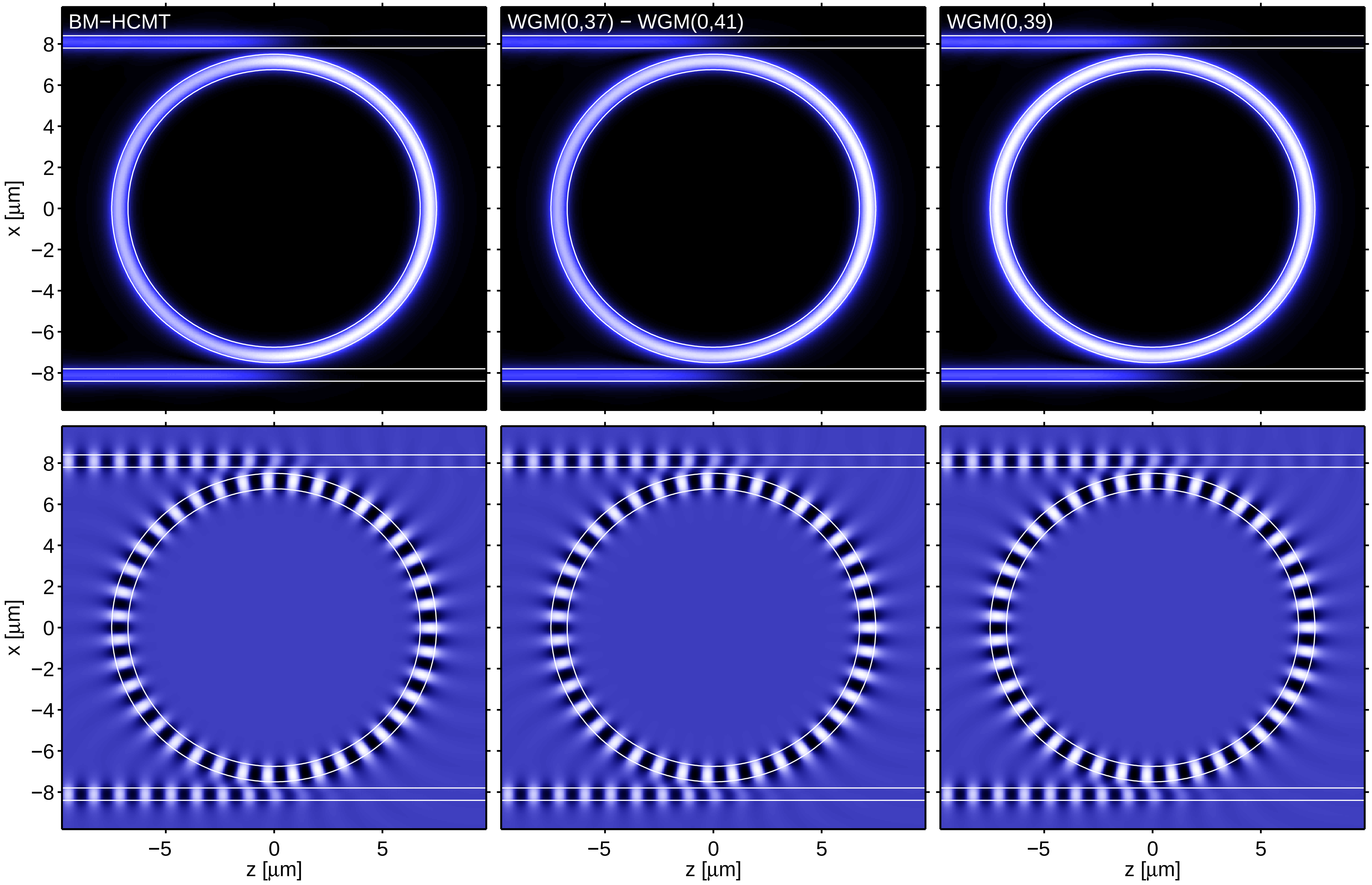
# *Single ring filter, transmission resonance*



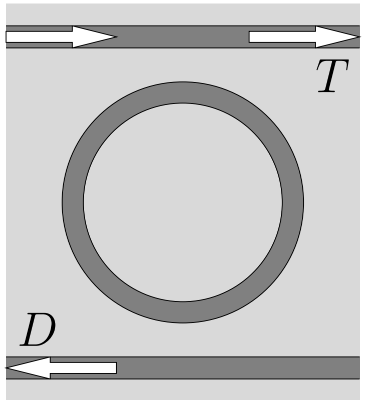
# *Single ring filter, transmission resonance*

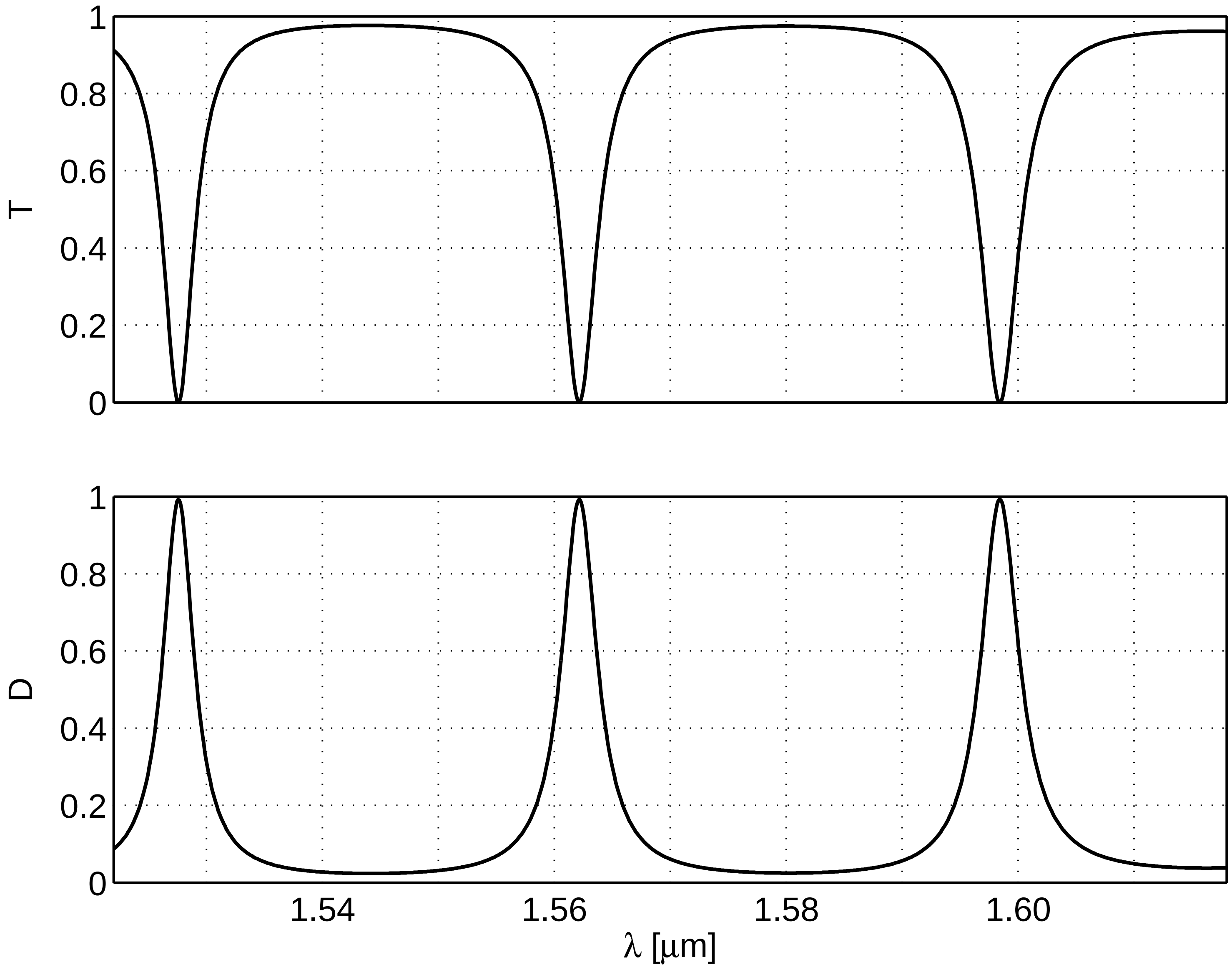


# *Single ring filter, transmission resonance*

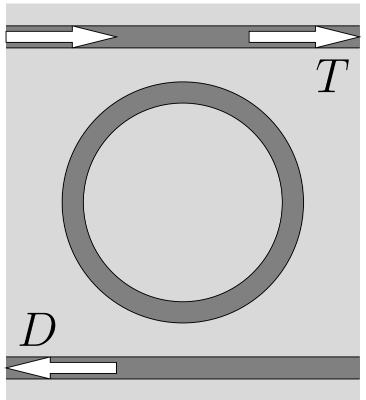


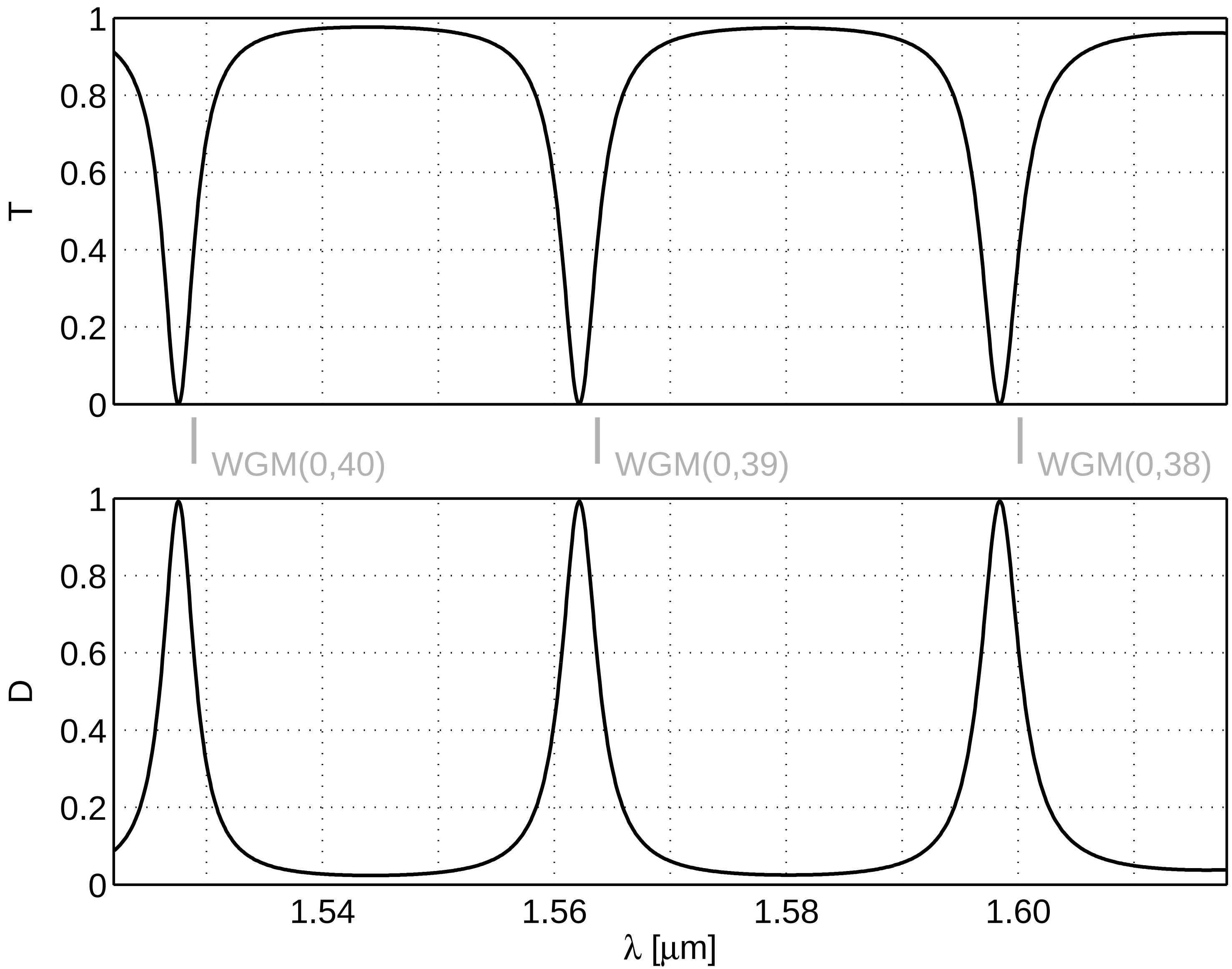
# *Single ring filter, resonance positions I*



# *Single ring filter, resonance positions I*



## *Supermodes*

Look for $\omega^{\mathrm{s}} \in \mathbb{C}$ where the system

$$\left\{ \begin{array}{r} \boldsymbol{\nabla} \times \boldsymbol{H} - \mathrm{i}\omega^{\mathrm{s}}\epsilon_0\epsilon\boldsymbol{E} = 0 \\ -\boldsymbol{\nabla} \times \boldsymbol{E} - \mathrm{i}\omega^{\mathrm{s}}\mu_0\boldsymbol{H} = 0 \end{array} \;\;\&\;\; \text{boundary conditions: “outgoing waves”} \right\}$$

permits nontrivial solutions $\boldsymbol{E}, \boldsymbol{H}$.

Look for $\omega^{\mathrm{s}} \in \mathbb{C}$ where the system

$$\left\{ \begin{array}{r} \boldsymbol{\nabla} \times \boldsymbol{H} - \mathrm{i}\omega^{\mathrm{s}}\epsilon_0\epsilon \boldsymbol{E} = 0 \\ -\boldsymbol{\nabla} \times \boldsymbol{E} - \mathrm{i}\omega^{\mathrm{s}}\mu_0 \boldsymbol{H} = 0 \end{array} \ \&\ \text{boundary conditions: “outgoing waves”} \right\}$$

permits nontrivial solutions $\boldsymbol{E}$, $\boldsymbol{H}$.

---

$$\begin{array}{r} \boldsymbol{\nabla} \times \boldsymbol{H} - \mathrm{i}\omega^{\mathrm{s}}\epsilon_0\epsilon \boldsymbol{E} = 0 \\ -\boldsymbol{\nabla} \times \boldsymbol{E} - \mathrm{i}\omega^{\mathrm{s}}\mu_0 \boldsymbol{H} = 0 \end{array} \quad \Bigg| \quad \cdot \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix}^{*}, \quad \iint_{\text{comp. domain}}$$

$\leftrightsquigarrow$

$$\iint \mathcal{A}(\boldsymbol{F}, \boldsymbol{G}; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z - \omega^{\mathrm{s}} \iint \mathcal{B}(\boldsymbol{F}, \boldsymbol{G}; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z = 0 \quad \text{for all } \boldsymbol{F}, \boldsymbol{G},$$

where

$$\mathcal{A}(\boldsymbol{F}, \boldsymbol{G}; \boldsymbol{E}, \boldsymbol{H}) = \boldsymbol{F}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) - \boldsymbol{G}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) \ ,$$

$$\mathcal{B}(\boldsymbol{F}, \boldsymbol{G}; \boldsymbol{E}, \boldsymbol{H}) = \mathrm{i}\,\epsilon_0\epsilon \boldsymbol{F}^* \cdot \boldsymbol{E} + \mathrm{i}\,\mu_0 \boldsymbol{G}^* \cdot \boldsymbol{H} \ .$$

## *HCMT supermode analysis*

- Insert $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}$,
- require

$$\iint \mathcal{A}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z - \omega^{\mathrm{s}} \iint \mathcal{B}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z = 0 \quad \text{for all } l\,,$$

- compute $A_{lk} = \iint \mathcal{A}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_k, \boldsymbol{H}_k)\, \mathrm{d}x\, \mathrm{d}z\,,$

$$B_{lk} = \iint \mathcal{B}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_k, \boldsymbol{H}_k)\, \mathrm{d}x\, \mathrm{d}z\,.$$

---

$$\sum_k A_{lk} a_k - \omega^{\mathrm{s}} B_{lk} a_k = 0 \quad \text{for all } l\,, \qquad \text{or} \qquad \mathsf{A}\boldsymbol{a} = \omega^{\mathrm{s}} \mathsf{B} \boldsymbol{a}\,.$$

## *HCMT supermode analysis*

- Insert $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}$,
- require

$$\iint \mathcal{A}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z - \omega^{\mathrm{s}} \iint \mathcal{B}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}, \boldsymbol{H})\, \mathrm{d}x\, \mathrm{d}z = 0 \ \text{ for all } \ l\,,$$

- compute

$$A_{lk} = \iint \mathcal{A}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_k, \boldsymbol{H}_k)\, \mathrm{d}x\, \mathrm{d}z\,,$$

$$B_{lk} = \iint \mathcal{B}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_k, \boldsymbol{H}_k)\, \mathrm{d}x\, \mathrm{d}z\,.$$

---

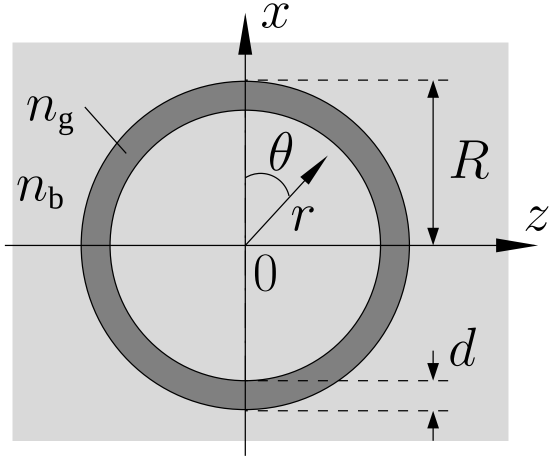
$$\sum_k A_{lk} a_k - \omega^{\mathrm{s}} B_{lk} a_k = 0 \ \text{ for all } \ l\,, \quad \text{or} \quad \mathsf{A}\boldsymbol{a} = \omega^{\mathrm{s}} \mathsf{B} \boldsymbol{a}\,.$$

$$\left\{ \omega,\ \lambda_{\mathrm{r}},\ Q,\ \Delta\lambda;\ \boldsymbol{E},\ \boldsymbol{H} \right\}^{\mathrm{s}}.$$

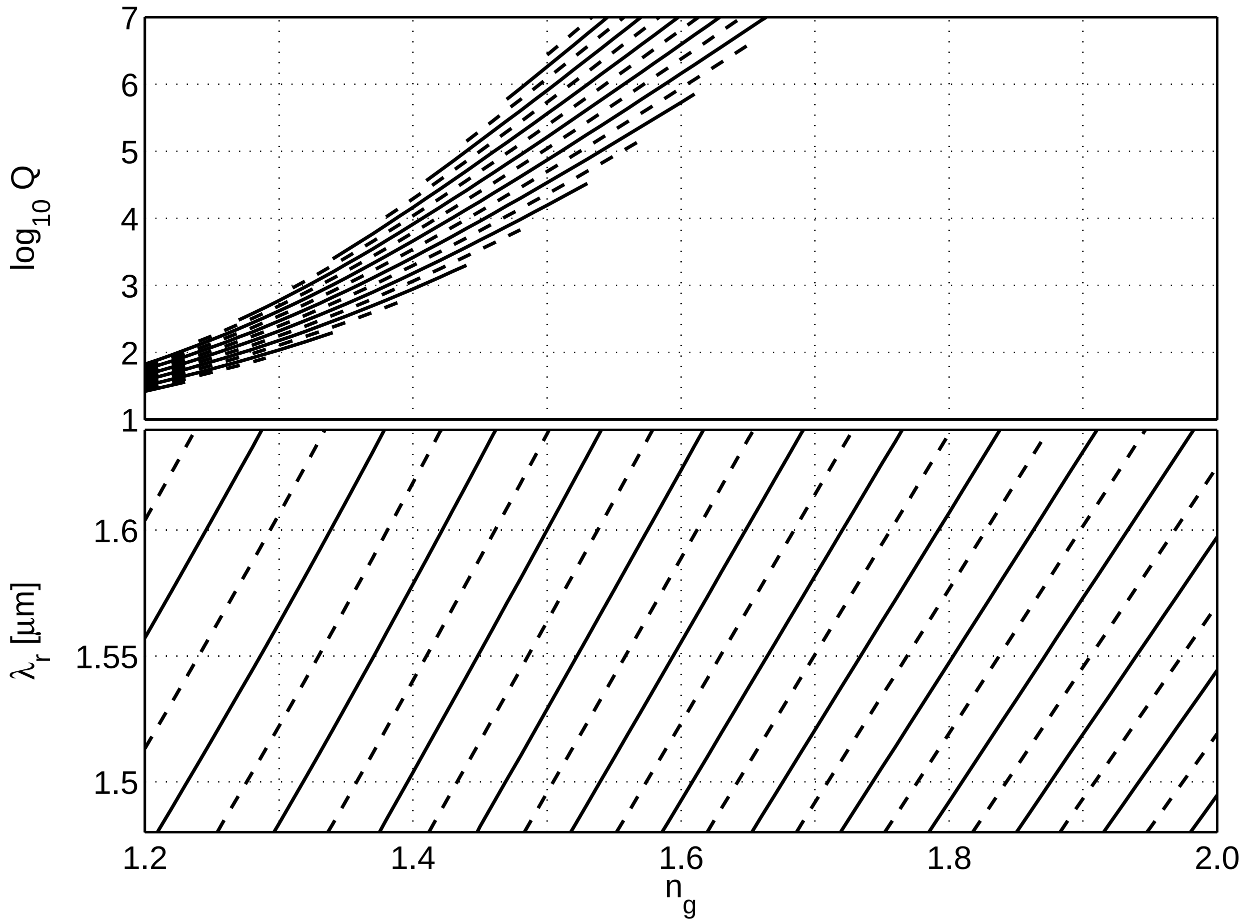
## *Further issues*

… plenty.

# *WGMs, small uniform perturbations*




TE, $R = 7.5\,\mu$m, $d = 0.75\,\mu$m
$n_{\mathrm{b}} = 1.0$.

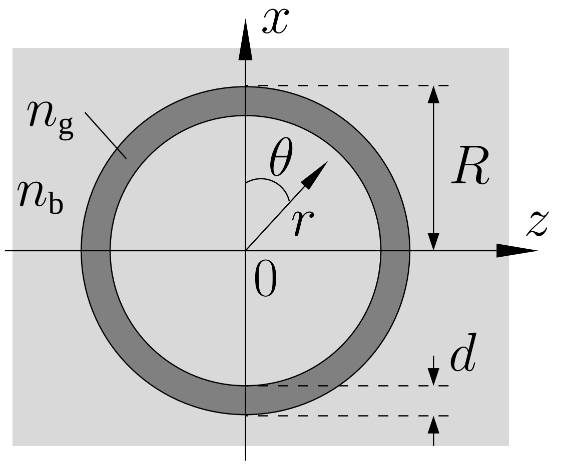



$$\epsilon_m \quad \leftrightsquigarrow \quad \mathrm{WGM}(\omega_m;\ \boldsymbol{E}_m, \boldsymbol{H}_m)\,,$$

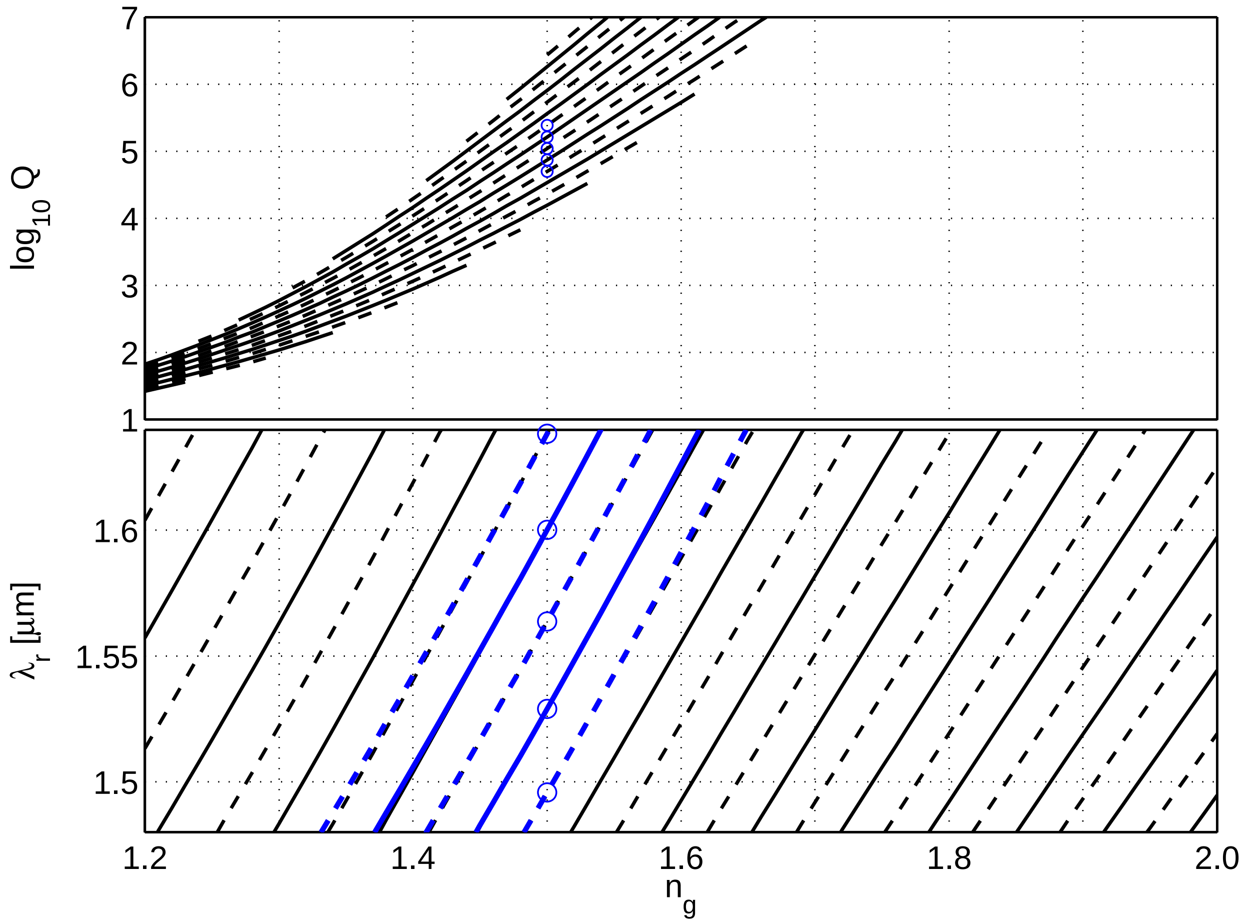
$$\epsilon = \epsilon_m + \Delta\epsilon \quad \leftrightsquigarrow \quad \mathrm{WGM}(\omega_m + \Delta\omega;\ \approx \boldsymbol{E}_m, \approx \boldsymbol{H}_m)\,,$$

$$\Delta\omega = -\frac{\omega_m \epsilon_0 \iint \Delta\epsilon |\boldsymbol{E}_m|^2 \,\mathrm{d}x\,\mathrm{d}z}{\iint \left(\epsilon_m \epsilon_0 |\boldsymbol{E}_m|^2 + \mu_0 |\boldsymbol{H}_m|^2\right) \mathrm{d}x\,\mathrm{d}z}\,.$$

# *WGMs, small uniform perturbations*




TE, $R = 7.5\,\mu$m, $d = 0.75\,\mu$m
$n_b = 1.0$.

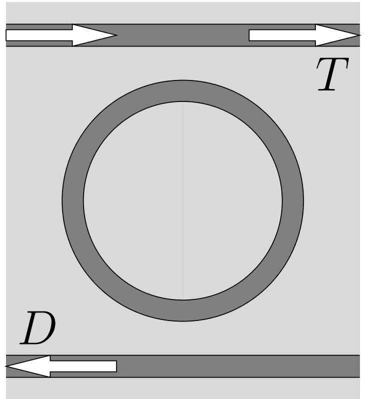


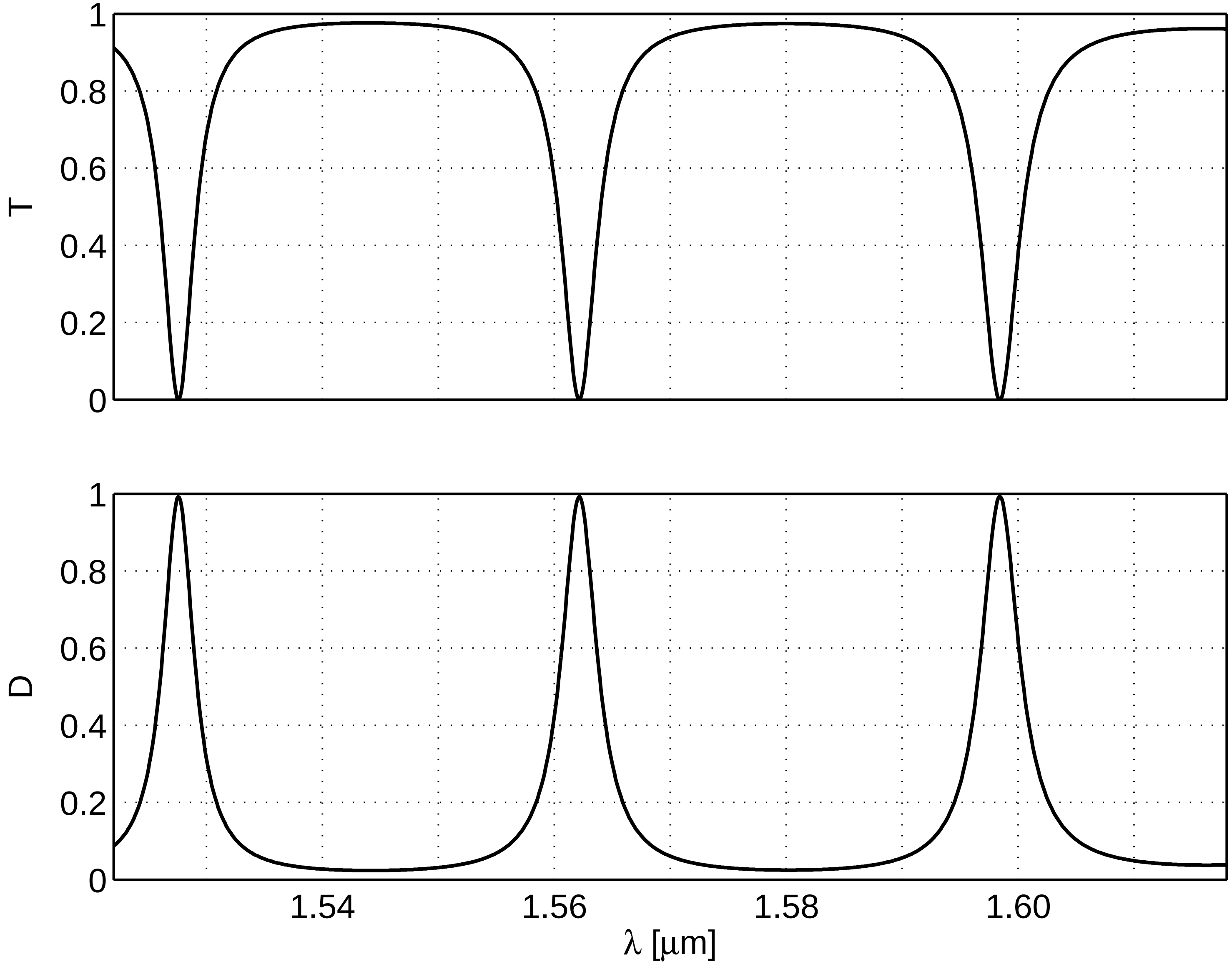

$$\epsilon_m \quad \longleftrightarrow \quad \mathrm{WGM}(\omega_m;\ \boldsymbol{E}_m, \boldsymbol{H}_m)\,,$$

$$\epsilon = \epsilon_m + \Delta\epsilon \quad \longleftrightarrow \quad \mathrm{WGM}(\omega_m + \Delta\omega;\ \approx \boldsymbol{E}_m, \approx \boldsymbol{H}_m)\,,$$

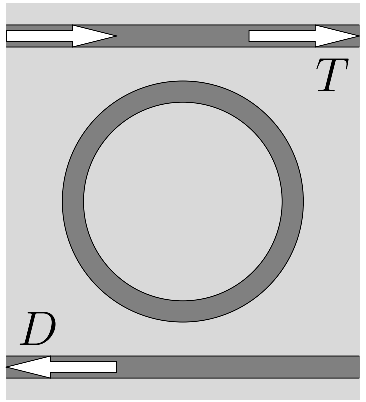
$$\Delta\omega = -\frac{\omega_m \epsilon_0 \iint \Delta\epsilon |\boldsymbol{E}_m|^2 \,\mathrm{d}x\,\mathrm{d}z}{\iint \left(\epsilon_m \epsilon_0 |\boldsymbol{E}_m|^2 + \mu_0 |\boldsymbol{H}_m|^2\right) \mathrm{d}x\,\mathrm{d}z}\,.$$

## *Single ring filter, resonance positions II*

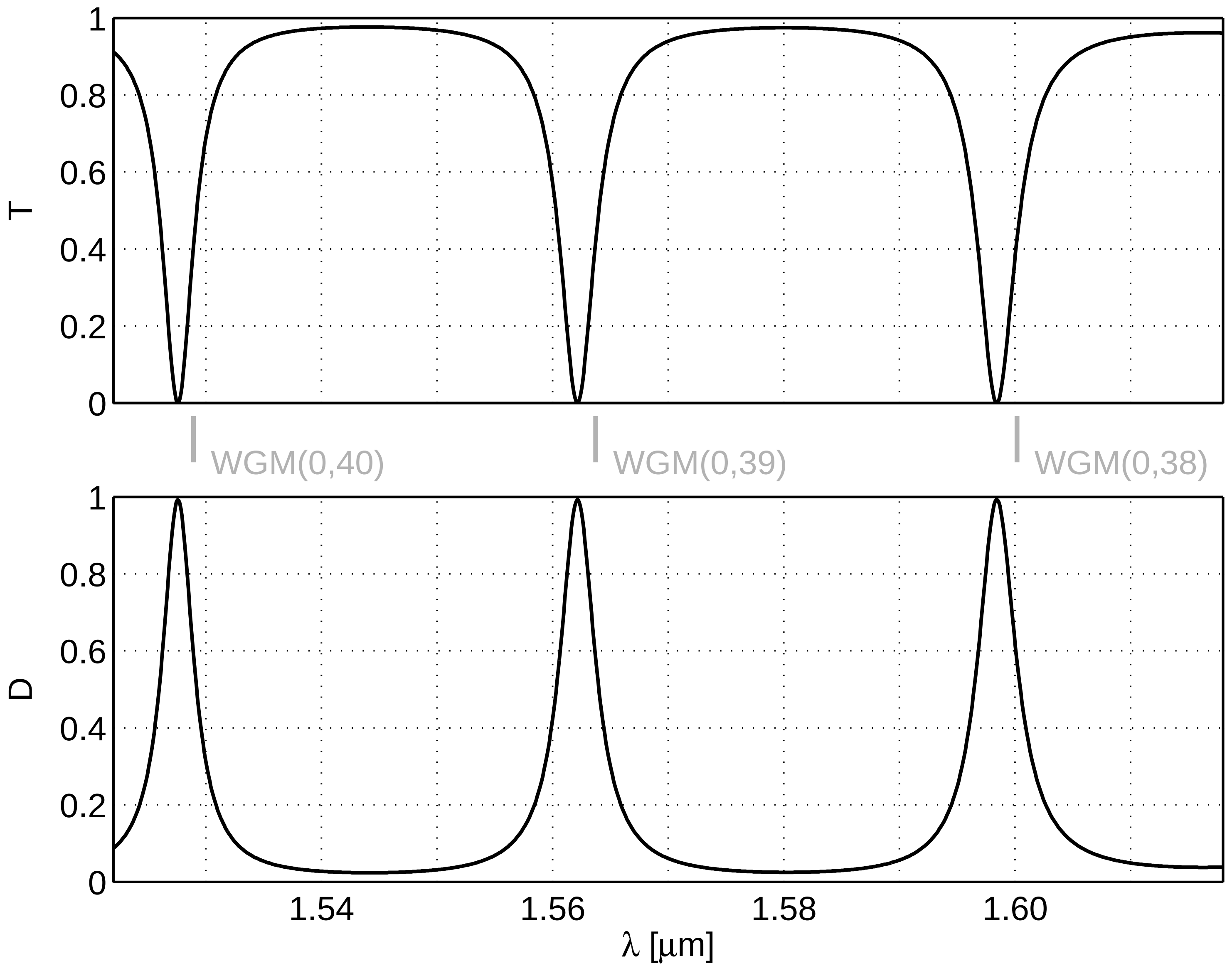


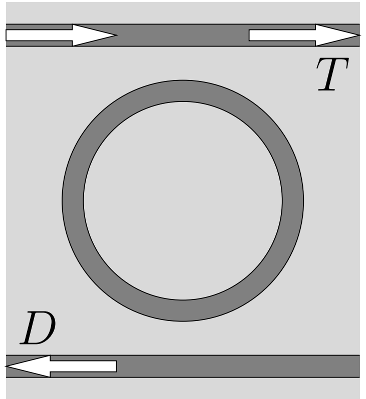
# *Single ring filter, resonance positions II*

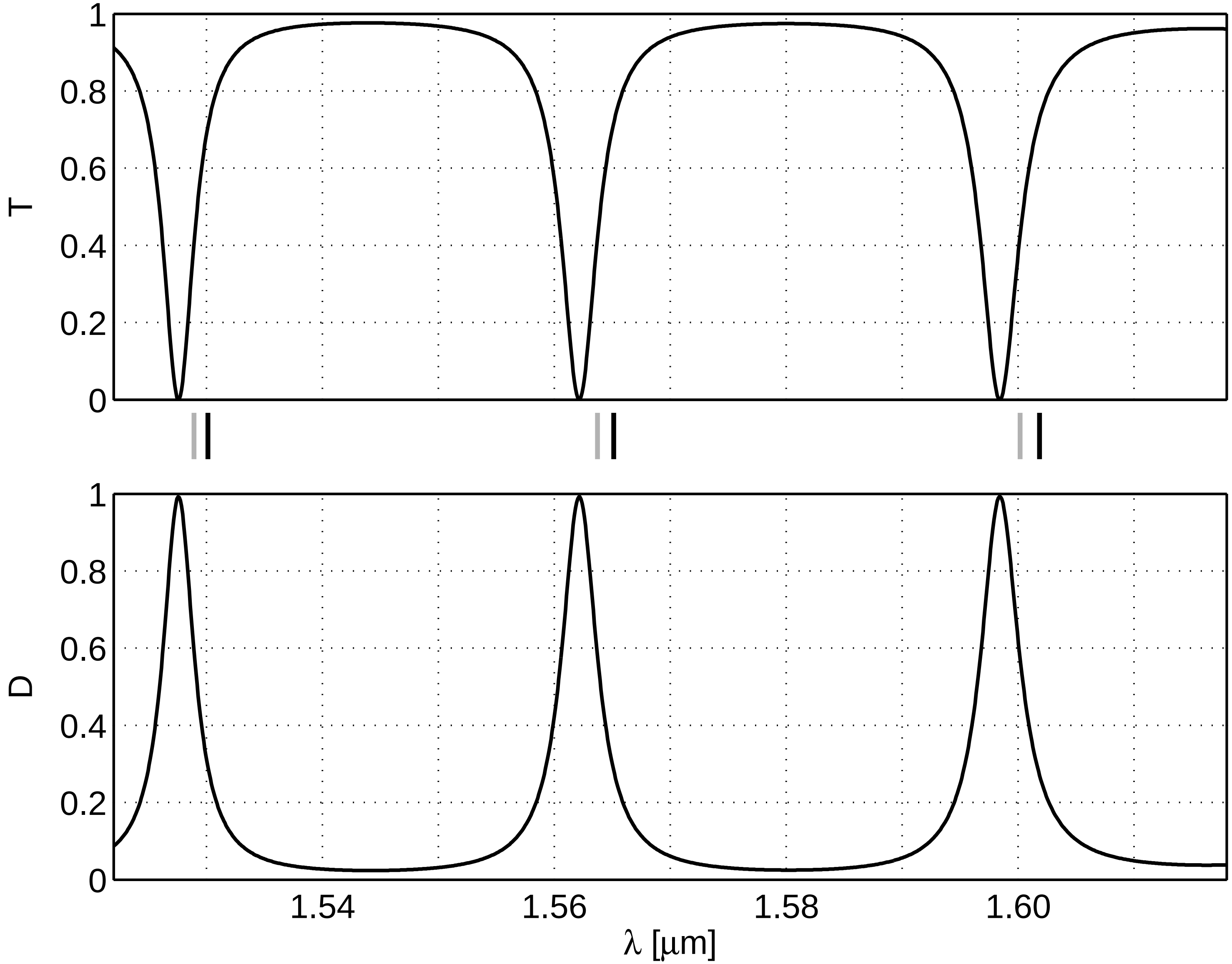



WGMs only



# *Single ring filter, resonance positions II*

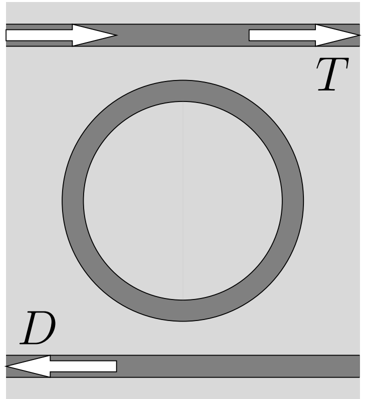



WGMs only

WGMs
& bus cores

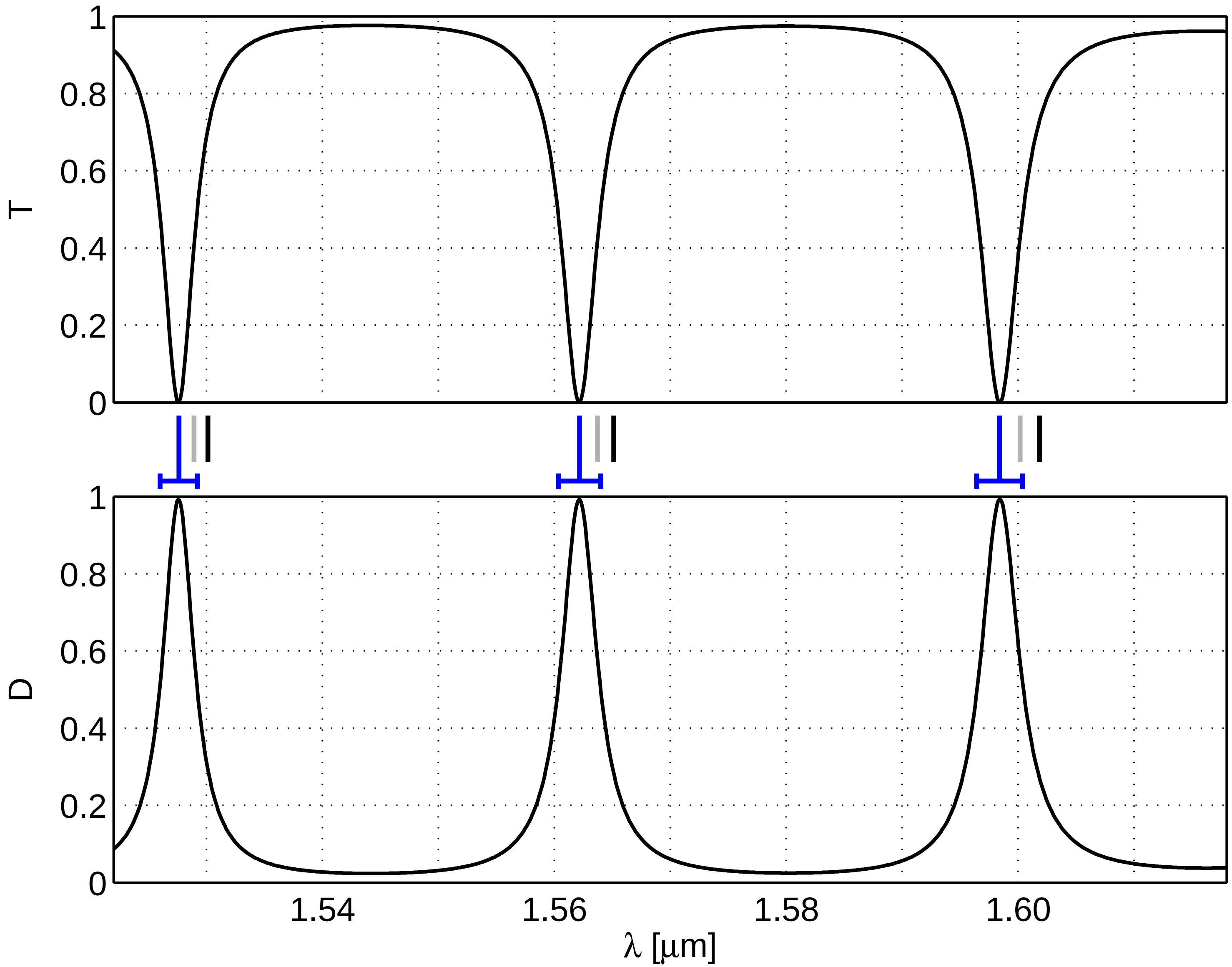
# *Single ring filter, resonance positions II*



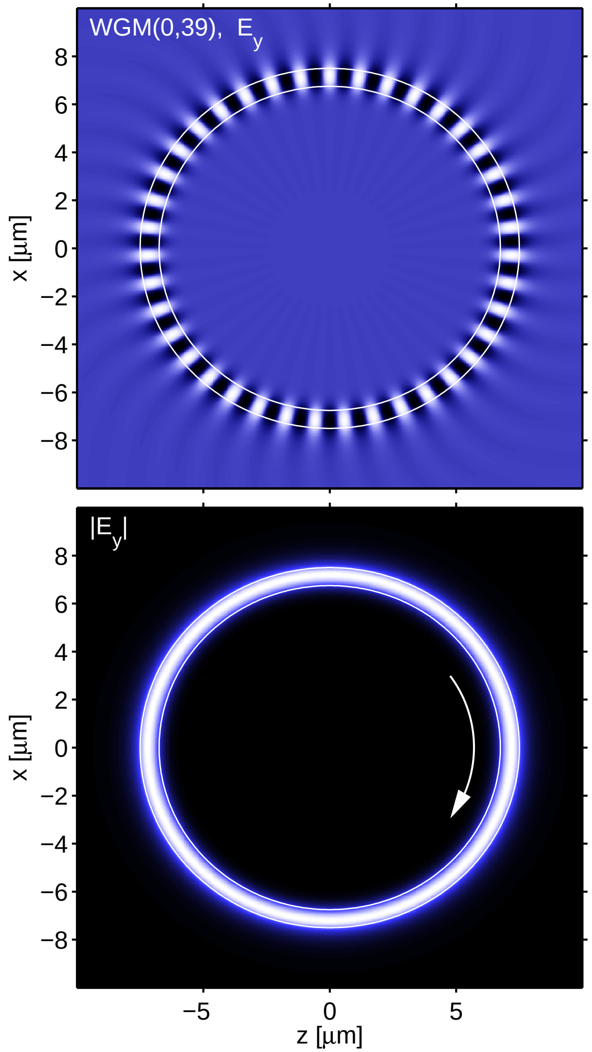

WGMs only

WGMs
& bus cores

WGMs
& bus fields

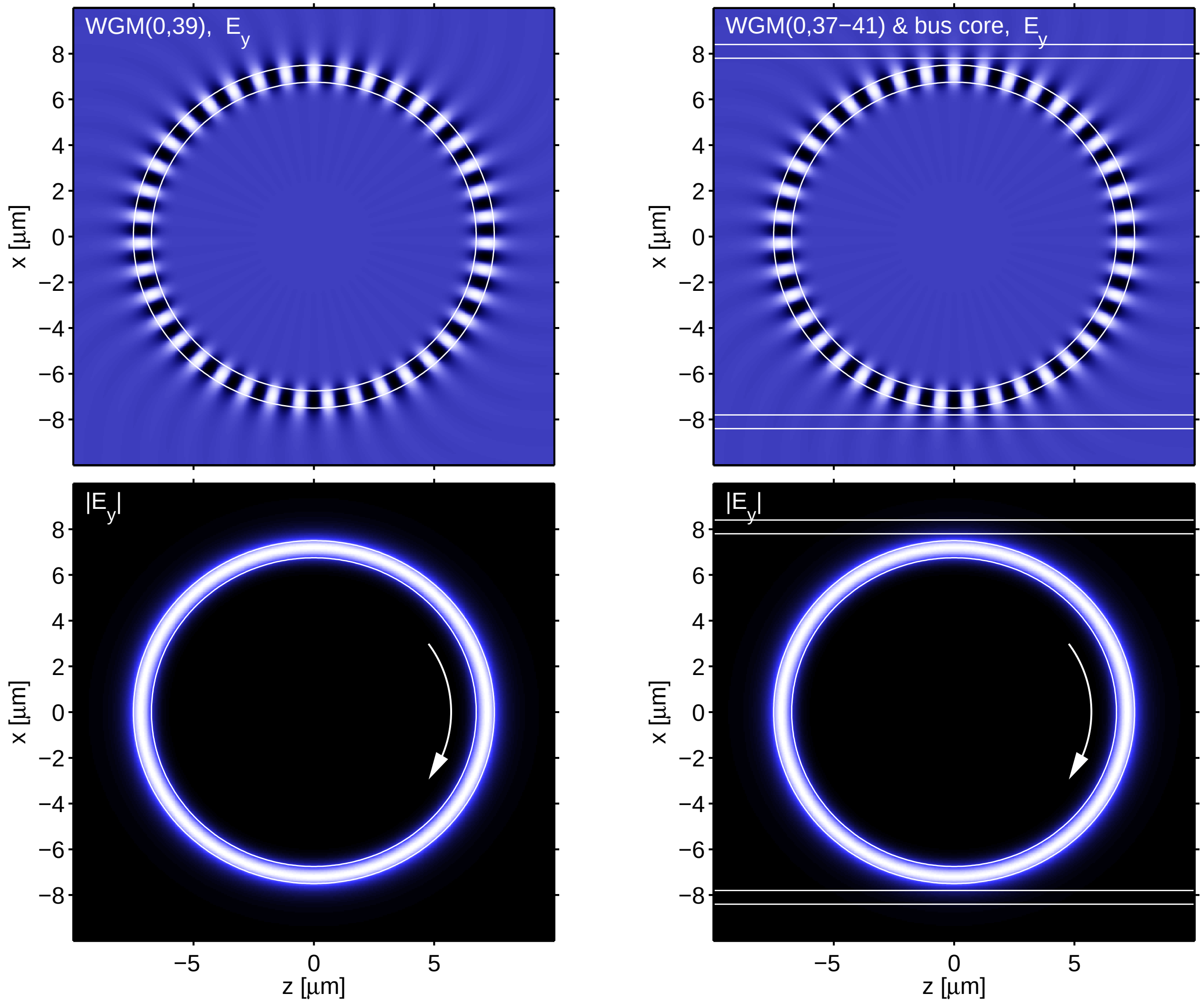


# *Single ring filter, unidirectional supermodes*




$\lambda_{\rm r} = 1.5637\,\mu$m,
$Q = 1.1 \cdot 10^5$,
$\Delta\lambda = 1.4 \cdot 10^{-5}\,\mu$m.

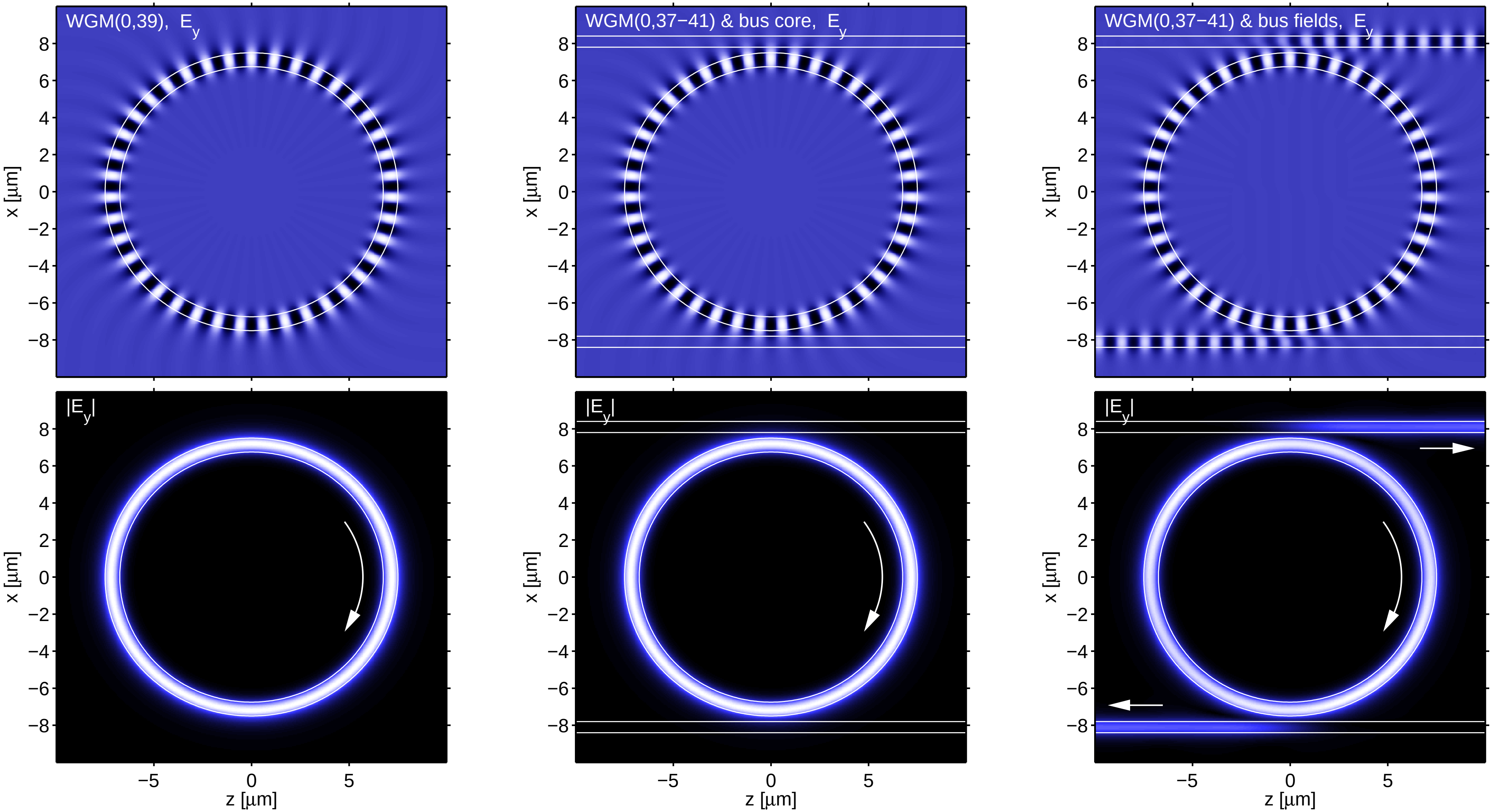
# *Single ring filter, unidirectional supermodes*




$\lambda_r = 1.5637\,\mu$m,
$Q = 1.1 \cdot 10^5$,
$\Delta\lambda = 1.4 \cdot 10^{-5}\,\mu$m.

$\lambda_r = 1.5651\,\mu$m,
$Q = 1.1 \cdot 10^5$,
$\Delta\lambda = 1.4 \cdot 10^{-5}\,\mu$m.

# *Single ring filter, unidirectional supermodes*

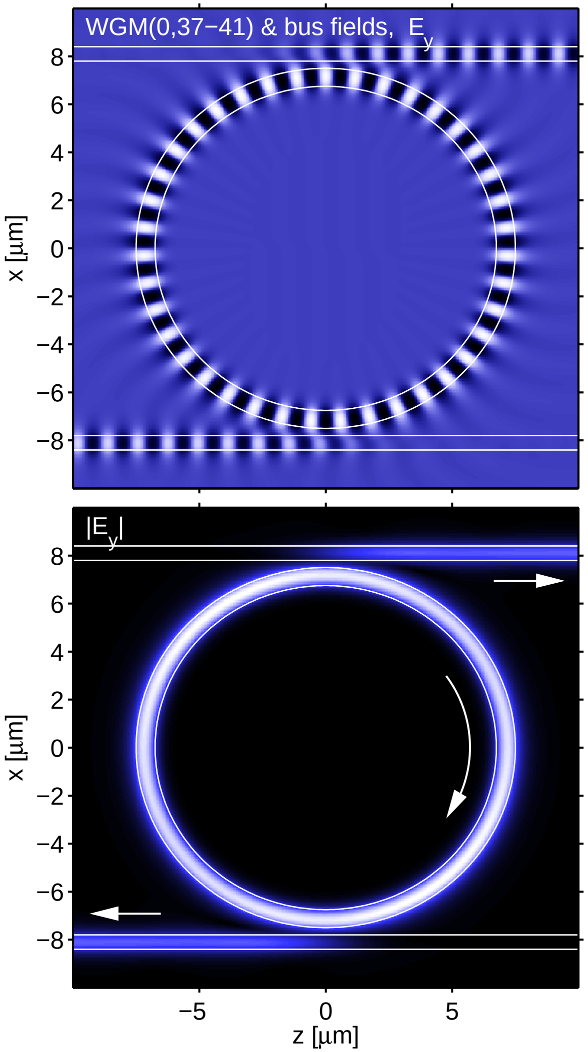



$\lambda_r = 1.5637\,\mu$m,
$Q = 1.1 \cdot 10^5$,
$\Delta\lambda = 1.4 \cdot 10^{-5}\,\mu$m.

$\lambda_r = 1.5651\,\mu$m,
$Q = 1.1 \cdot 10^5$,
$\Delta\lambda = 1.4 \cdot 10^{-5}\,\mu$m.

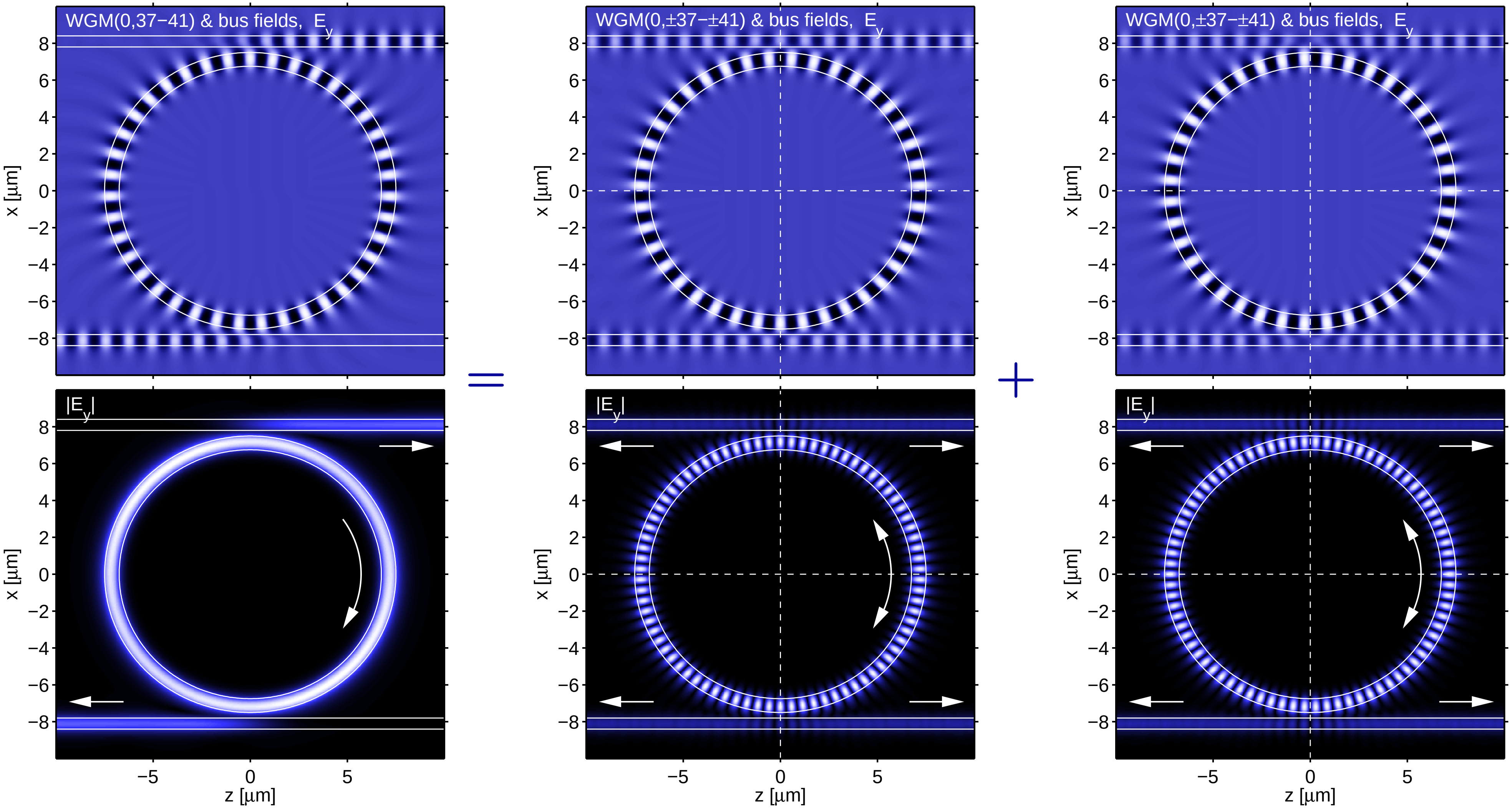
$\lambda_r = 1.5622\,\mu$m,
$Q = 4.3 \cdot 10^2$,
$\Delta\lambda = 3.7 \cdot 10^{-3}\,\mu$m.

# *Single ring filter, bidirectional supermodes*




$\lambda_\mathrm{r} = 1.56219\,\mu\mathrm{m}$,
$Q = 4.3 \cdot 10^2$,
$\Delta\lambda = 3.7 \cdot 10^{-3}\,\mu\mathrm{m}$.

# *Single ring filter, bidirectional supermodes*

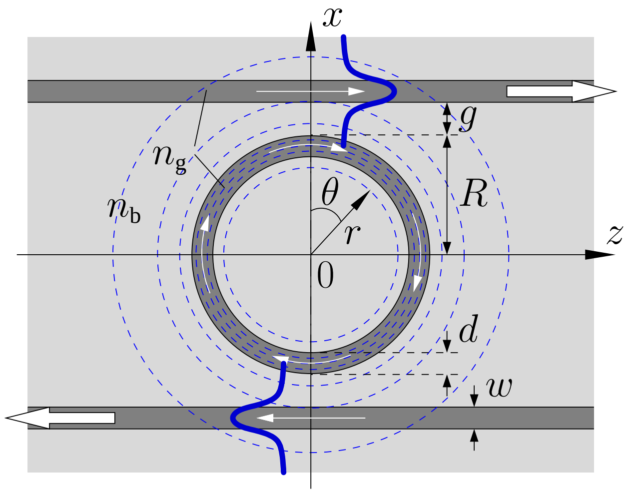



$\lambda_{\mathrm{r}} = 1.56219\,\mu\mathrm{m}$,
$Q = 4.3 \cdot 10^2$,
$\Delta\lambda = 3.7 \cdot 10^{-3}\,\mu\mathrm{m}$.

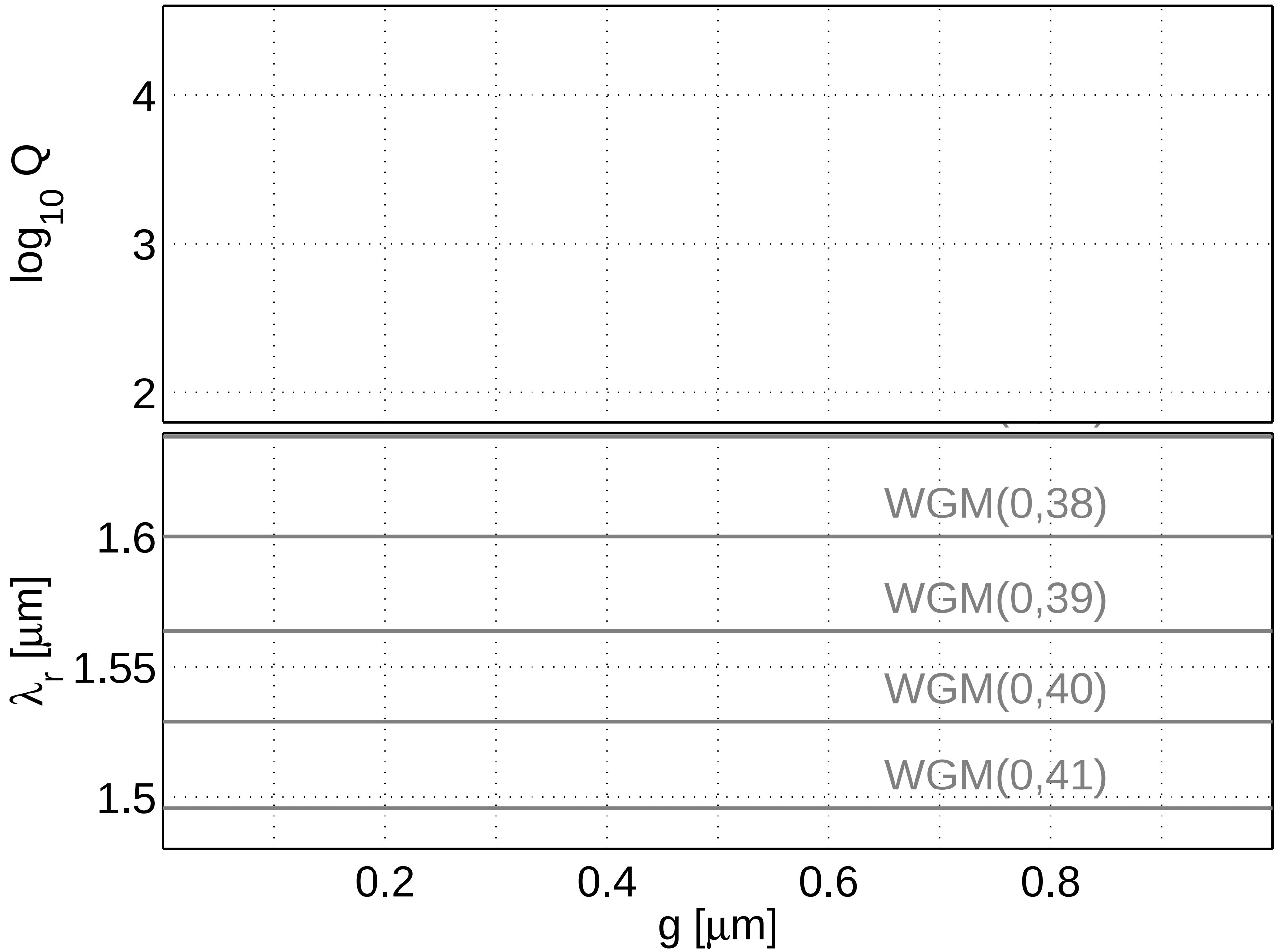
$\lambda_{\mathrm{r}} = 1.56223\,\mu\mathrm{m}$,
$Q = 4.4 \cdot 10^2$,
$\Delta\lambda = 3.5 \cdot 10^{-3}\,\mu\mathrm{m}$.

$\lambda_{\mathrm{r}} = 1.56215\,\mu\mathrm{m}$,
$Q = 4.0 \cdot 10^2$,
$\Delta\lambda = 3.9 \cdot 10^{-3}\,\mu\mathrm{m}$.

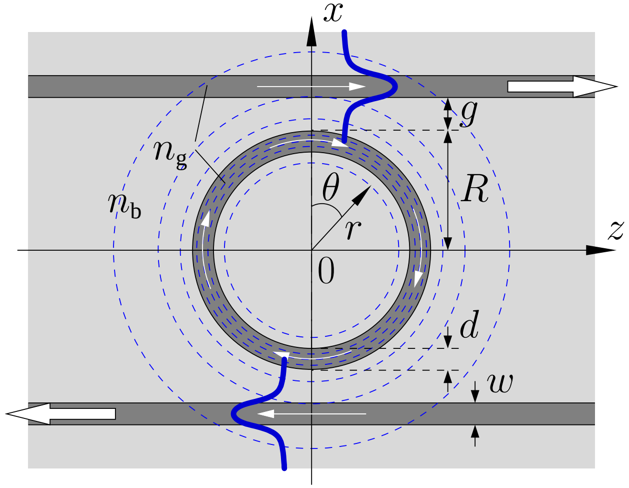
# *Single ring filter, supermodes vs. gap*



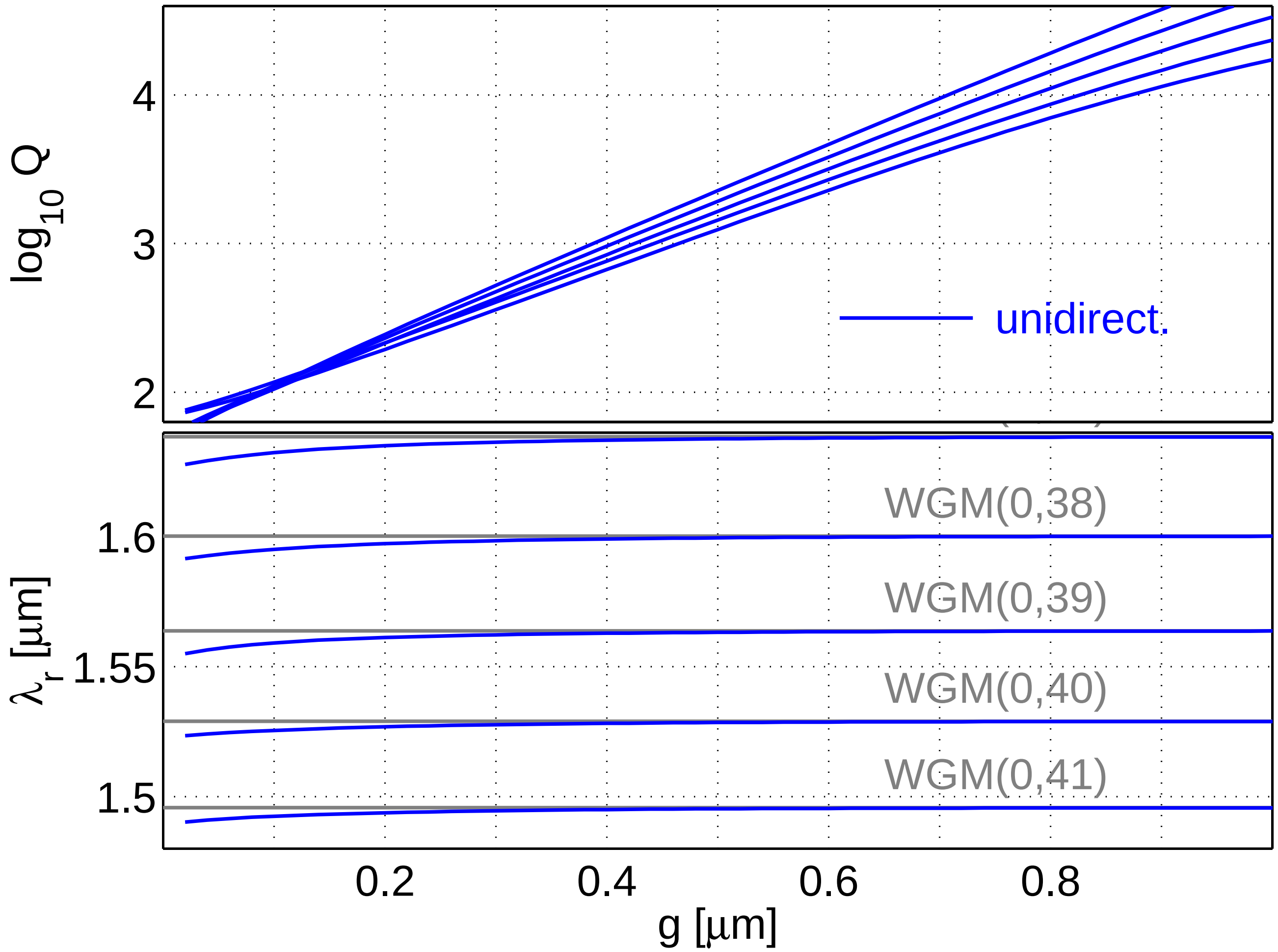

TE,
$R = 7.5\,\mu$m, $d = 0.75\,\mu$m,
$w = 0.6\,\mu$m,
$n_g = 1.5$, $n_b = 1.0$.



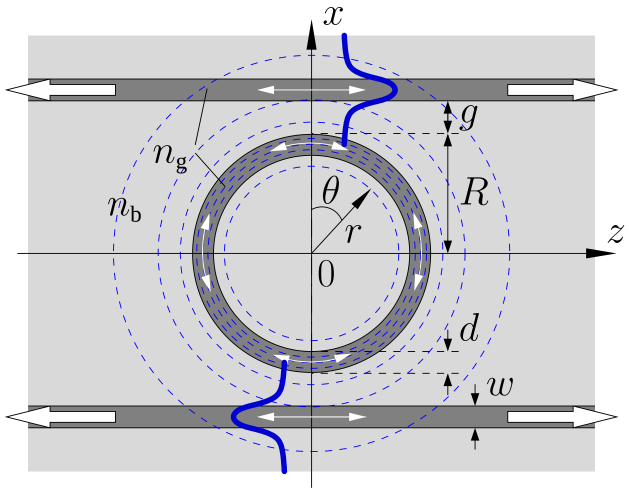
# *Single ring filter, supermodes vs. gap*



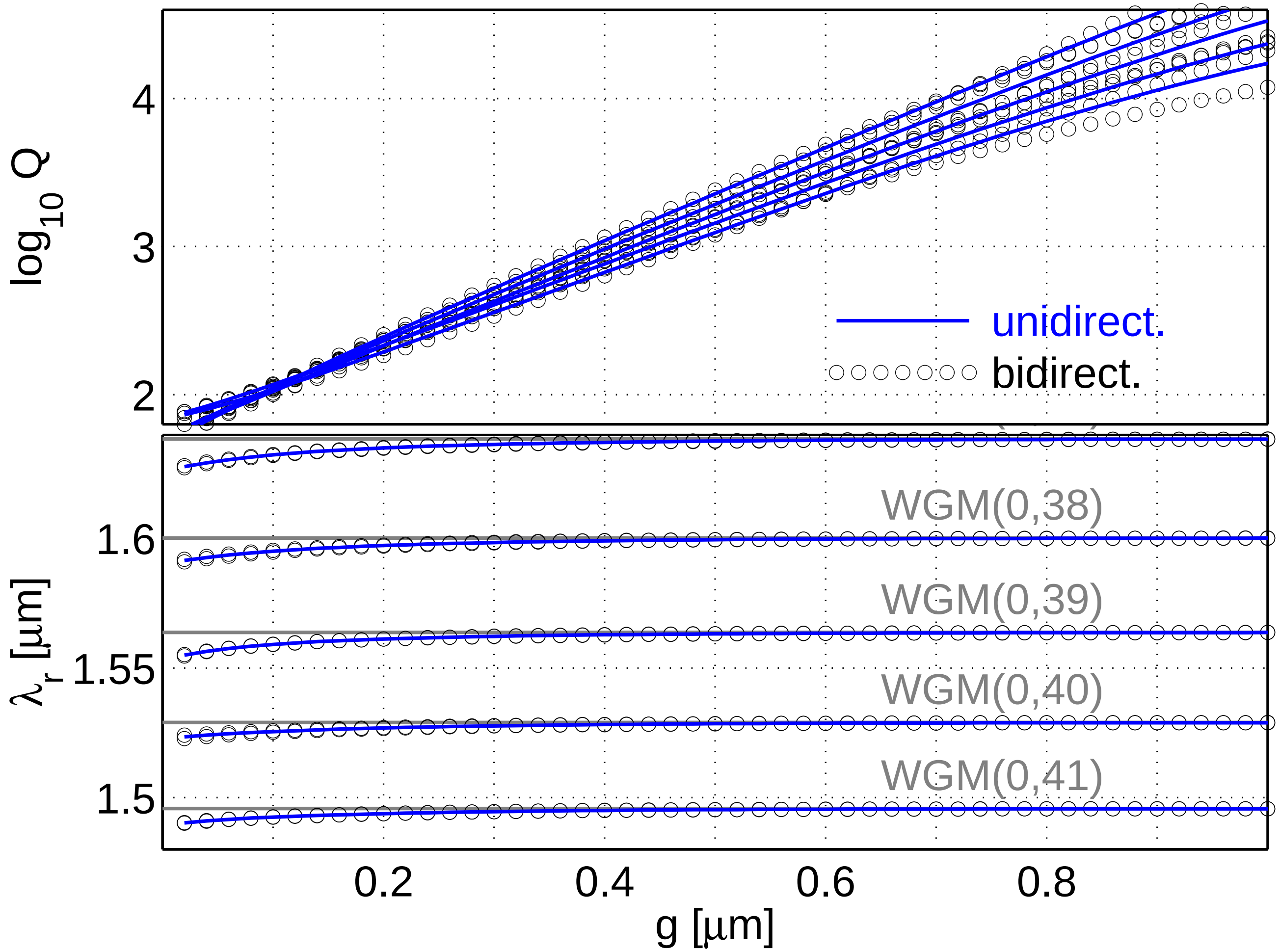

TE,
$R = 7.5\,\mu$m, $d = 0.75\,\mu$m,
$w = 0.6\,\mu$m,
$n_g = 1.5$, $n_b = 1.0$.



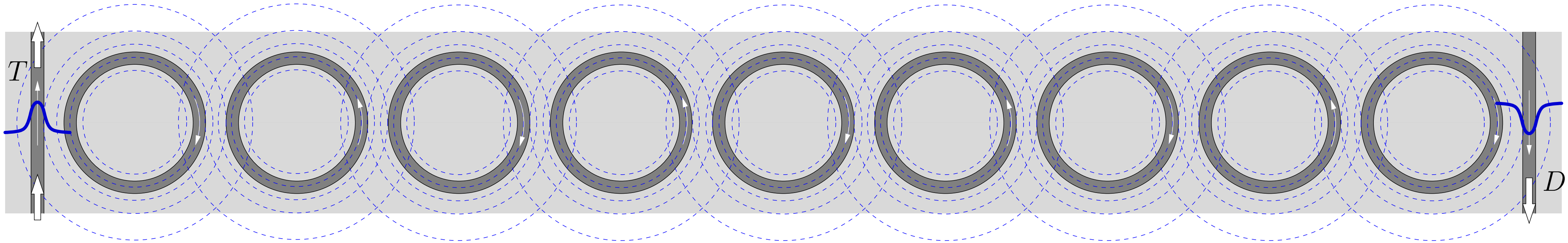
# *Single ring filter, supermodes vs. gap*



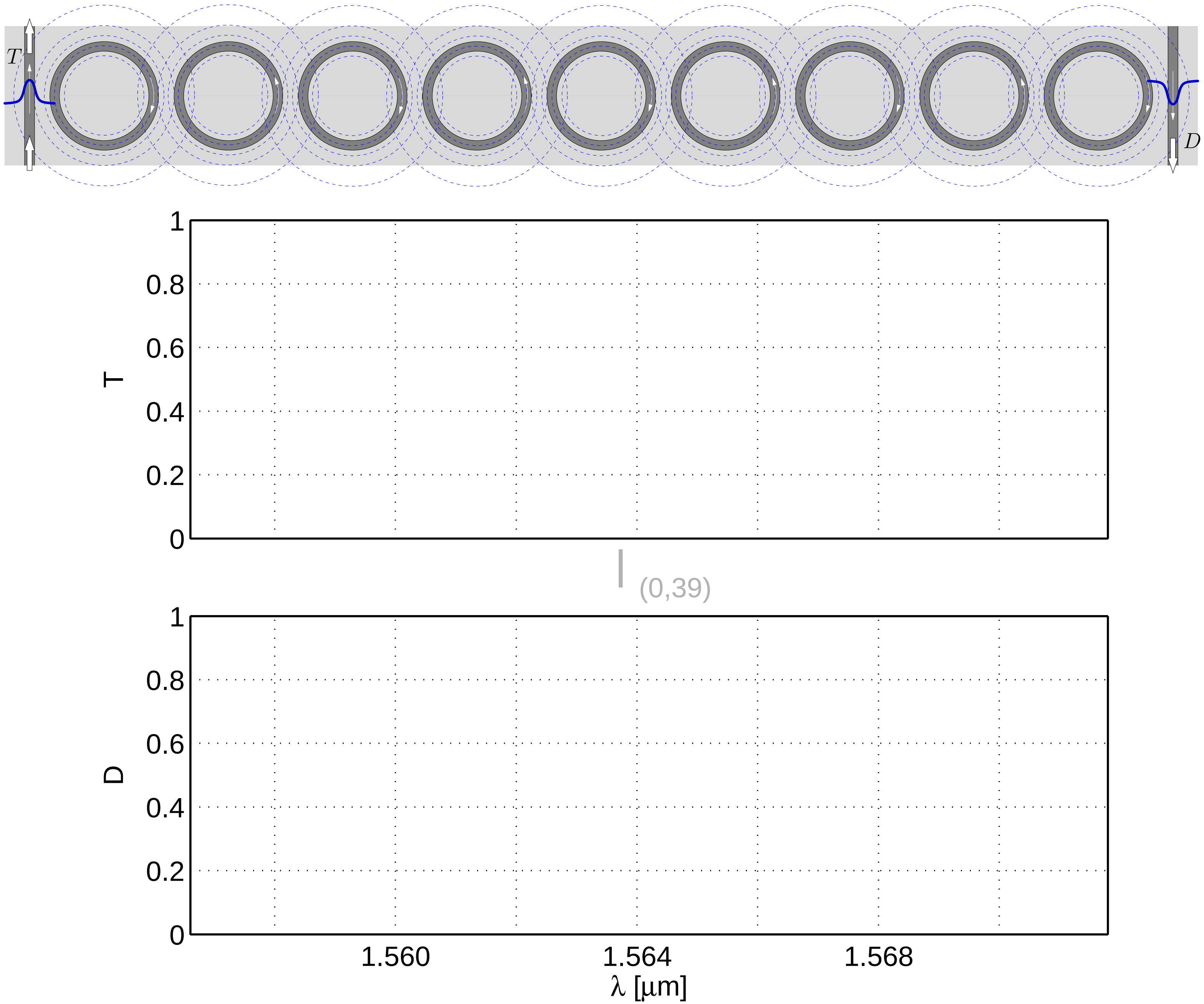

TE,
$R = 7.5\,\mu\text{m},\ d = 0.75\,\mu\text{m},$
$w = 0.6\,\mu\text{m},$
$n_\text{g} = 1.5,\ n_\text{b} = 1.0.$



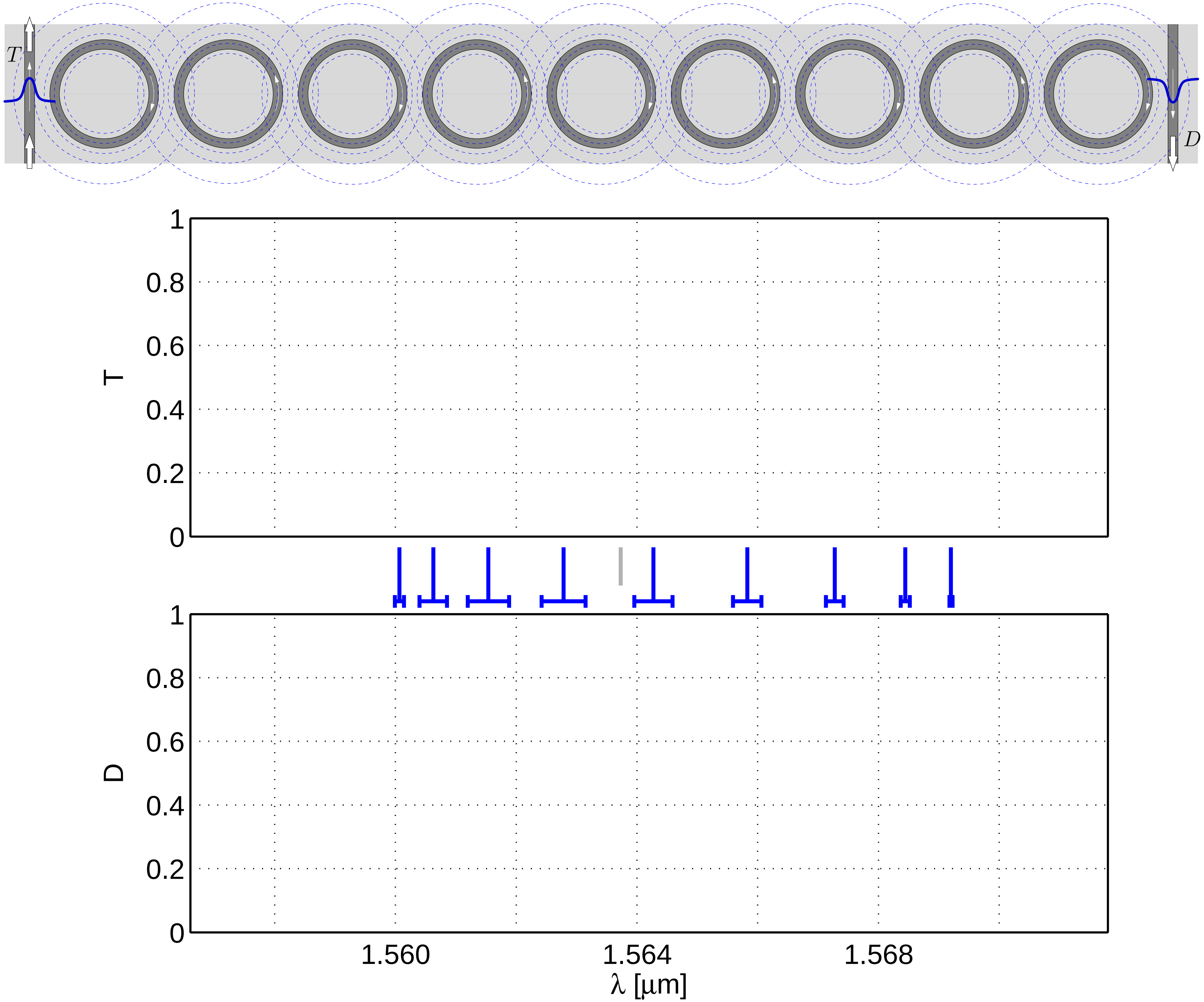
# CROW



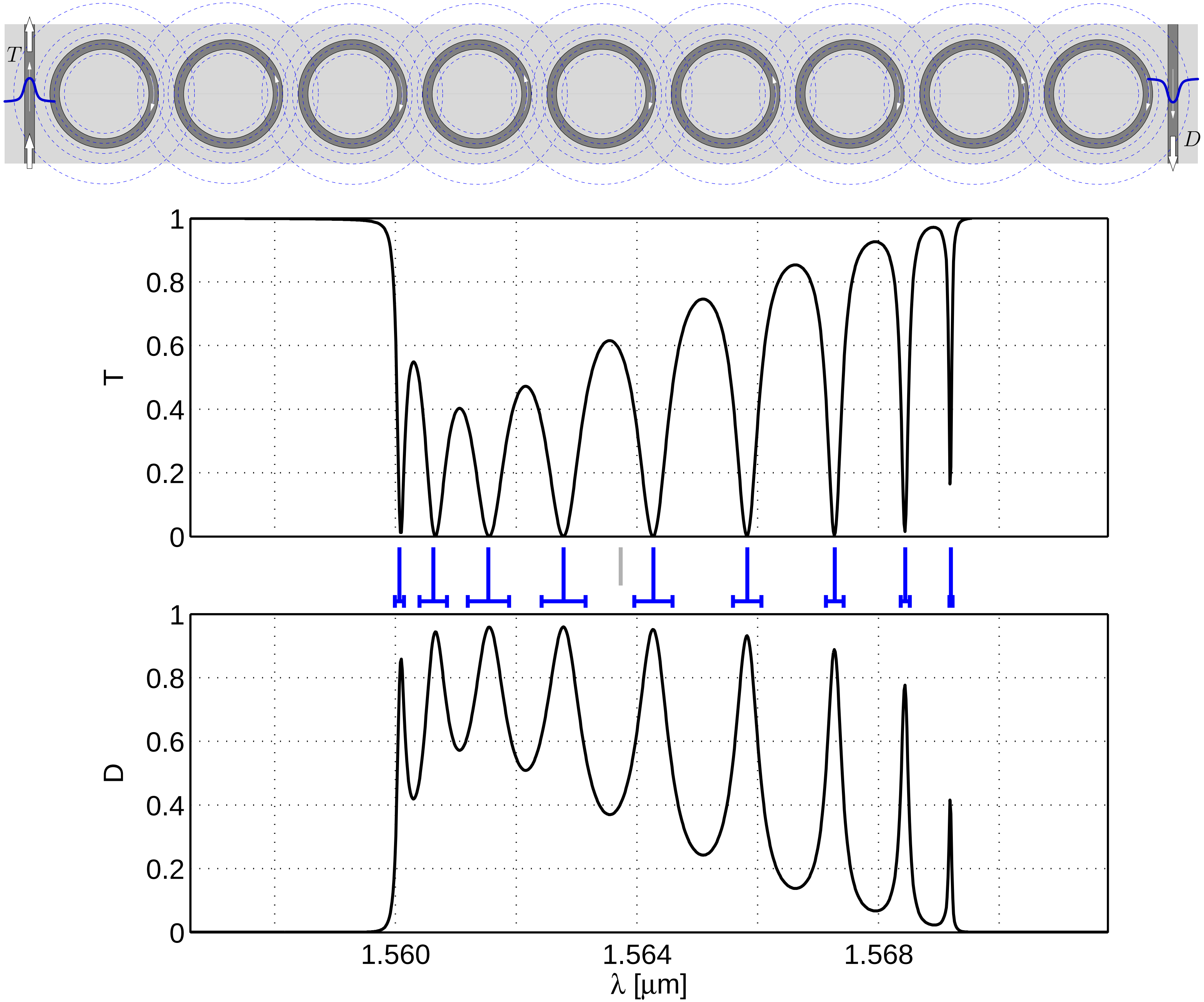
# *CROW, spectral response I*



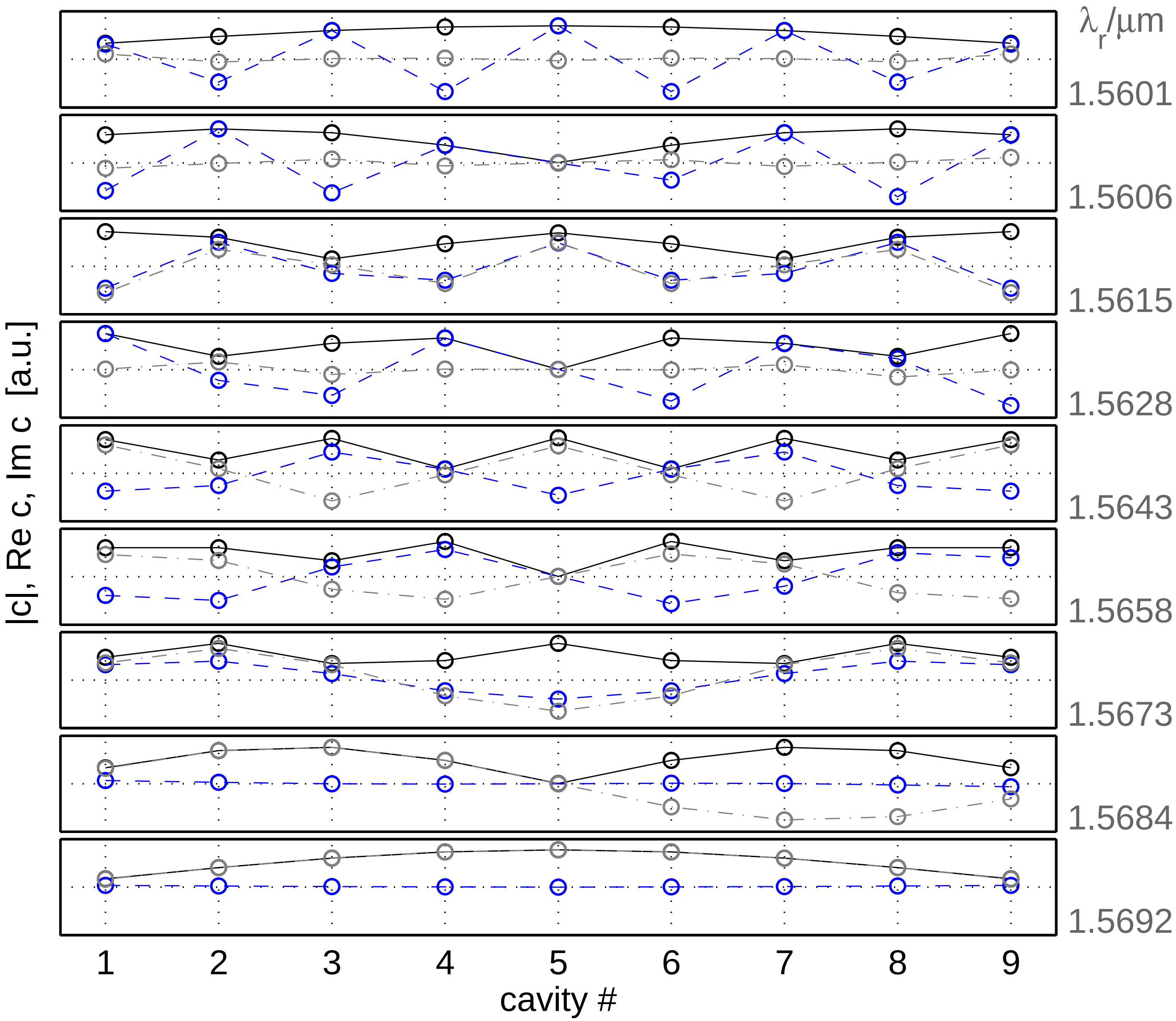
# *CROW, spectral response I*



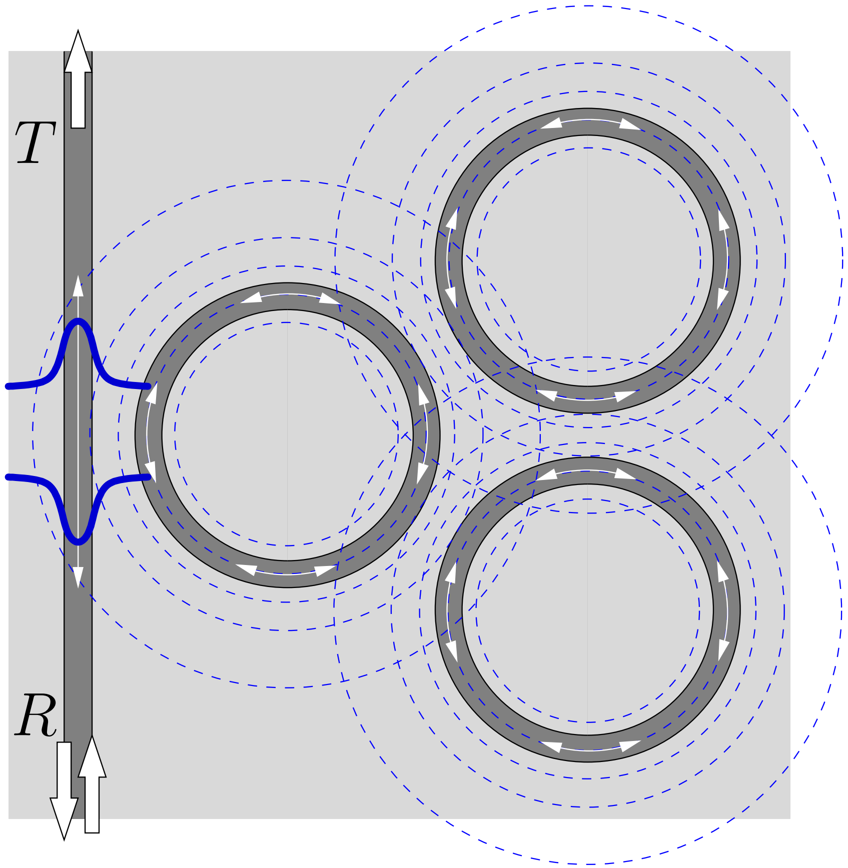
# CROW, spectral response I



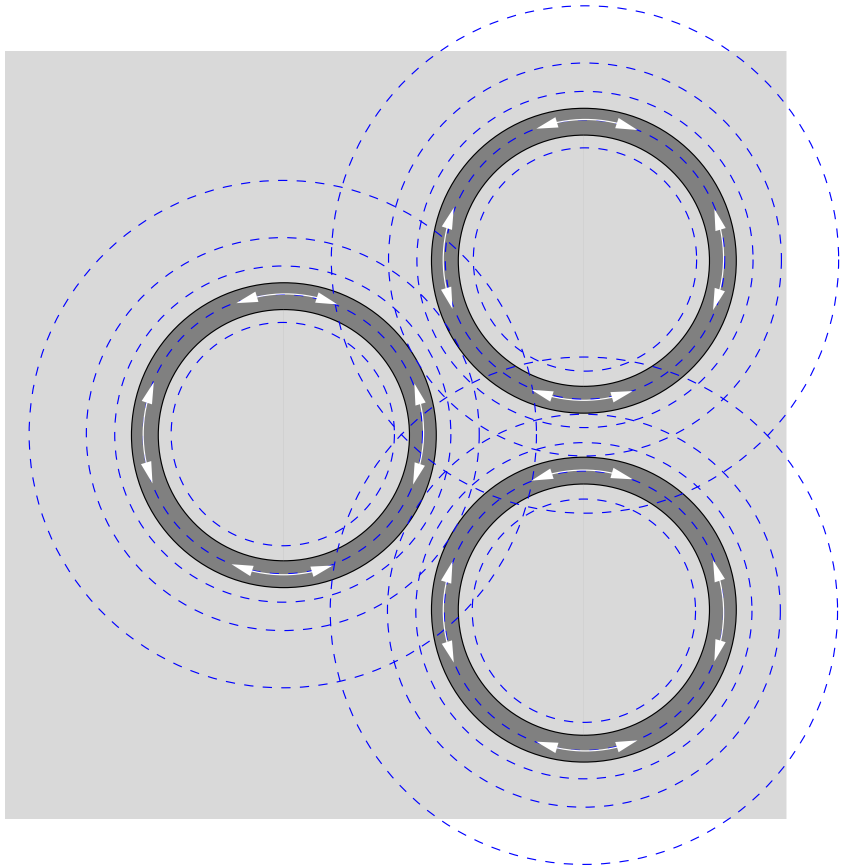
# CROW, supermode pattern



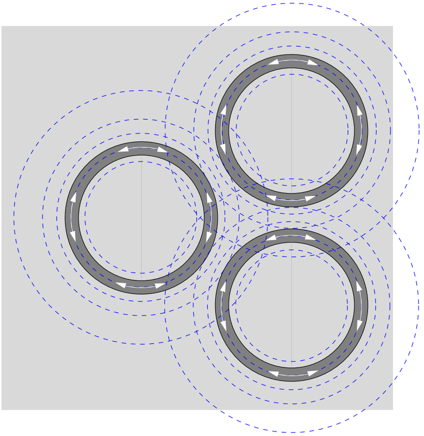
# *Three-ring molecule*



# *Three-ring molecule*

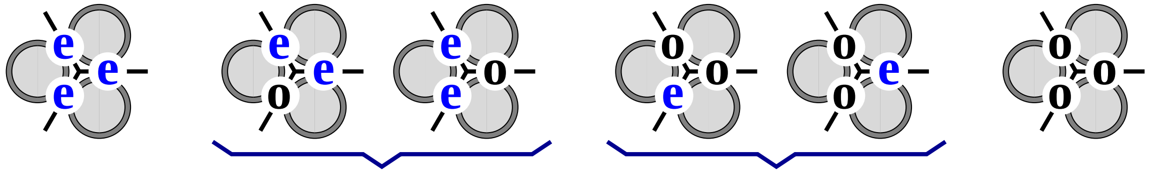


## *Three-ring molecule, supermodes*

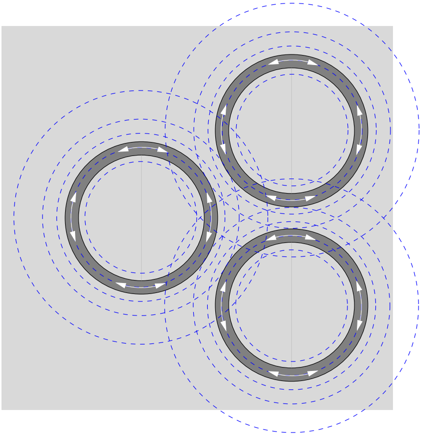


Template: $3 \times \mathrm{WGM}(0, \pm 39) \rightsquigarrow$ 6 supermodes.

Symmetries:

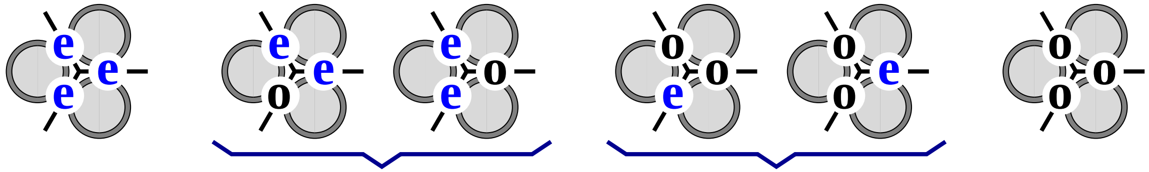


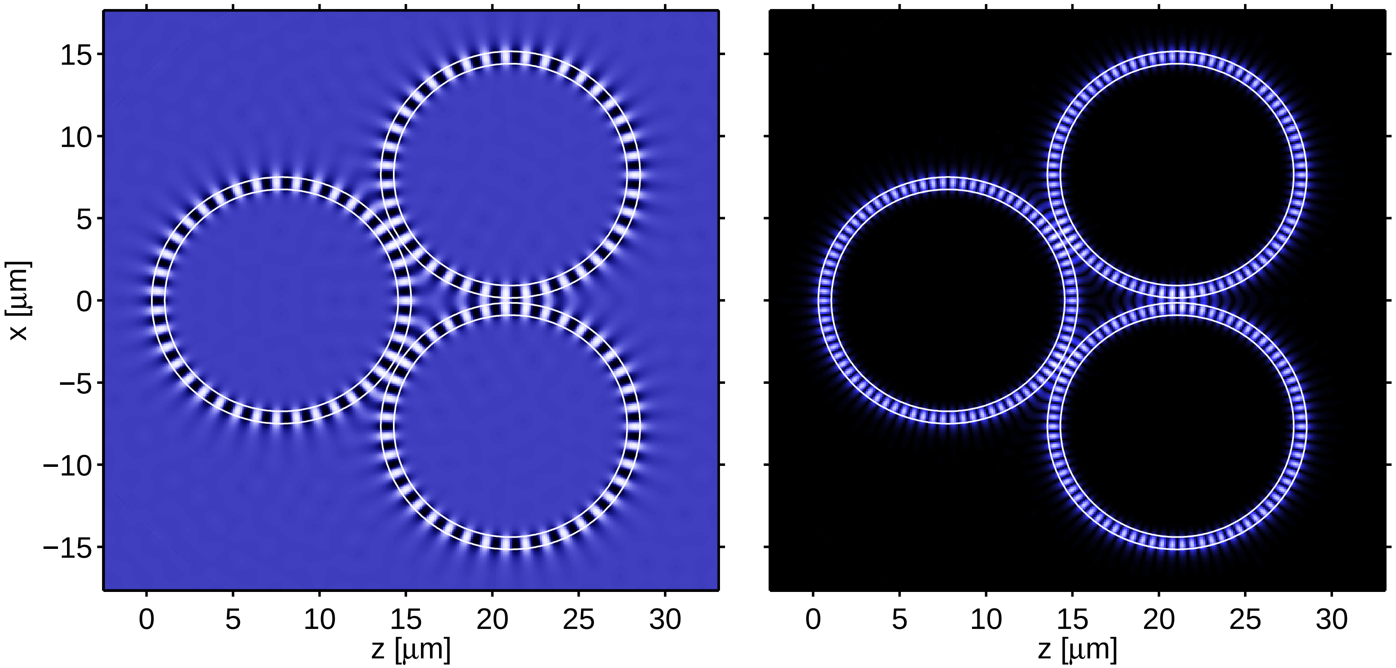
# *Three-ring molecule, supermodes*



Template: $3 \times \mathrm{WGM}(0, \pm 39)$ → 6 supermodes.

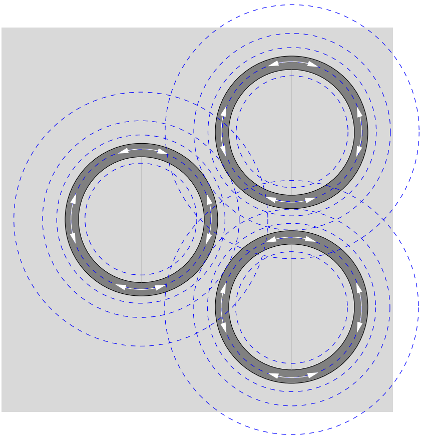
Symmetries:





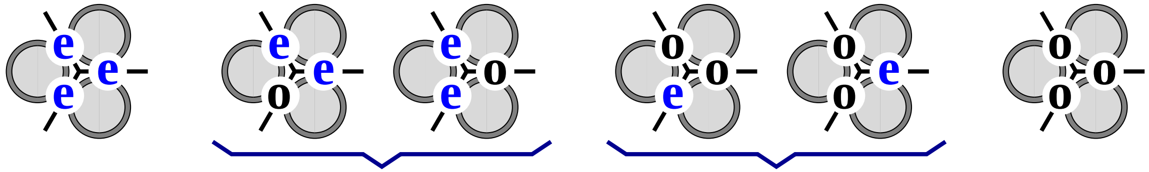

$\lambda_{\mathrm{r}} = 1.56946\,\mu\mathrm{m}$,
$Q = 1.3 \cdot 10^5$,
$\Delta\lambda = 1.1 \cdot 10^{-5}\,\mu\mathrm{m}$.

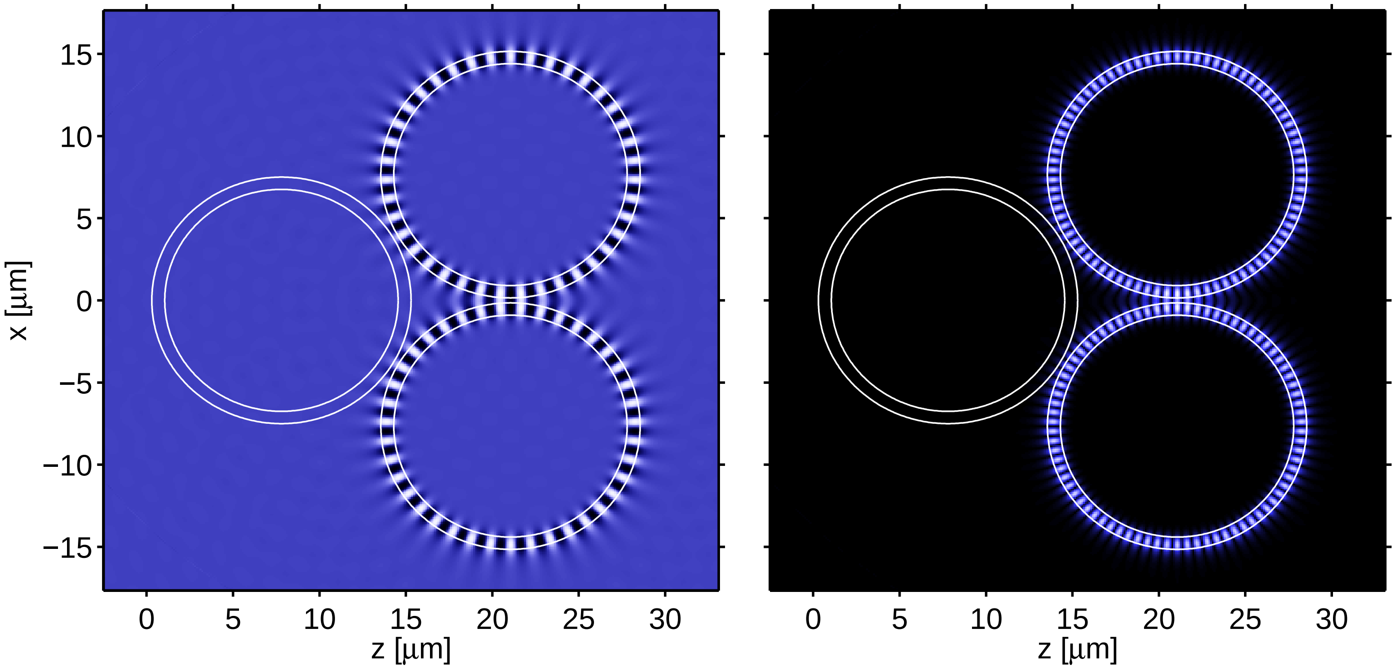
# *Three-ring molecule, supermodes*



Template: $3 \times \mathrm{WGM}(0, \pm 39) \rightsquigarrow$ 6 supermodes.

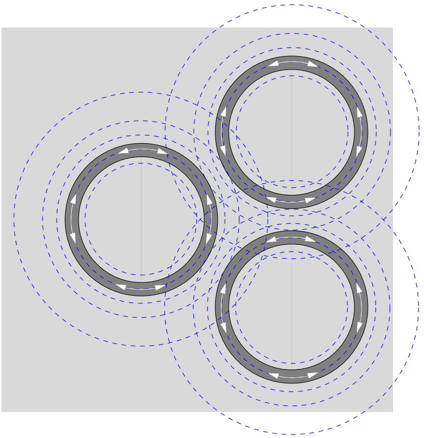
Symmetries:





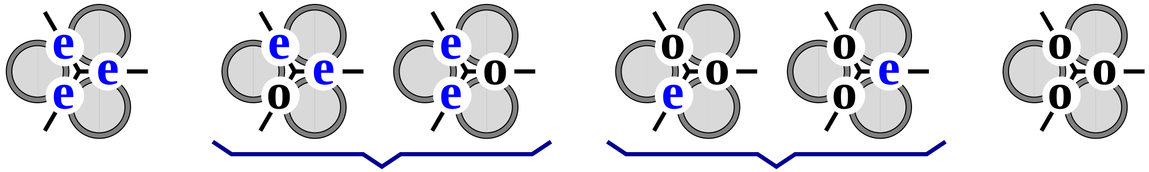

$\lambda_{\mathrm{r}} = 1.56715\,\mu\mathrm{m}$,
$Q = 1.2 \cdot 10^5$,
$\Delta\lambda = 1.3 \cdot 10^{-5}\,\mu\mathrm{m}$.

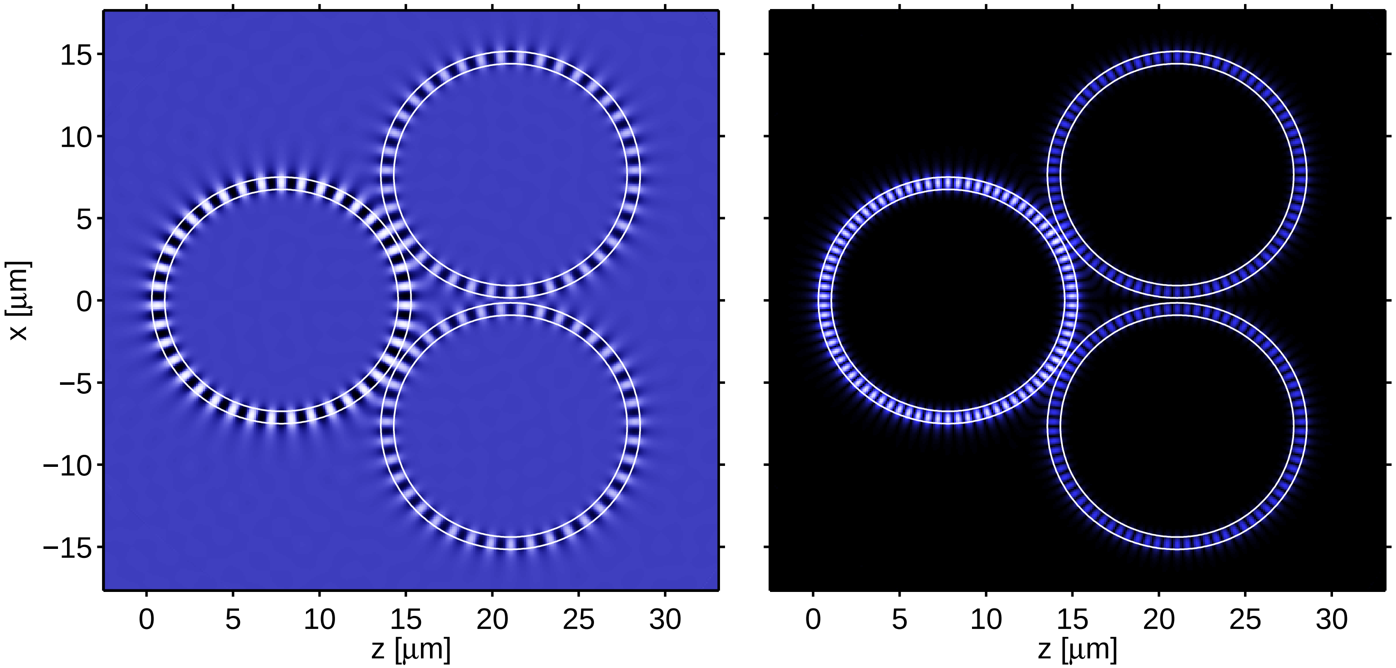
# *Three-ring molecule, supermodes*



Template: $3 \times \mathrm{WGM}(0, \pm 39) \rightsquigarrow$ 6 supermodes.

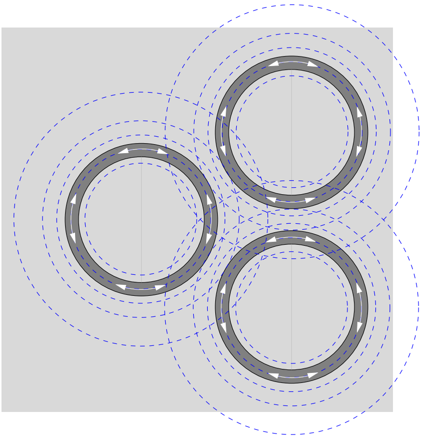
Symmetries:



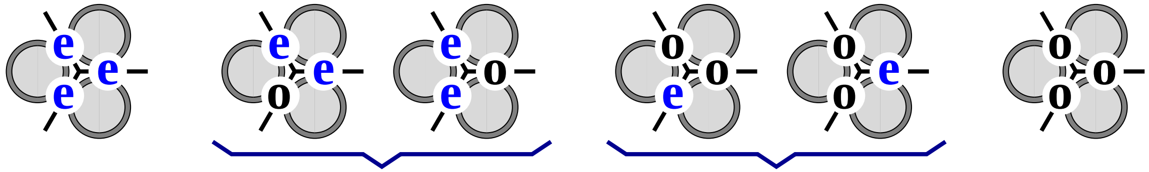



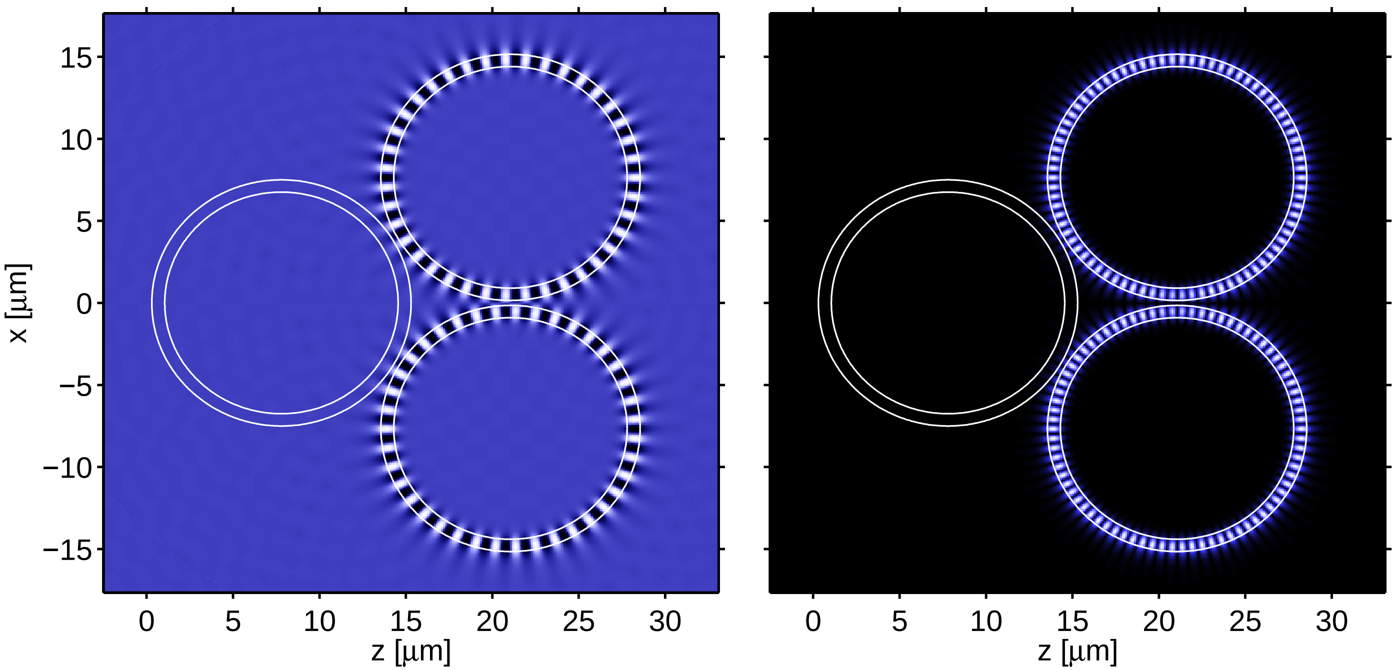

$\lambda_{\mathrm{r}} = 1.56714\,\mu\mathrm{m}$,
$Q = 0.9 \cdot 10^5$,
$\Delta\lambda = 1.7 \cdot 10^{-5}\,\mu\mathrm{m}$.

# *Three-ring molecule, supermodes*



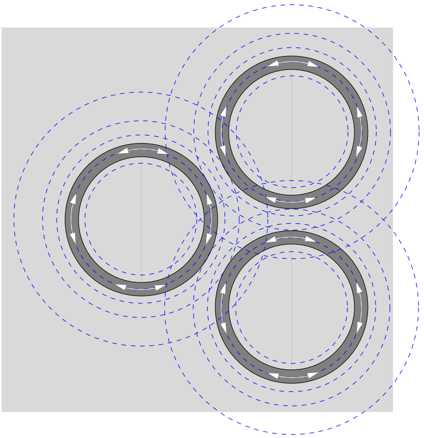
Template: $3 \times \mathrm{WGM}(0, \pm 39) \rightsquigarrow$ 6 supermodes.

Symmetries: 

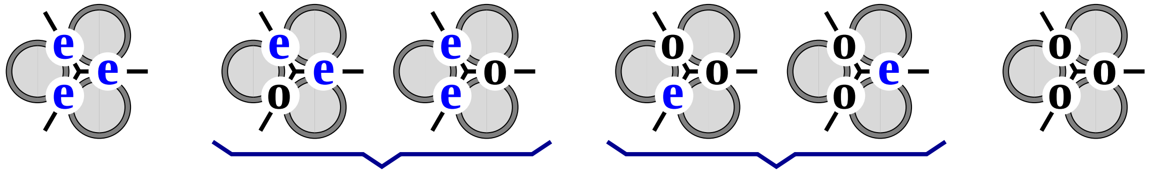

$\lambda_{\mathrm{r}} = 1.56235\,\mu\mathrm{m}$,
$Q = 1.0 \cdot 10^5$,
$\Delta\lambda = 1.6 \cdot 10^{-5}\,\mu\mathrm{m}$.

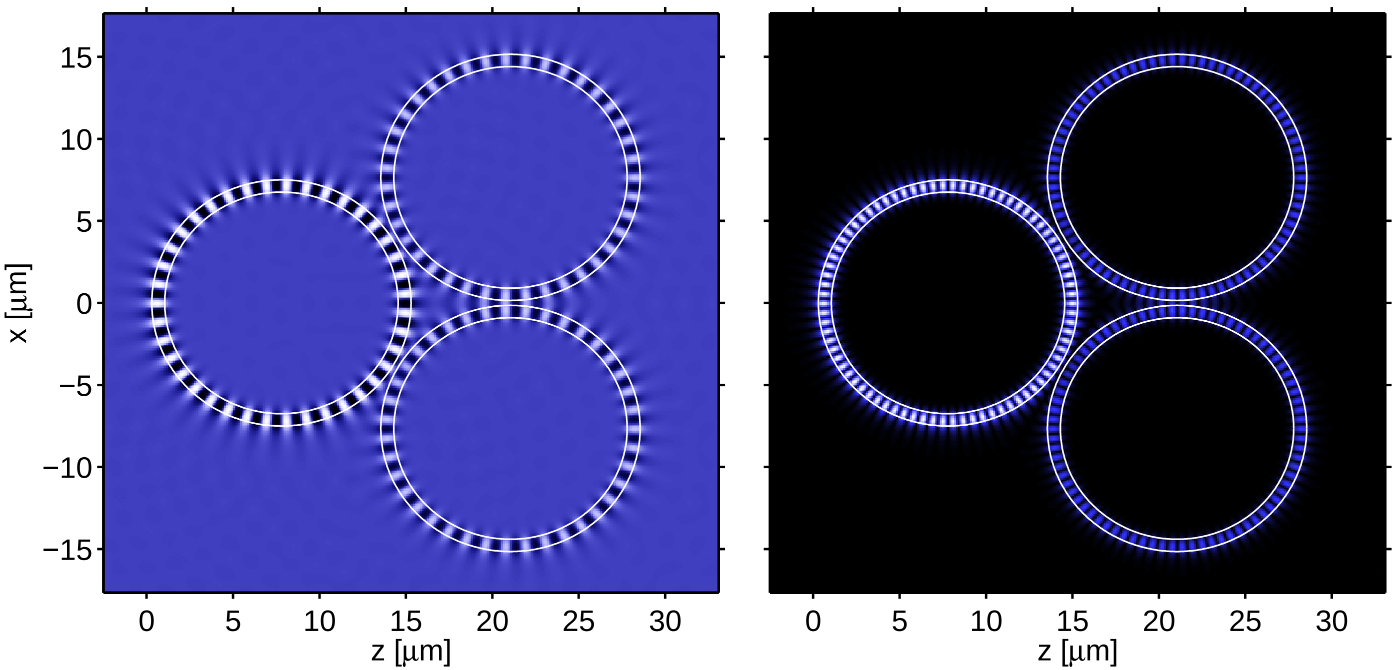
# *Three-ring molecule, supermodes*



Template: $3 \times \mathrm{WGM}(0, \pm 39)$ → 6 supermodes.

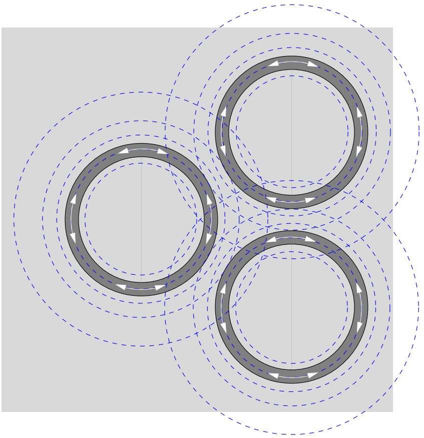
Symmetries:



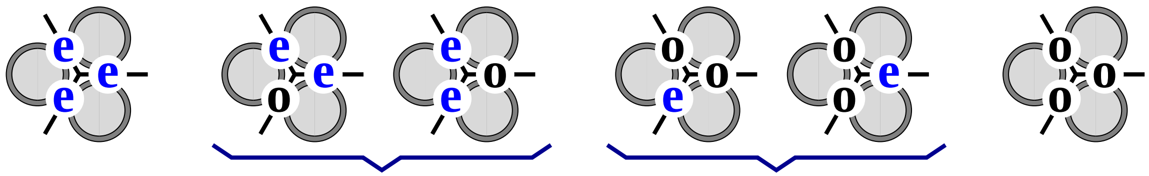

$\lambda_{\mathrm{r}} = 1.56234\,\mu\mathrm{m}$,
$Q = 1.0 \cdot 10^5$,
$\Delta\lambda = 1.5 \cdot 10^{-5}\,\mu\mathrm{m}$.

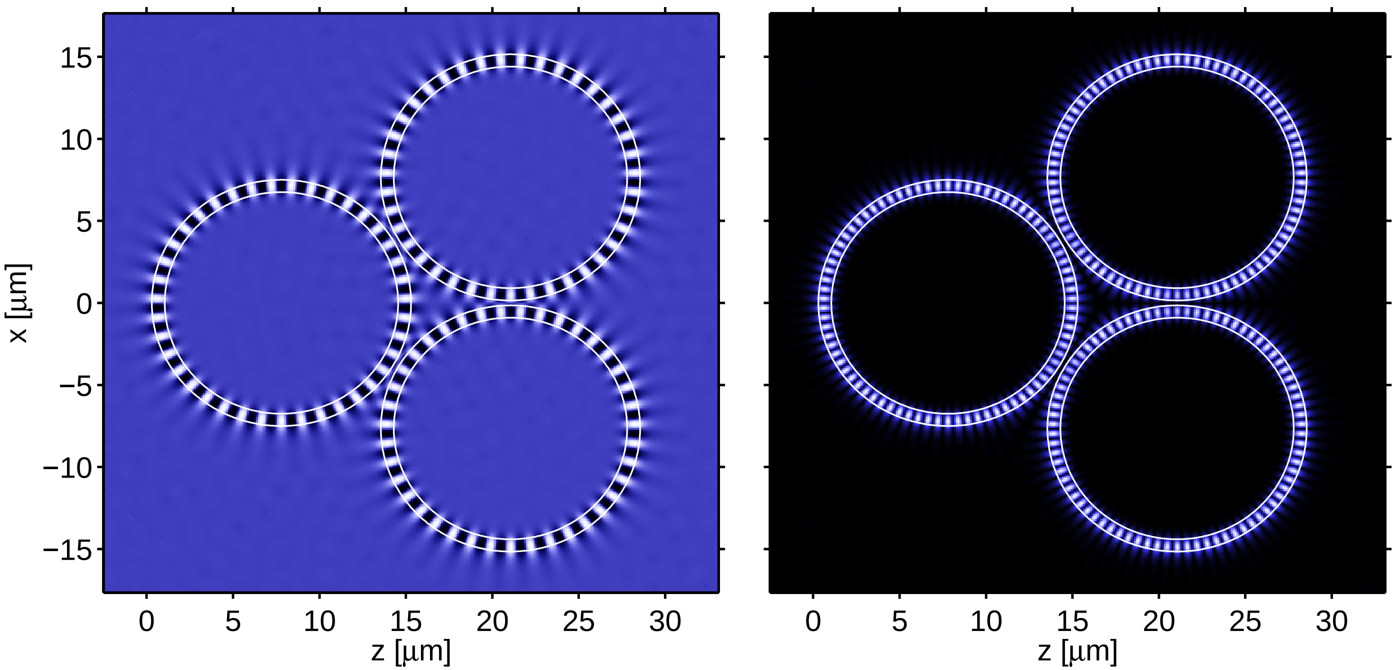
# *Three-ring molecule, supermodes*



Template: $3 \times \mathrm{WGM}(0, \pm 39)$ ⤳ 6 supermodes.

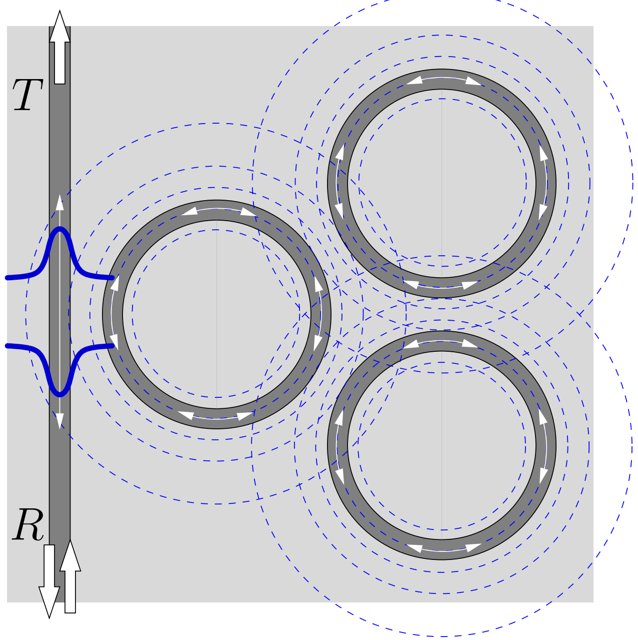
Symmetries:

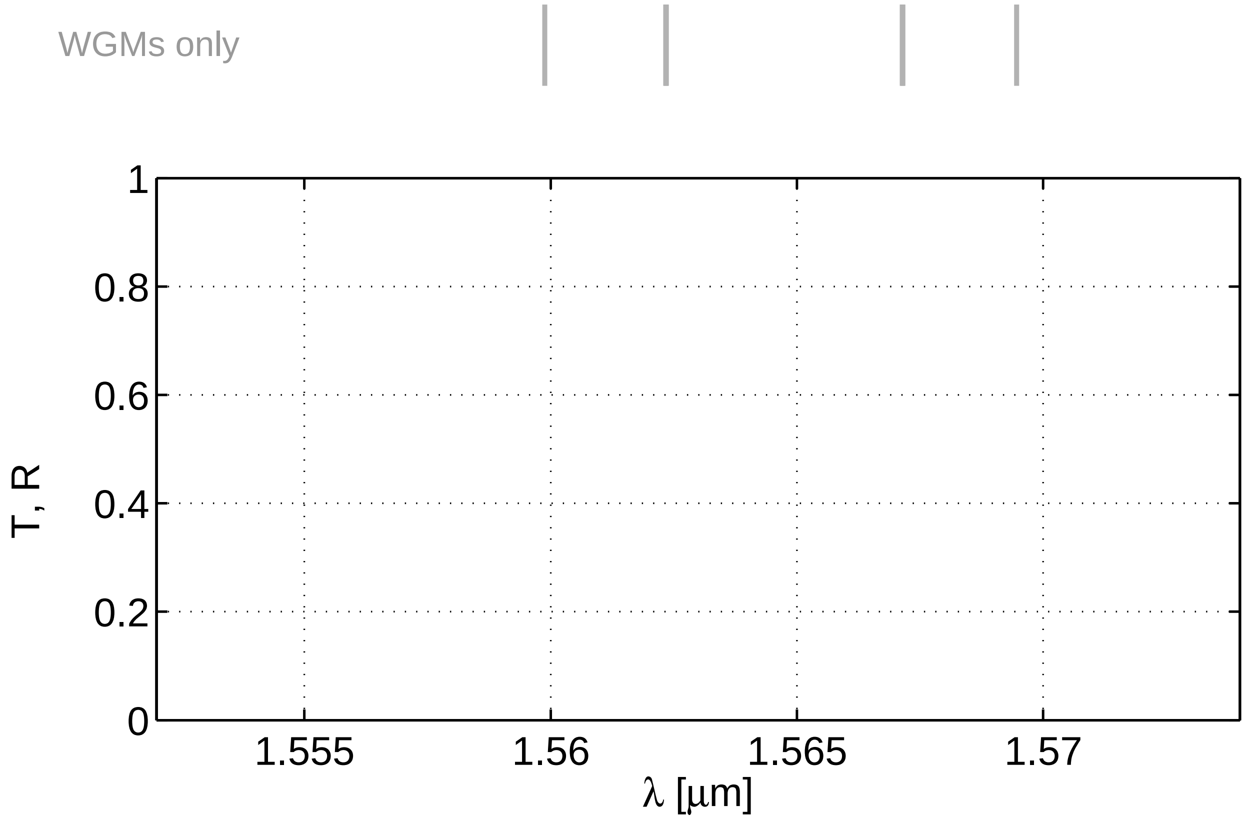




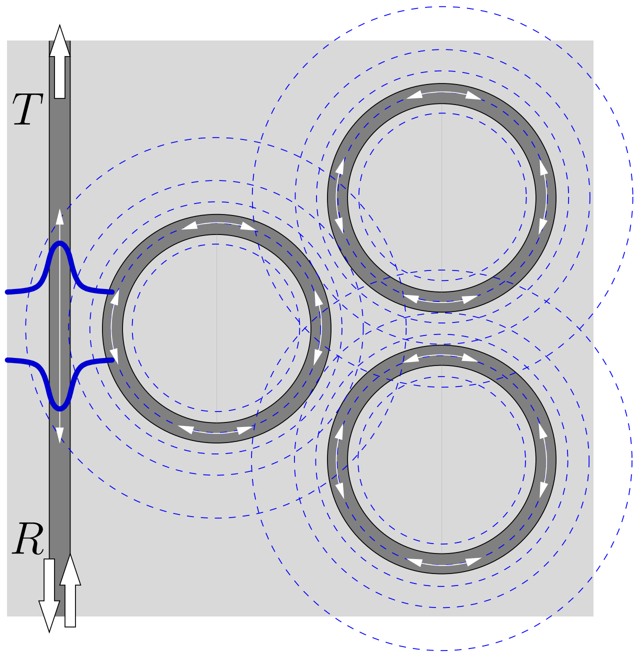

$\lambda_\mathrm{r} = 1.55988\,\mu\mathrm{m}$,
$Q = 1.2 \cdot 10^5$,
$\Delta\lambda = 1.3 \cdot 10^{-5}\,\mu\mathrm{m}$.

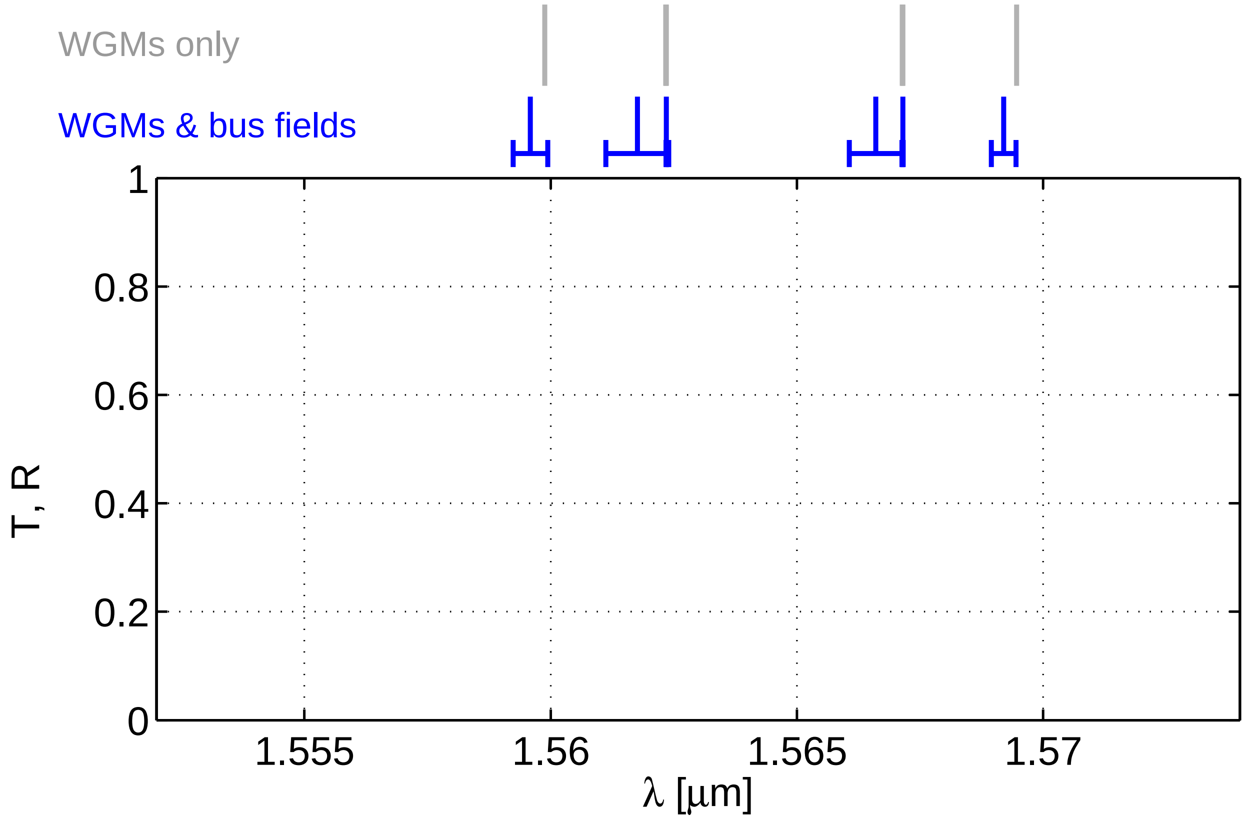
# *Three-ring molecule, excitation*



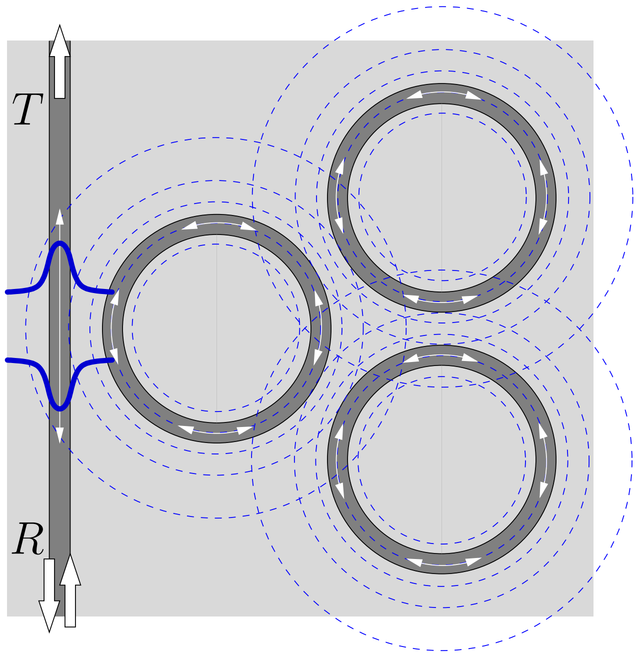



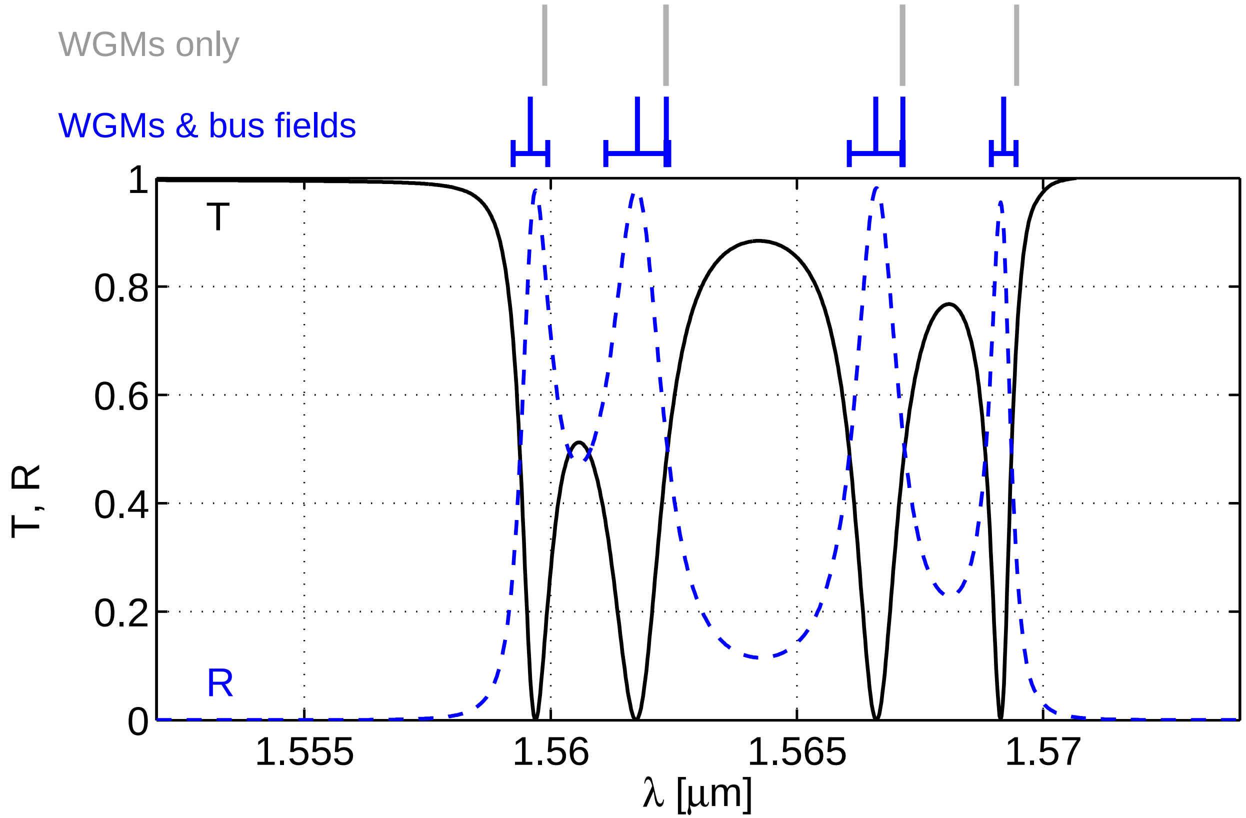
# *Three-ring molecule, excitation*



# *Three-ring molecule, excitation*

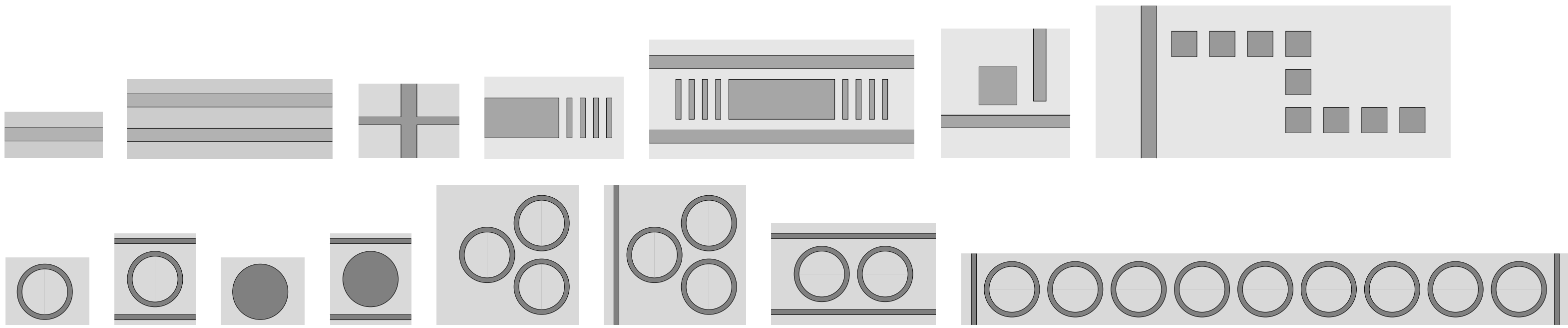


## *Concluding remarks*

Hybrid analytical / numerical Coupled Mode Theory, HCMT:

- an ab-initio, quantitative, quite general CMT variant, alternatively
- a numerical (FEM) approach with highly specialized base functions,
- configurations with localized resonances: demonstrated,
- extension to 3-D (todo): numerical basis fields, still moderate effort,
- very close to common ways of reasoning in integrated optics,
- reasonably versatile:



. . .

— *supplementary material* —

## *Fast evaluation of spectral properties*

Time consuming: evaluation of modal "overlaps" $K_{lk}$ in $\mathsf{K}$:

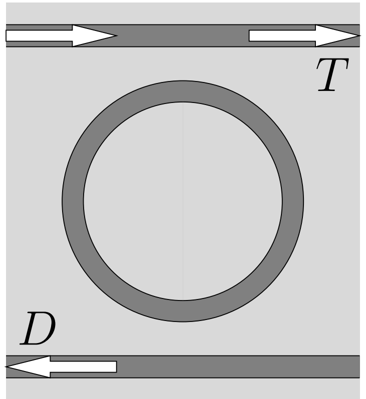
$$K_{lk} = \iint \mathcal{K}(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_k, \boldsymbol{H}_k)\,\mathrm{d}x\,\mathrm{d}z\,.$$

All properties of the modal basis fields change but slowly with $\lambda$;
rapid spectral variations are due to the *solution* of the linear system involving $\mathsf{K}$.

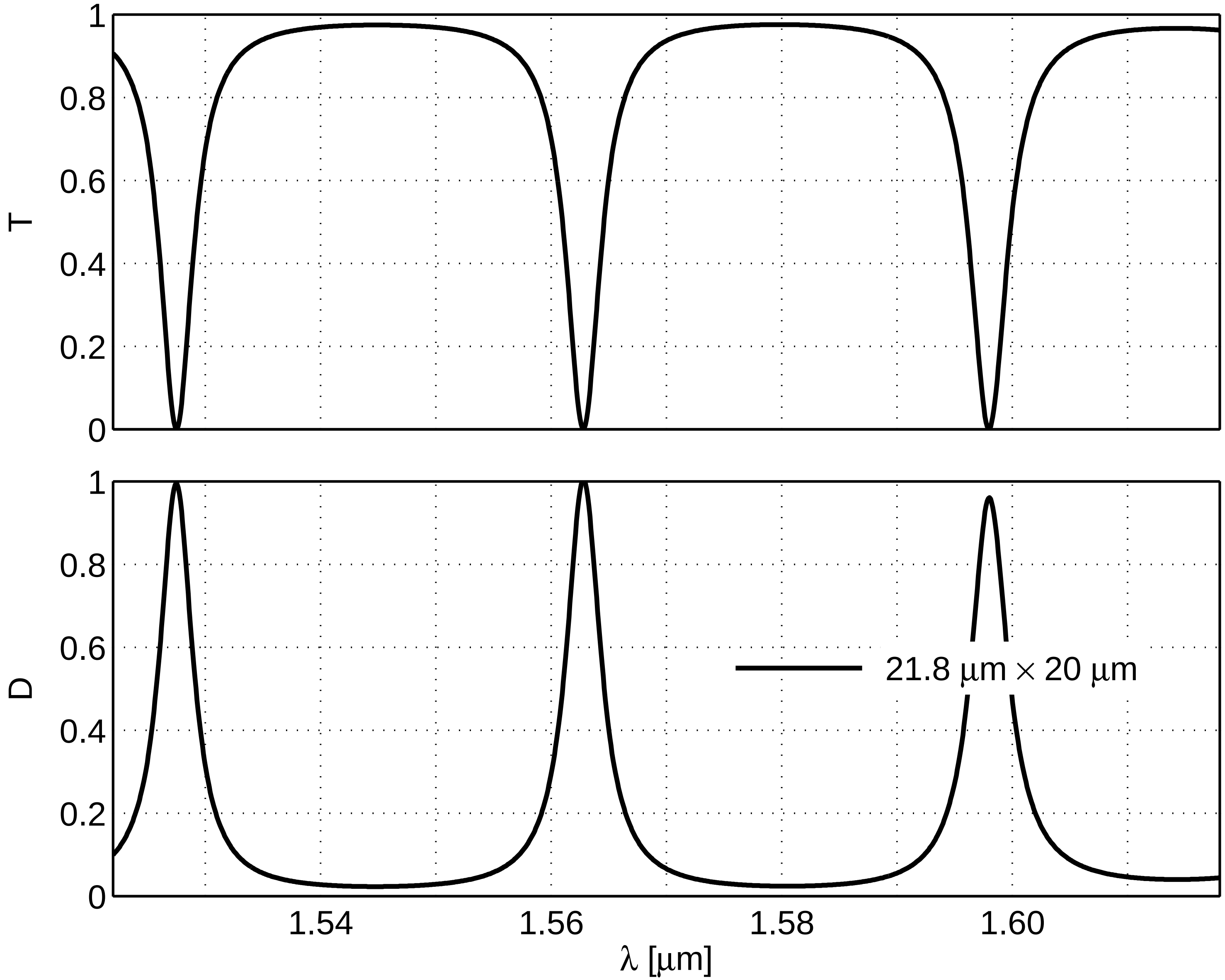
⇝ Interpolate $\mathsf{K}(\lambda)$:

- Interval of interest $\lambda \in [\lambda_a, \lambda_b]$, $\quad \lambda_0 := \frac{3}{4}\lambda_a + \frac{1}{4}\lambda_b,\ \ \lambda_1 := \frac{1}{4}\lambda_a + \frac{3}{4}\lambda_b,$
- compute only $\mathsf{K}_0 = \mathsf{K}(\lambda_0)$ and $\mathsf{K}_1 = \mathsf{K}(\lambda_1)$ directly,
- interpolate $\mathsf{K}_\mathrm{i}(\lambda) = \mathsf{K}_0 + \dfrac{\lambda - \lambda_0}{\lambda_1 - \lambda_0}\,(\mathsf{K}_1 - \mathsf{K}_0)$,
- solve for $\boldsymbol{a}(\lambda)$ with $\mathsf{K}_\mathrm{i}(\lambda)$.

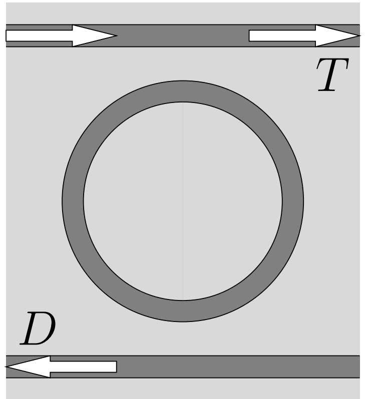
# *Computational window*



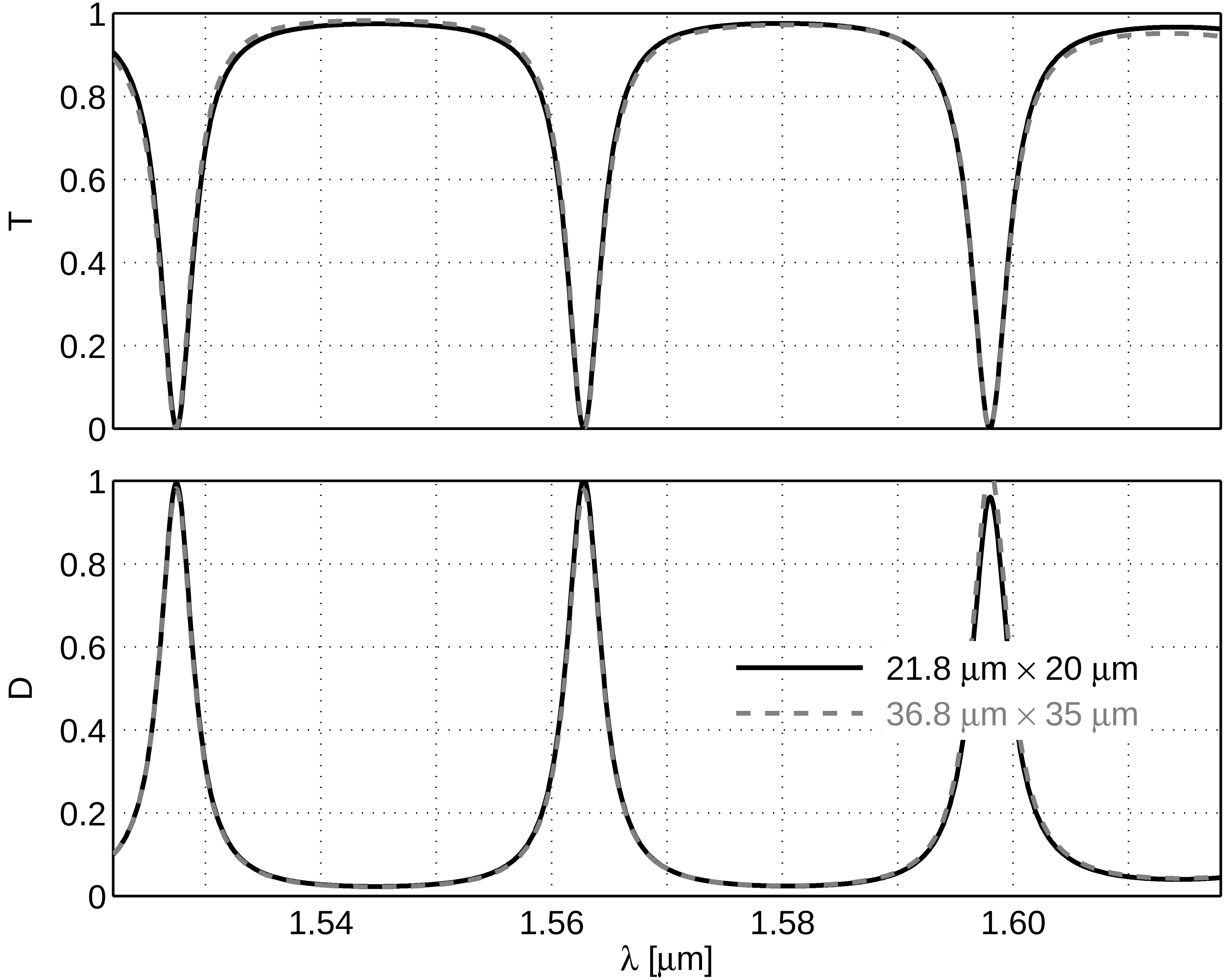

$R = 7.5\,\mu\text{m}$



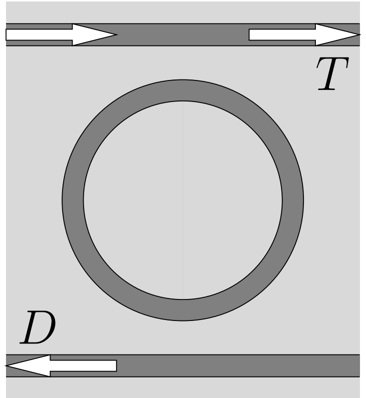
## *Computational window*



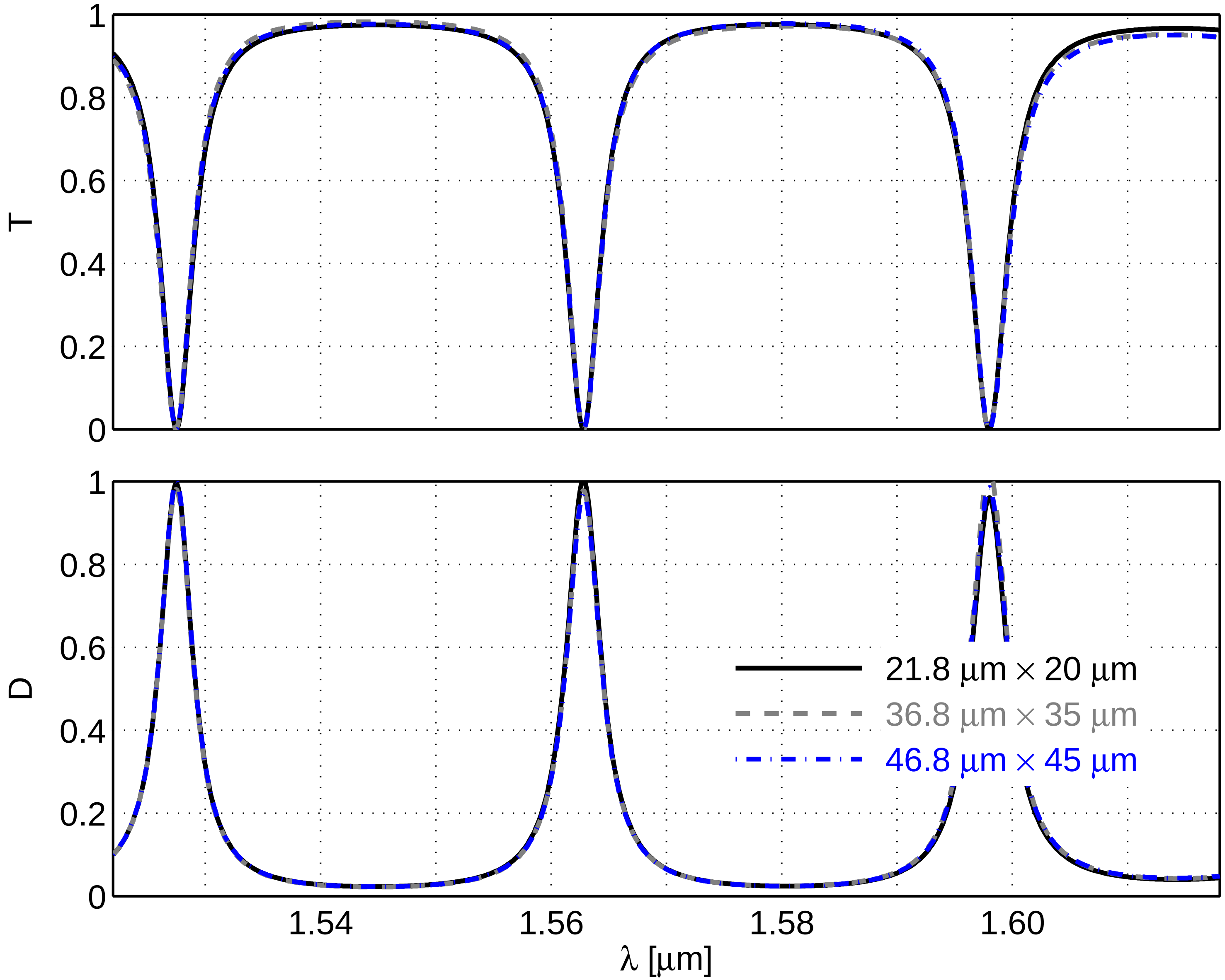

$R = 7.5\,\mu\mathrm{m}$



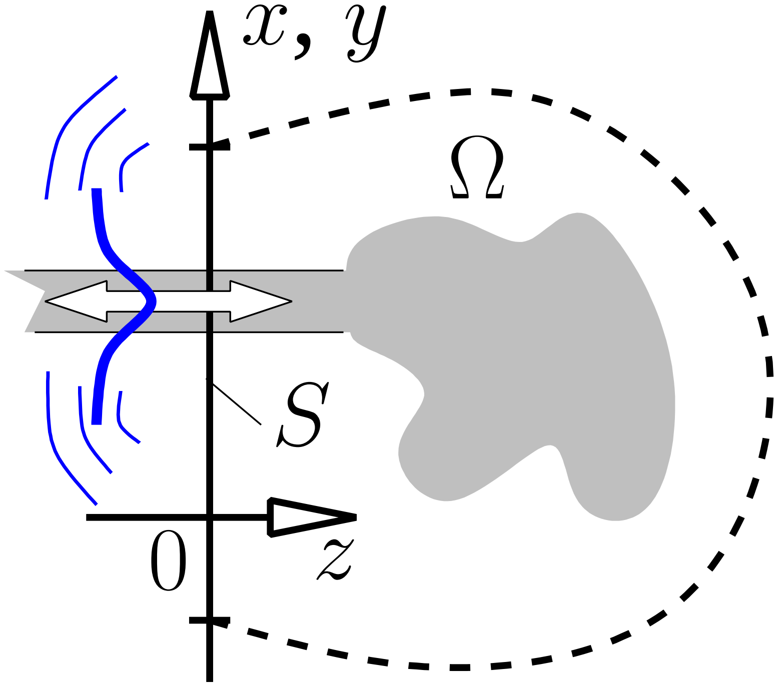
## *Computational window*




$R = 7.5\,\mu\text{m}$



## *Abstract scattering problem*



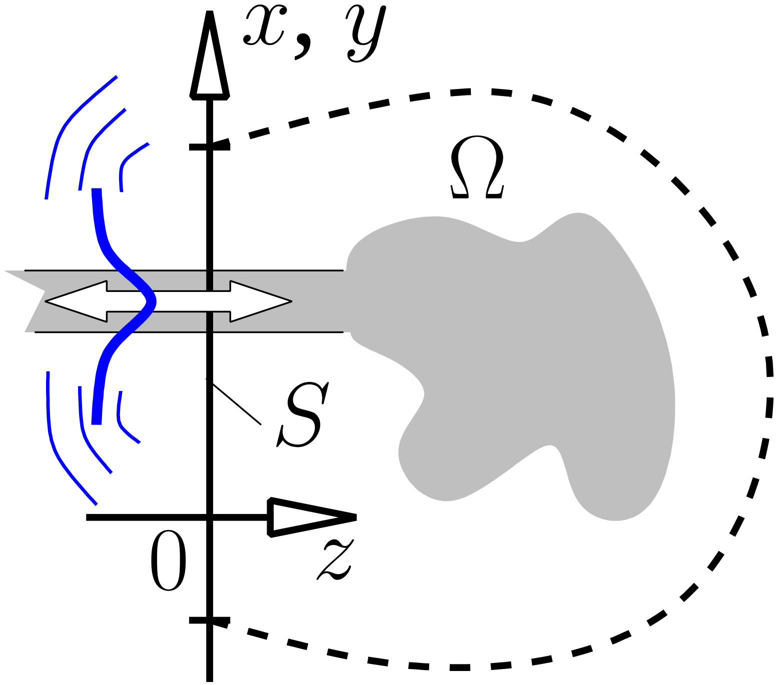

$\Omega$: domain of interest,

$$\left.\begin{aligned} \nabla \times \boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E} &= 0 \\ -\nabla \times \boldsymbol{E} - \mathrm{i}\omega\mu_0\boldsymbol{H} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

for given frequency $\omega$, permittivity $\epsilon = n^2$,

$S$: an exemplary port plane,
waveguides enter $\Omega$ through $S$.

## *Abstract scattering problem*




$\Omega$: domain of interest,

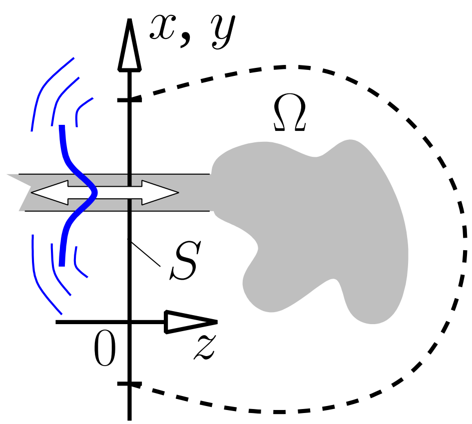
$$\left.\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E} &= 0 \\ -\boldsymbol{\nabla} \times \boldsymbol{E} - \mathrm{i}\omega\mu_0\boldsymbol{H} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

for given frequency $\omega$, permittivity $\epsilon = n^2$,

$S$: an exemplary port plane,
waveguides enter $\Omega$ through $S$.

*Variational form including suitable boundary conditions ?*

## *Boundary conditions*




Ingredients:

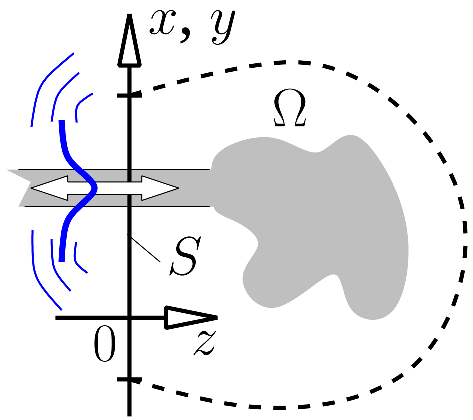
- Complete set of normal modes on $S$,
  $(\tilde{\boldsymbol{E}}_m, \pm\tilde{\boldsymbol{H}}_m)(x, y)$ $\longleftrightarrow$ propagation along $\pm z$.
- Product on $S$: $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iint_S (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{e}_z \, \mathrm{d}x \, \mathrm{d}y$.
- Modal orthogonality properties $\langle \tilde{\boldsymbol{E}}_l, \tilde{\boldsymbol{H}}_k \rangle = \delta_{lk} N_k$, $N_k = \langle \tilde{\boldsymbol{E}}_k, \tilde{\boldsymbol{H}}_k \rangle$.

---

"Any" electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ on $S$ can be expanded as

$$\boldsymbol{E} = \sum_m e_m \tilde{\boldsymbol{E}}_m, \quad e_m = \frac{1}{N_m} \langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m \rangle, \quad \boldsymbol{H} = \sum_m h_m \tilde{\boldsymbol{H}}_m, \quad h_m = \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle,$$

## *Boundary conditions*




Ingredients:

- Complete set of normal modes on $S$,
  $(\tilde{\boldsymbol{E}}_m, \pm\tilde{\boldsymbol{H}}_m)(x, y) \longleftrightarrow$ propagation along $\pm z$.
- Product on $S$: $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iint_S (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{e}_z \, \mathrm{d}x \, \mathrm{d}y$.
- Modal orthogonality properties $\langle \tilde{\boldsymbol{E}}_l, \tilde{\boldsymbol{H}}_k \rangle = \delta_{lk} N_k, \;\; N_k = \langle \tilde{\boldsymbol{E}}_k, \tilde{\boldsymbol{H}}_k \rangle$.

---

"Any" electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ on $S$ can be expanded as

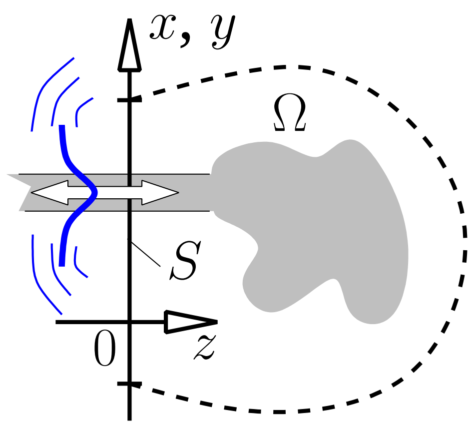
$$\boldsymbol{E} = \sum_m e_m \tilde{\boldsymbol{E}}_m, \;\; e_m = \frac{1}{N_m} \langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m \rangle, \;\; \boldsymbol{H} = \sum_m h_m \tilde{\boldsymbol{H}}_m, \;\; h_m = \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle,$$

or

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_m f_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix} + \sum_m b_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ -\tilde{\boldsymbol{H}}_m \end{pmatrix}, \qquad \begin{aligned} f_m &= (e_m + h_m)/2, \\ b_m &= (e_m - h_m)/2 \end{aligned}$$

(transverse components only).

# *Transparent influx boundary conditions (TIBCs)*



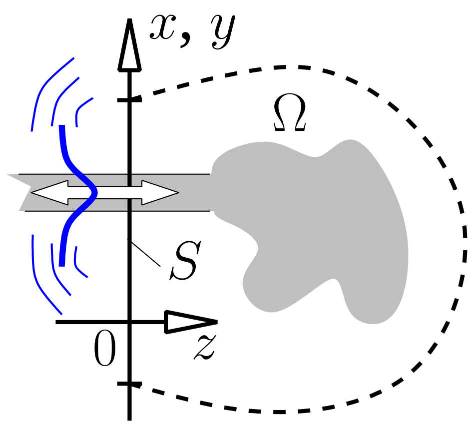

... on $S$ for inhomogeneous exterior, incoming waveguides:

$$\boldsymbol{E} = \sum_m 2F_m \tilde{\boldsymbol{E}}_m - \sum_m \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle \tilde{\boldsymbol{E}}_m \,,$$

$$\boldsymbol{H} = \sum_m 2F_m \tilde{\boldsymbol{H}}_m - \sum_m \frac{1}{N_m} \langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m \rangle \tilde{\boldsymbol{H}}_m \,;$$

$F_m$: influx, given coefficients of incoming waves; $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}_{\text{inc}} = \sum_m F_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix}.$

# *Transparent influx boundary conditions (TIBCs)*




... on $S$ for inhomogeneous exterior, incoming waveguides:

$$\boldsymbol{E} = \sum_m 2F_m \tilde{\boldsymbol{E}}_m - \sum_m \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle \tilde{\boldsymbol{E}}_m \,,$$

$$\boldsymbol{H} = \sum_m 2F_m \tilde{\boldsymbol{H}}_m - \sum_m \frac{1}{N_m} \langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m \rangle \tilde{\boldsymbol{H}}_m \,;$$

$F_m$: influx, given coefficients of incoming waves; $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}_{\text{inc}} = \sum_m F_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix}.$

↪ For a general field of the form $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_m f_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix} + \sum_m b_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ -\tilde{\boldsymbol{H}}_m \end{pmatrix}$

the TIBCs require $f_m = F_m$, while $b_m$ can be arbitrary.

## *Frequency domain Maxwell equations, variational form*

Consider the functional

$$\mathcal{L}(\boldsymbol{E},\boldsymbol{H}) = \iiint_\Omega \left\{\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) + \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^2 + \mathrm{i}\omega\mu_0\boldsymbol{H}^2\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(cf. e.g. C. Vassallo. *Optical Waveguide Concepts*. Elsevier, Amsterdam, 1991).

First variation:

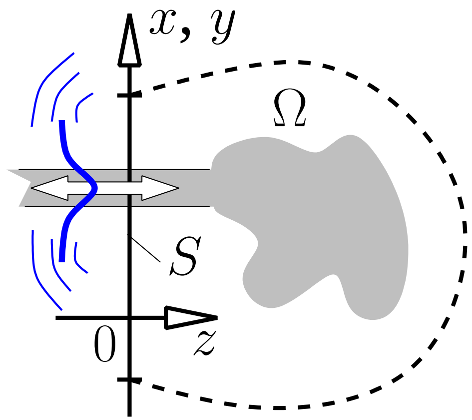
$$\begin{aligned}\delta\mathcal{L}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) \;&=\; \iiint_\Omega \Big\{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}) \\ &\qquad + 2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E} + \mathrm{i}\omega\mu_0\boldsymbol{H})\Big\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\quad - \iint_{\partial\Omega} \{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H} + (\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.\end{aligned}$$

---

Stationarity $\delta\mathcal{L}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) = 0$ for arbitrary $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ implies

- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the Maxwell equations in $\Omega$
- and that transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish on $\partial\Omega$.

## *Variational form of the scattering problem*

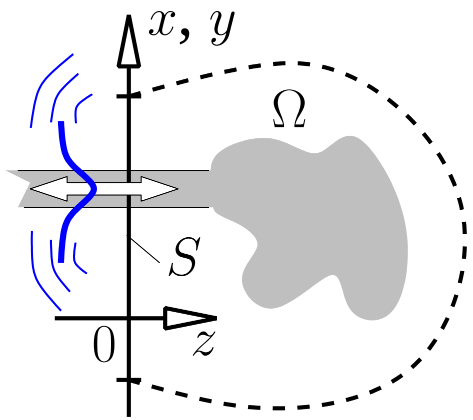



. . . based on the functional:

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) = \iiint_\Omega \left\{\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) + \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^2 + \mathrm{i}\omega\mu_0\boldsymbol{H}^2\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$- \sum_m 2F_m\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle - \langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\right\}$$
$$+ \sum_m \frac{1}{2N_m}\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle^2 - \langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle^2\right\}$$

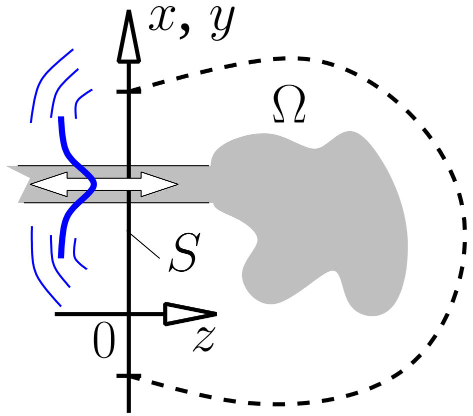
## *Variational form of the scattering problem, first variation*

$$
\begin{aligned}
\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) \quad = \quad & \iiint_\Omega \Big\{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}-\mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}) \\
& \qquad +2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}+\mathrm{i}\omega\mu_0\boldsymbol{H})\Big\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
+ \quad & \Big\langle \boldsymbol{E}-\sum_m 2F_m\tilde{\boldsymbol{E}}_m+\sum_m\frac{1}{N_m}\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle\tilde{\boldsymbol{E}}_m,\ \delta\boldsymbol{H}\Big\rangle \\
- \quad & \Big\langle \delta\boldsymbol{E},\ \boldsymbol{H}-\sum_m 2F_m\tilde{\boldsymbol{H}}_m+\sum_m\frac{1}{N_m}\langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\tilde{\boldsymbol{H}}_m\Big\rangle \\
- \quad & \iint_{\partial\Omega\backslash S}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H}+(\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\ \mathrm{d}A\,.
\end{aligned}
$$



## *Variational form of the scattering problem, first variation*

$$
\begin{aligned}
\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) \quad = \quad & \iiint_\Omega \big\{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}-\mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}) \\
& \qquad + 2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}+\mathrm{i}\omega\mu_0\boldsymbol{H})\big\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
+ \quad & \Big\langle \boldsymbol{E}-\sum_m 2F_m\tilde{\boldsymbol{E}}_m+\sum_m\frac{1}{N_m}\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle\tilde{\boldsymbol{E}}_m,\,\delta\boldsymbol{H}\Big\rangle \\
- \quad & \Big\langle \delta\boldsymbol{E},\,\boldsymbol{H}-\sum_m 2F_m\tilde{\boldsymbol{H}}_m+\sum_m\frac{1}{N_m}\langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\tilde{\boldsymbol{H}}_m\Big\rangle \\
- \quad & \iint_{\partial\Omega\backslash S}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H}+(\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.
\end{aligned}
$$




Stationarity $\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H})=0$ for arbitrary $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ implies

- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the Maxwell equations in $\Omega$,
- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy TIBCs on $S$,
- and that transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish on $\partial\Omega\backslash S$.

## *Variational HCMT scheme*

$$(\boldsymbol{E}, \boldsymbol{H}) = \sum_k a_k (\boldsymbol{E}_k, \boldsymbol{H}_k)$$

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{H}) \longrightarrow \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

## *Variational HCMT scheme*

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) \xrightarrow{\;(\boldsymbol{E},\boldsymbol{H}) = \sum_k a_k(\boldsymbol{E}_k,\boldsymbol{H}_k)\;} \mathcal{F}_{\rm r}(\boldsymbol{a})$$

Restricted functional:

$$\mathcal{F}_{\rm r}(\boldsymbol{a}) = \sum_{l,k} a_l F_{lk} a_k + \sum_l R_l a_l + \sum_{l,k} a_l B_{lk} a_k \,,$$

$$F_{lk} = \iiint_\Omega \left\{\boldsymbol{E}_l \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}_k) + \boldsymbol{H}_l \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}_k) \right.$$
$$\left. - \,\mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_l \cdot \boldsymbol{E}_k + \mathrm{i}\omega\mu_0 \boldsymbol{H}_l \cdot \boldsymbol{H}_k \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \,,$$

$$R_l = -\sum_m 2F_m \left\{ \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}_l\rangle - \langle \boldsymbol{E}_l, \tilde{\boldsymbol{H}}_m\rangle \right\},$$

$$B_{lk} = \sum_m \frac{1}{2N_m} \left\{ \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}_l\rangle \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}_k\rangle - \langle \boldsymbol{E}_l, \tilde{\boldsymbol{H}}_m\rangle \langle \boldsymbol{E}_k, \tilde{\boldsymbol{H}}_m\rangle \right\},$$

$+$ contributions $R$, $B$ from other port planes.

## ***Variational HCMT scheme***

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) \xrightarrow{\quad (\boldsymbol{E},\boldsymbol{H}) = \sum_k a_k(\boldsymbol{E}_k,\boldsymbol{H}_k) \quad} \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Restricted functional:

$$\mathcal{F}_{\mathrm{r}}(\boldsymbol{a}) = \boldsymbol{a}\cdot \mathsf{M}\boldsymbol{a} + \boldsymbol{R}\cdot\boldsymbol{a}.$$

## *Variational HCMT scheme*

$$(\boldsymbol{E}, \boldsymbol{H}) = \sum_k a_k (\boldsymbol{E}_k, \boldsymbol{H}_k)$$

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{H}) \longrightarrow \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Restricted functional:

$\mathcal{F}_{\mathrm{r}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \mathsf{M}\boldsymbol{a} + \boldsymbol{R} \cdot \boldsymbol{a}.$

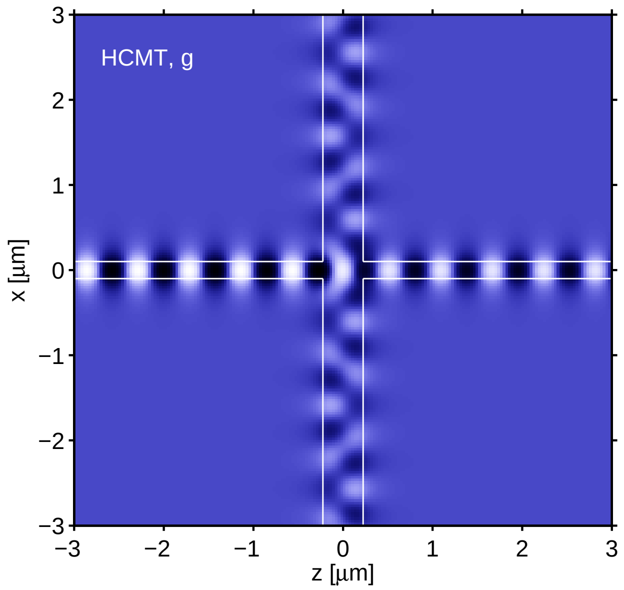
Require $\delta\mathcal{F}_{\mathrm{r}} = \delta\boldsymbol{a} \cdot \left(\left(\mathsf{M} + \mathsf{M}^{\mathsf{T}}\right)\boldsymbol{a} + \boldsymbol{R}\right) = 0$ for all $\delta\boldsymbol{a}$,

$\left(\mathsf{M} + \mathsf{M}^{\mathsf{T}}\right)\boldsymbol{a} + \boldsymbol{R} = 0,$

$\boldsymbol{a},$

$f_m,\ b_m,\ u_m,\ d_m,\ \ \boldsymbol{E},\ \boldsymbol{H}.$

## *Comments*




HCMT scheme based on the variational form of the guided wave scattering problem:

- Expansions at the TIBC ports reduce to single terms due to modal orthogonality.
- Bidirectional basis fields are required for all channels in the field templates.

Alternative functional:

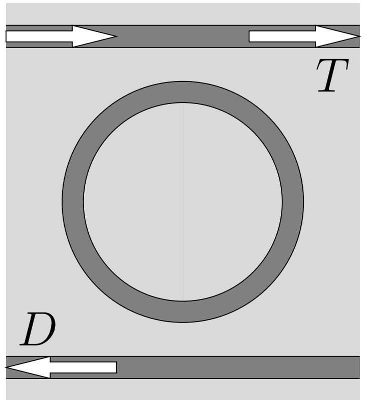
$$\mathcal{C}(\boldsymbol{E}, \boldsymbol{H}) = \iiint_\Omega \{\boldsymbol{E}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) - \boldsymbol{H}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}^* \cdot \boldsymbol{E} + \mathrm{i}\omega\mu_0 \boldsymbol{H}^* \cdot \boldsymbol{H}\} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

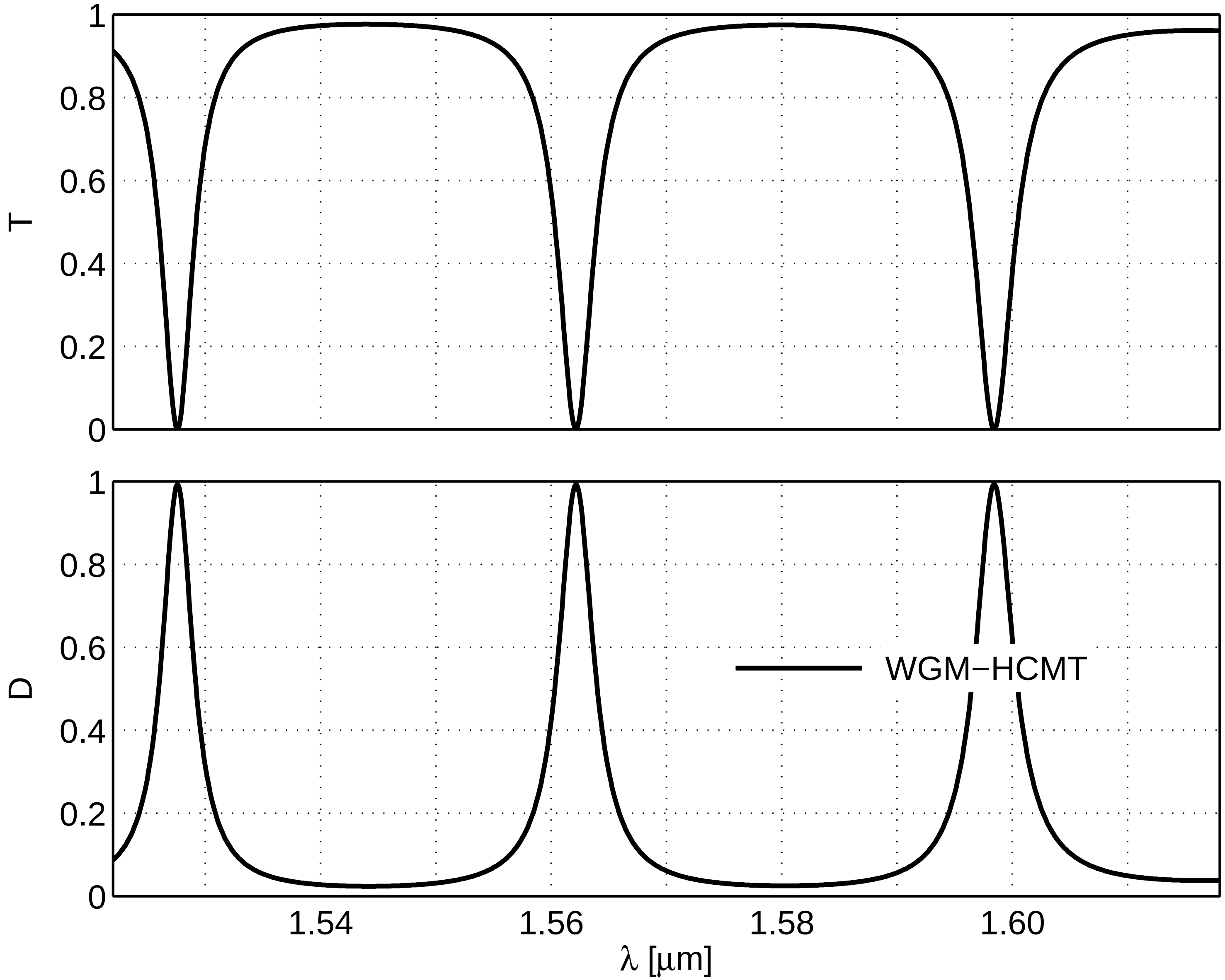
?

Extend $\mathcal{C}$ by boundary integrals such that

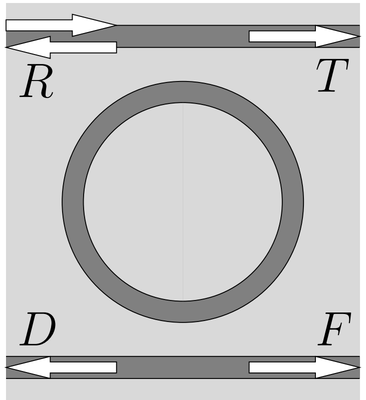
- the boundary terms in $\delta\mathcal{C}$ cancel ⟷ the Galerkin scheme could be viewed as a variational restriction of $\mathcal{C}$.
- TIBCs are satisfied as natural boundary conditions if $\mathcal{C}$ becomes stationary ⟷ variational scheme with complex conjugate fields.

# *Single ring filter, transmission, bidirectional template*

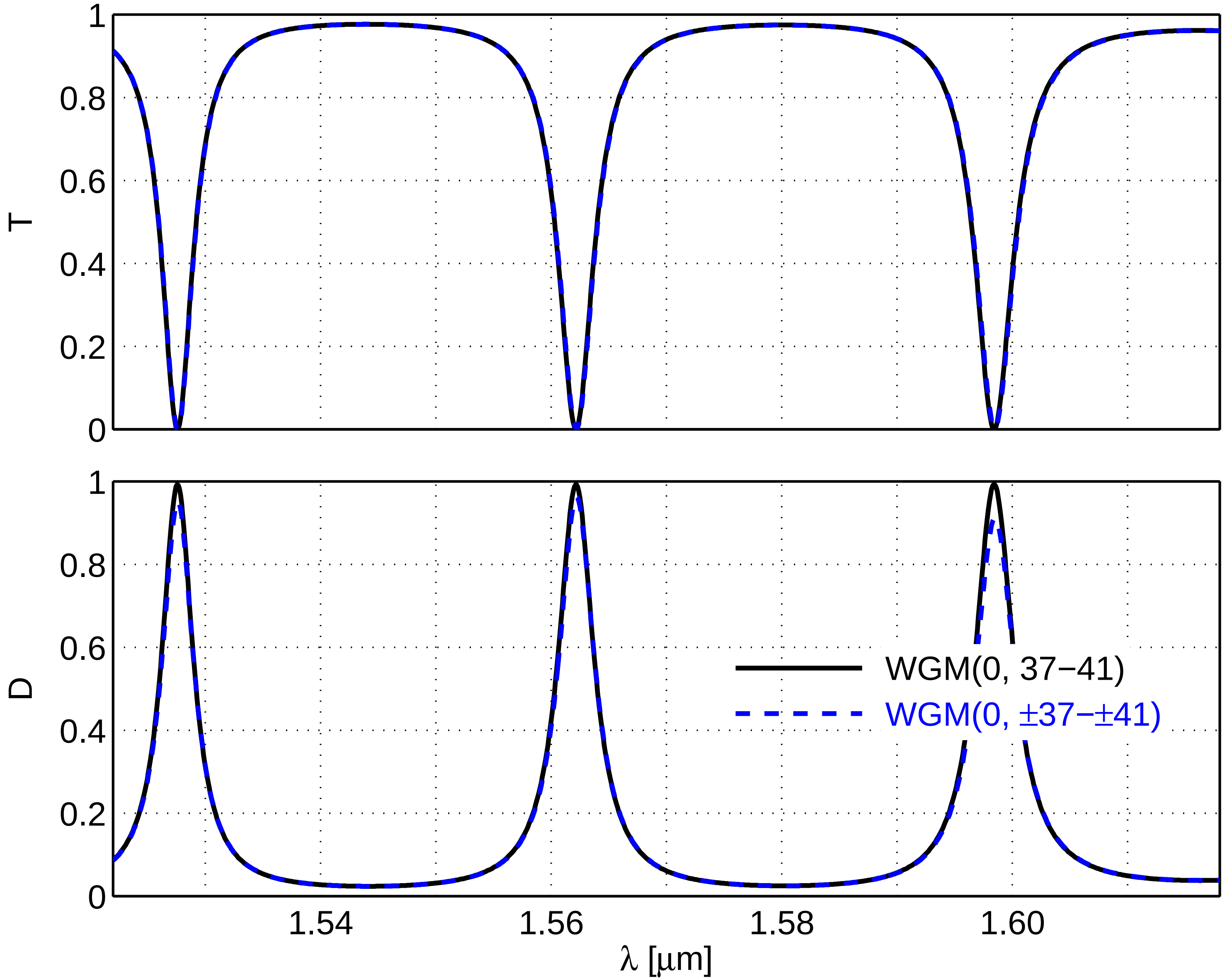


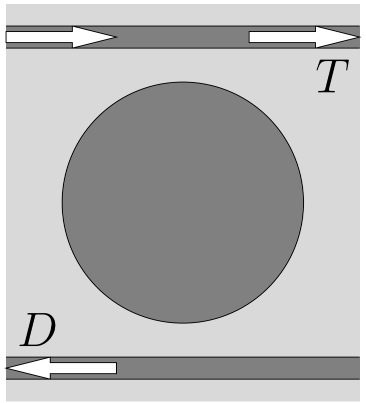
# *Single ring filter, transmission, bidirectional template*



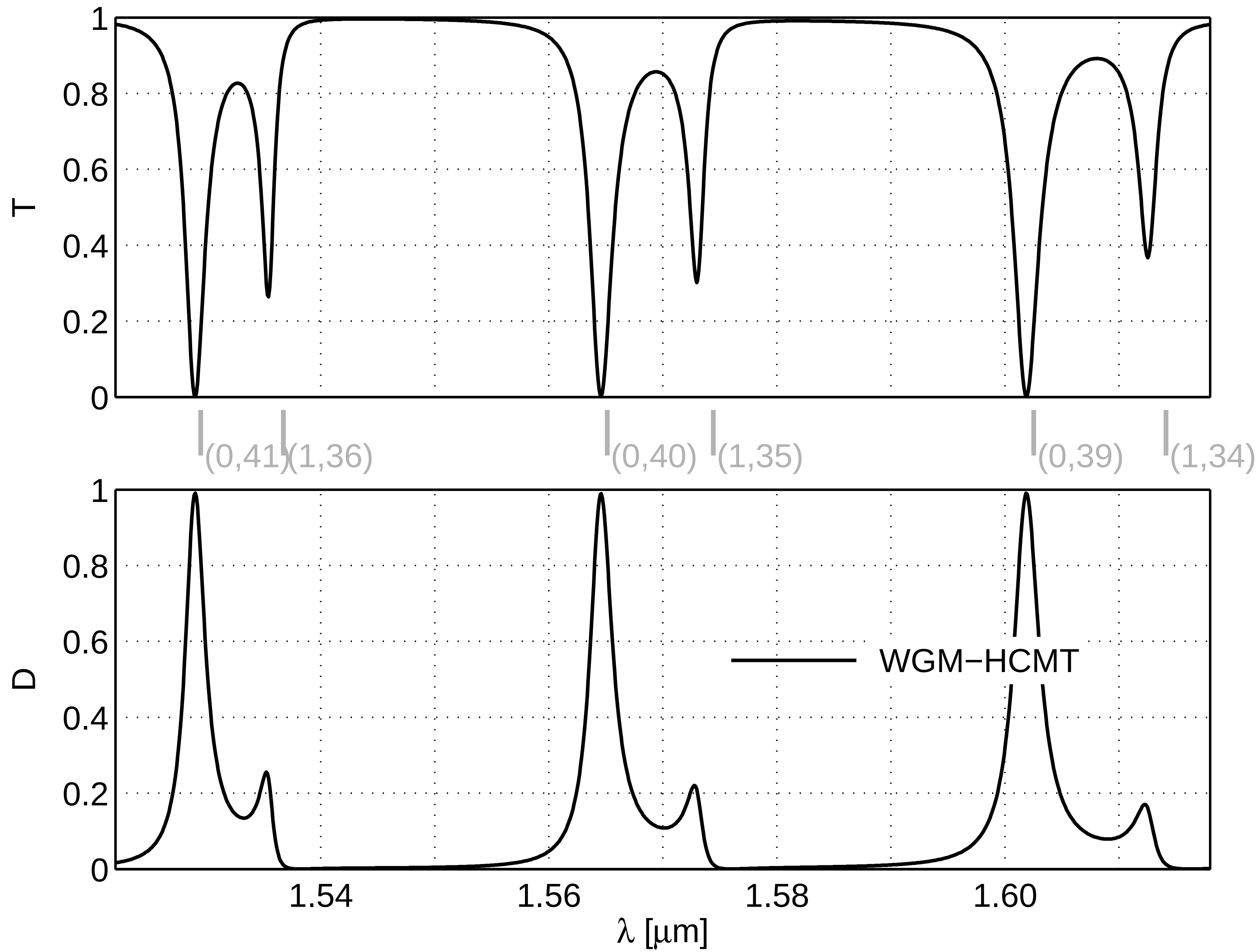

$$R < 10^{-4},$$
$$F < 10^{-4}.$$



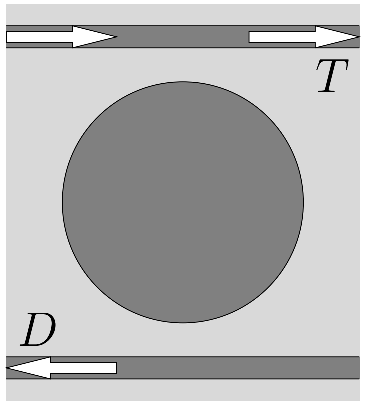
# *Micro-disk resonator, spectral response*



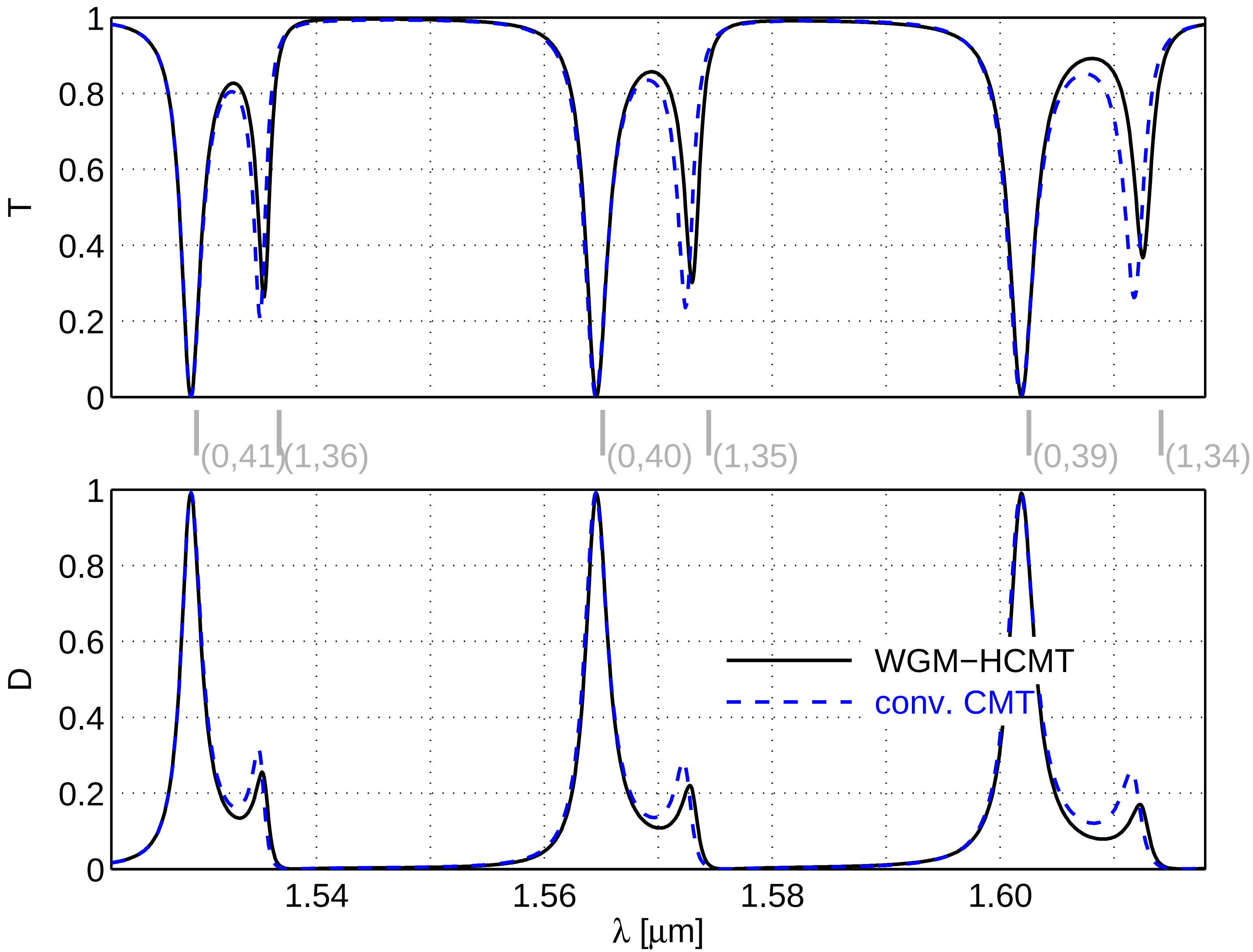

WGMs only



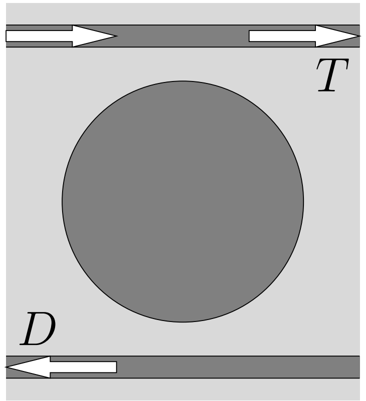
# *Micro-disk resonator, spectral response*



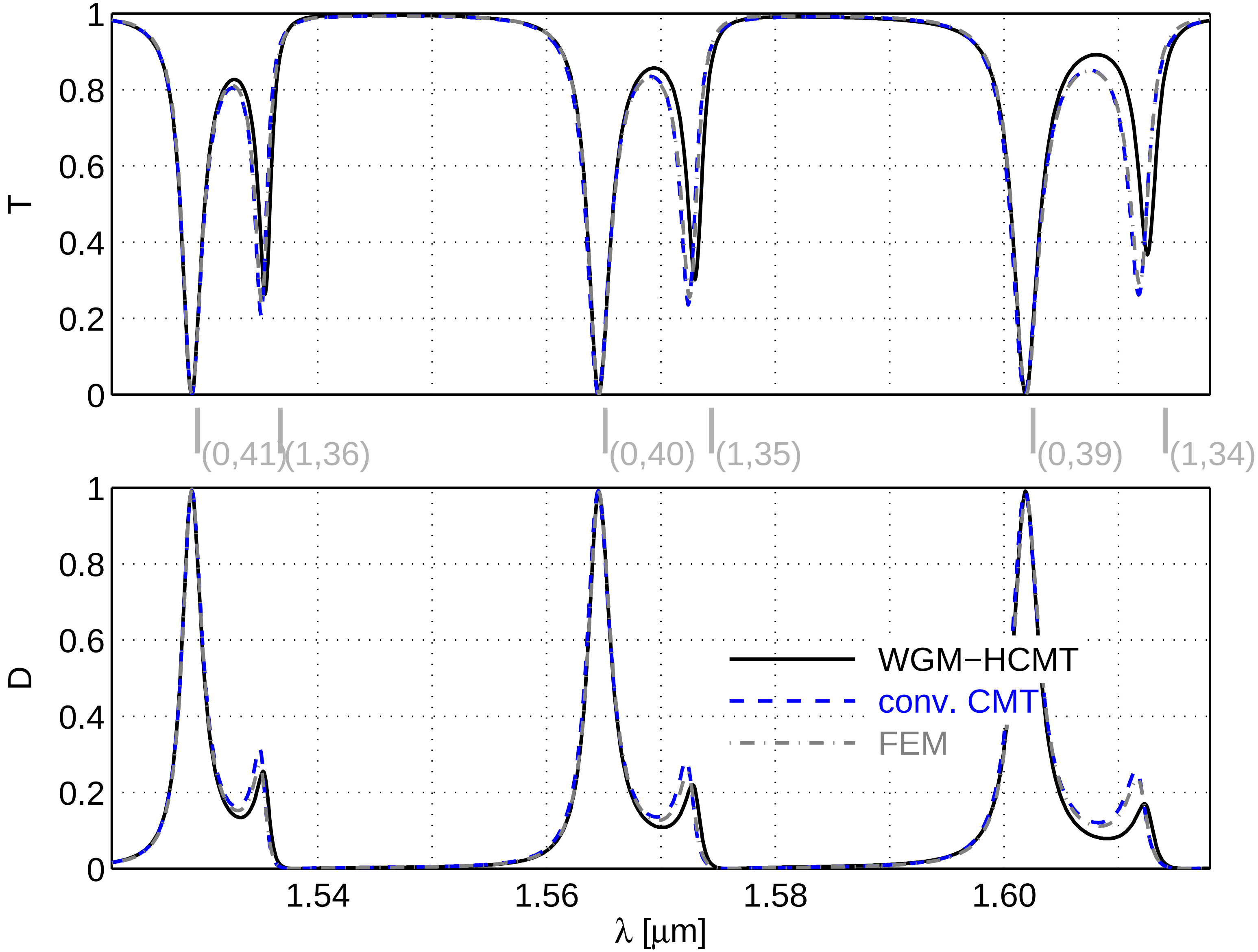

WGMs only



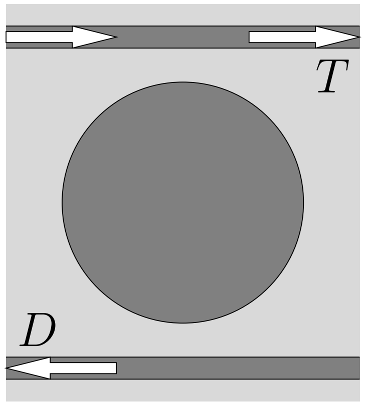
# *Micro-disk resonator, spectral response*



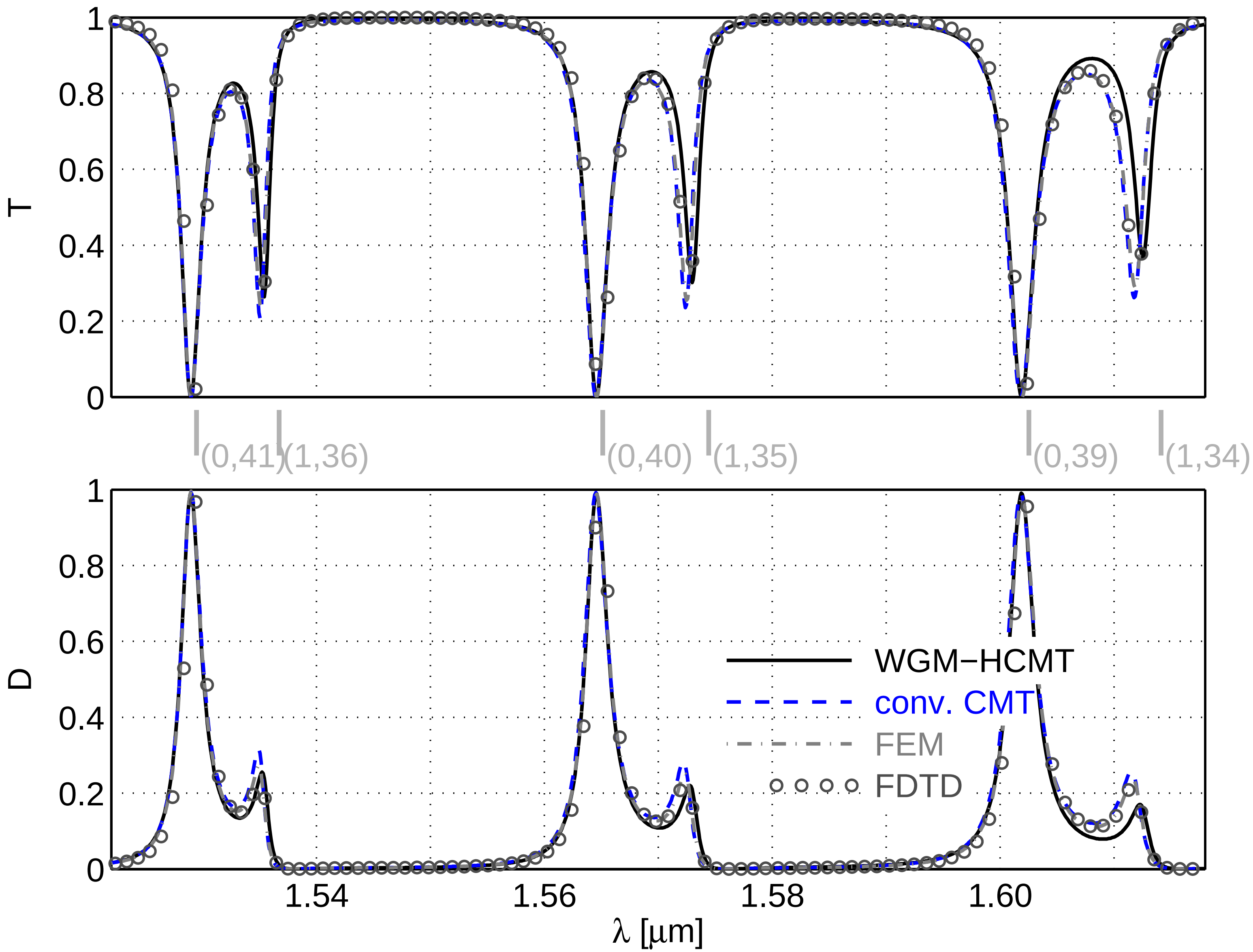

WGMs only



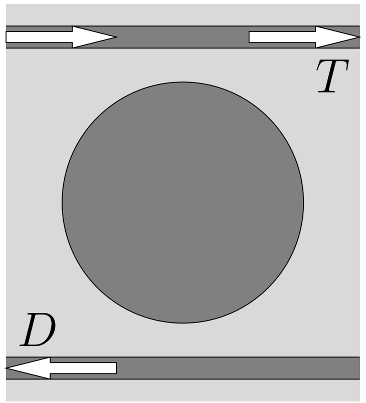
## *Micro-disk resonator, spectral response*




WGMs only

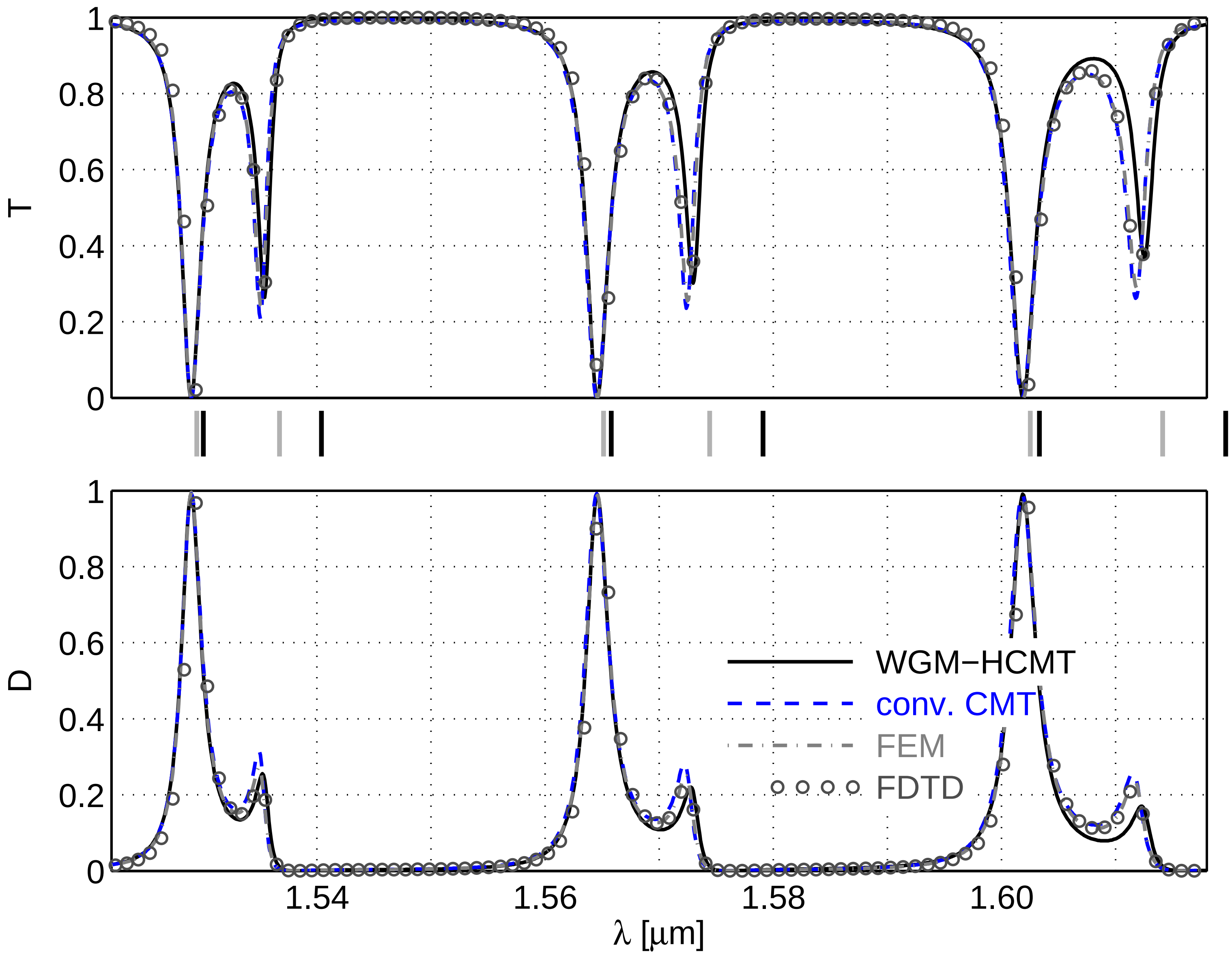


# *Micro-disk resonator, spectral response*

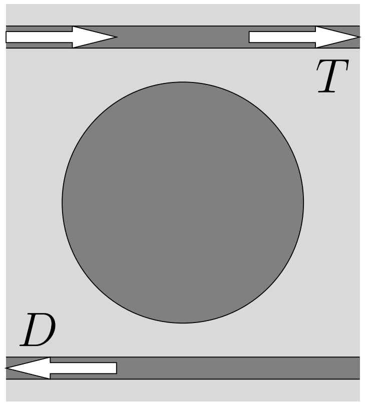


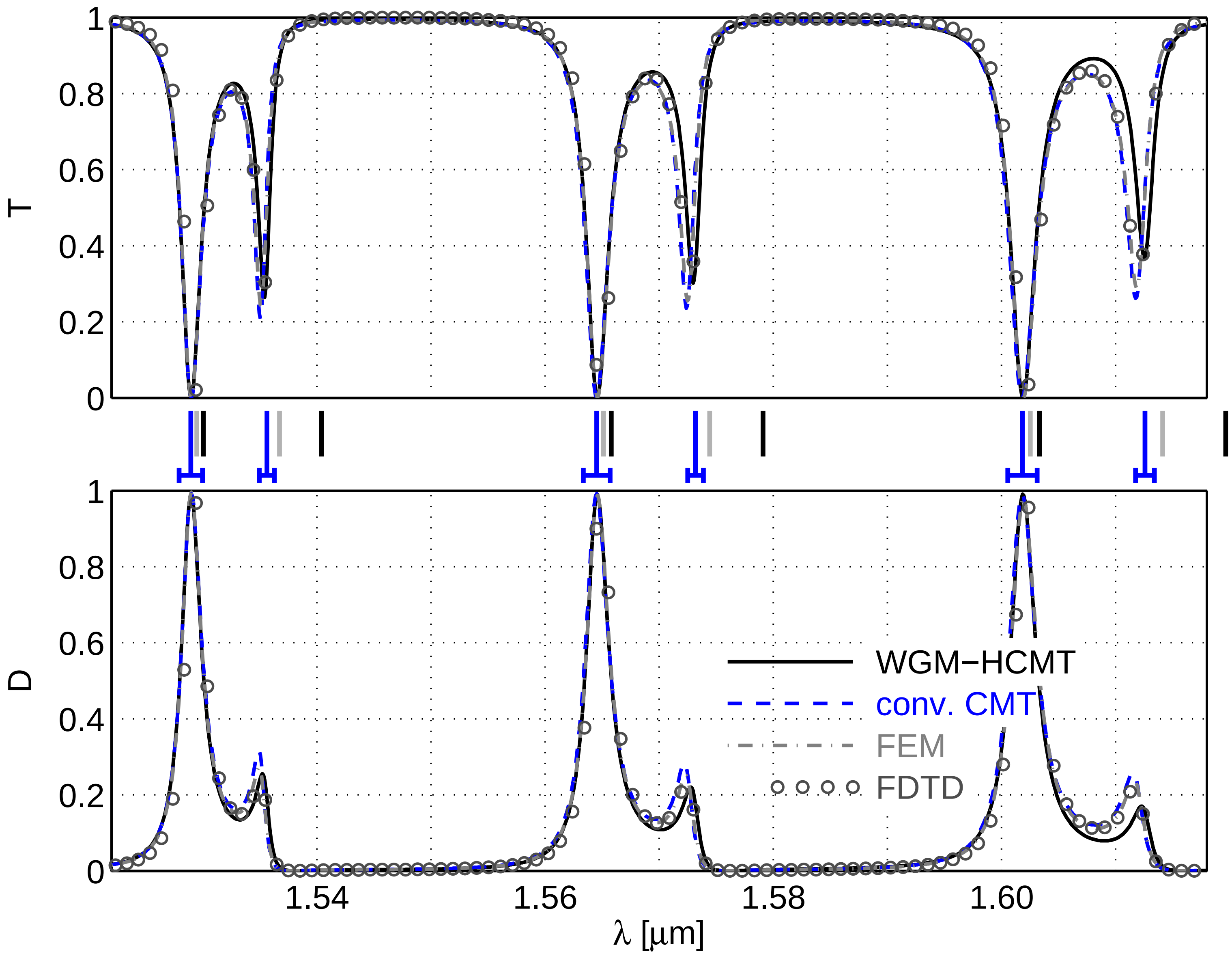

WGMs only

WGMs
& bus cores



# *Micro-disk resonator, spectral response*



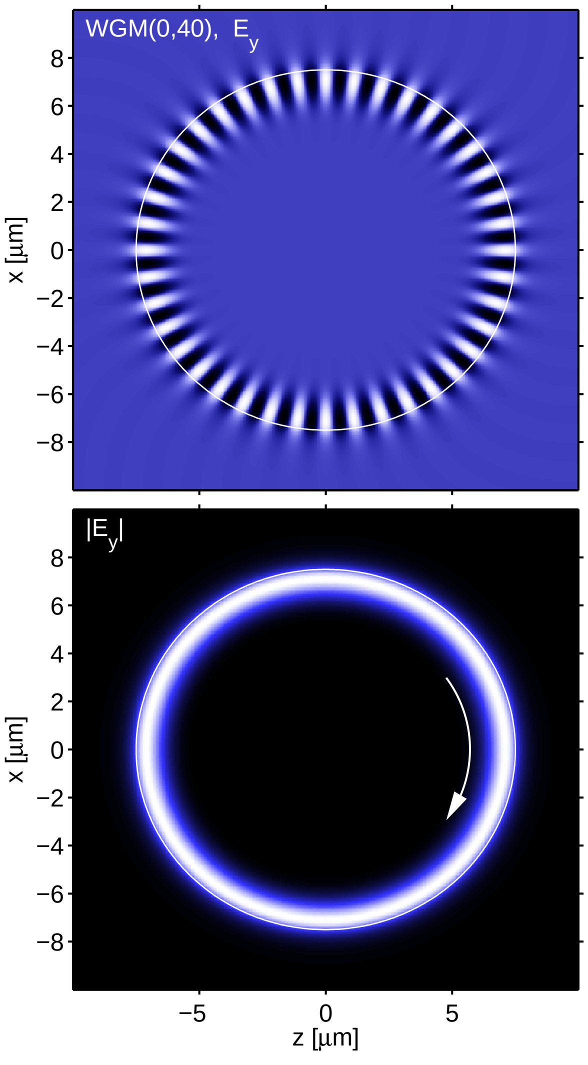




WGMs only

WGMs
& bus cores

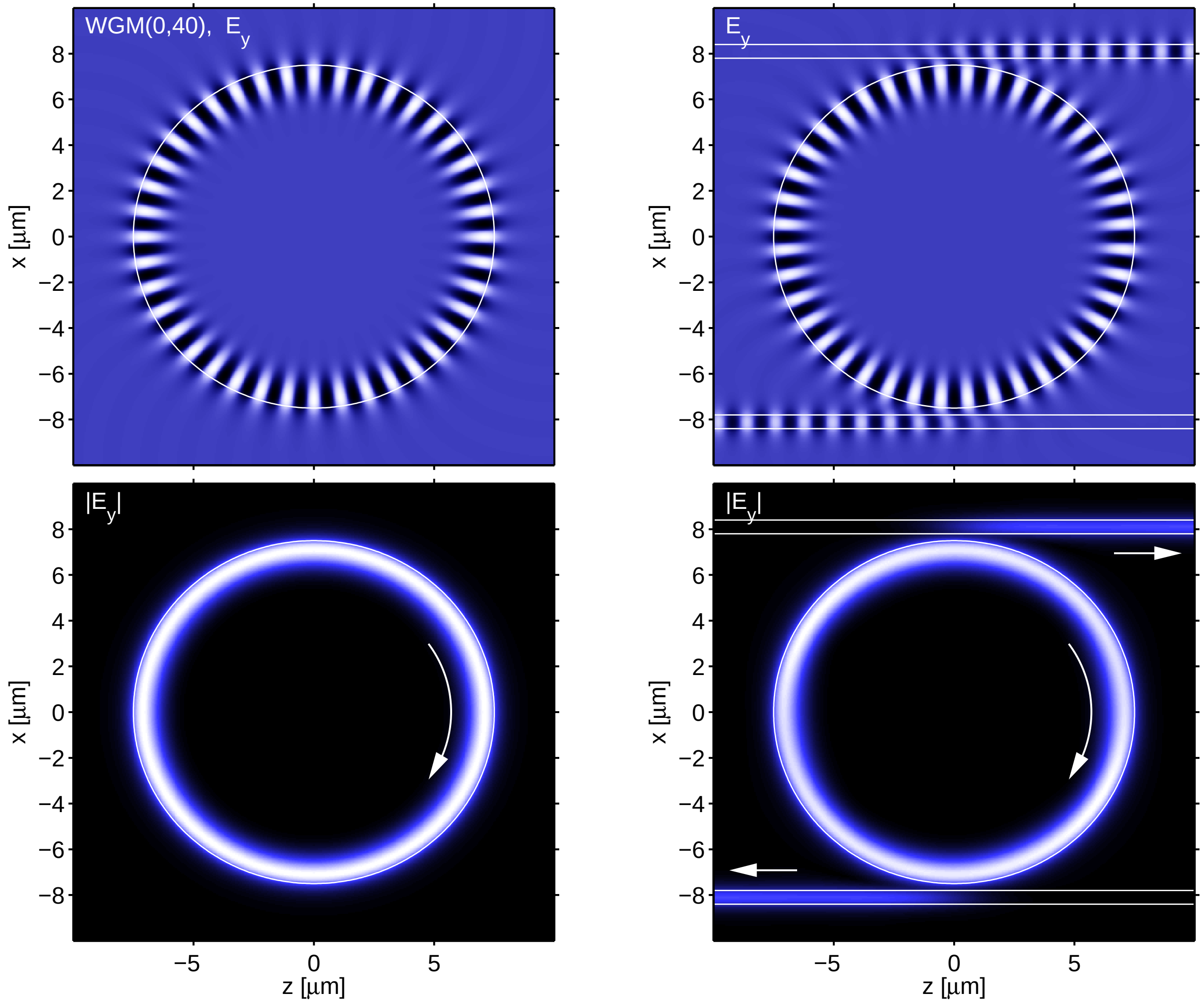
WGMs
& bus fields

# *Micro-disk, resonant fields (0)*




$\lambda_{\rm r} = 1.56514\,\mu$m,
$Q = 8.2 \cdot 10^5$,
$\Delta\lambda = 1.9 \cdot 10^{-6}\,\mu$m.

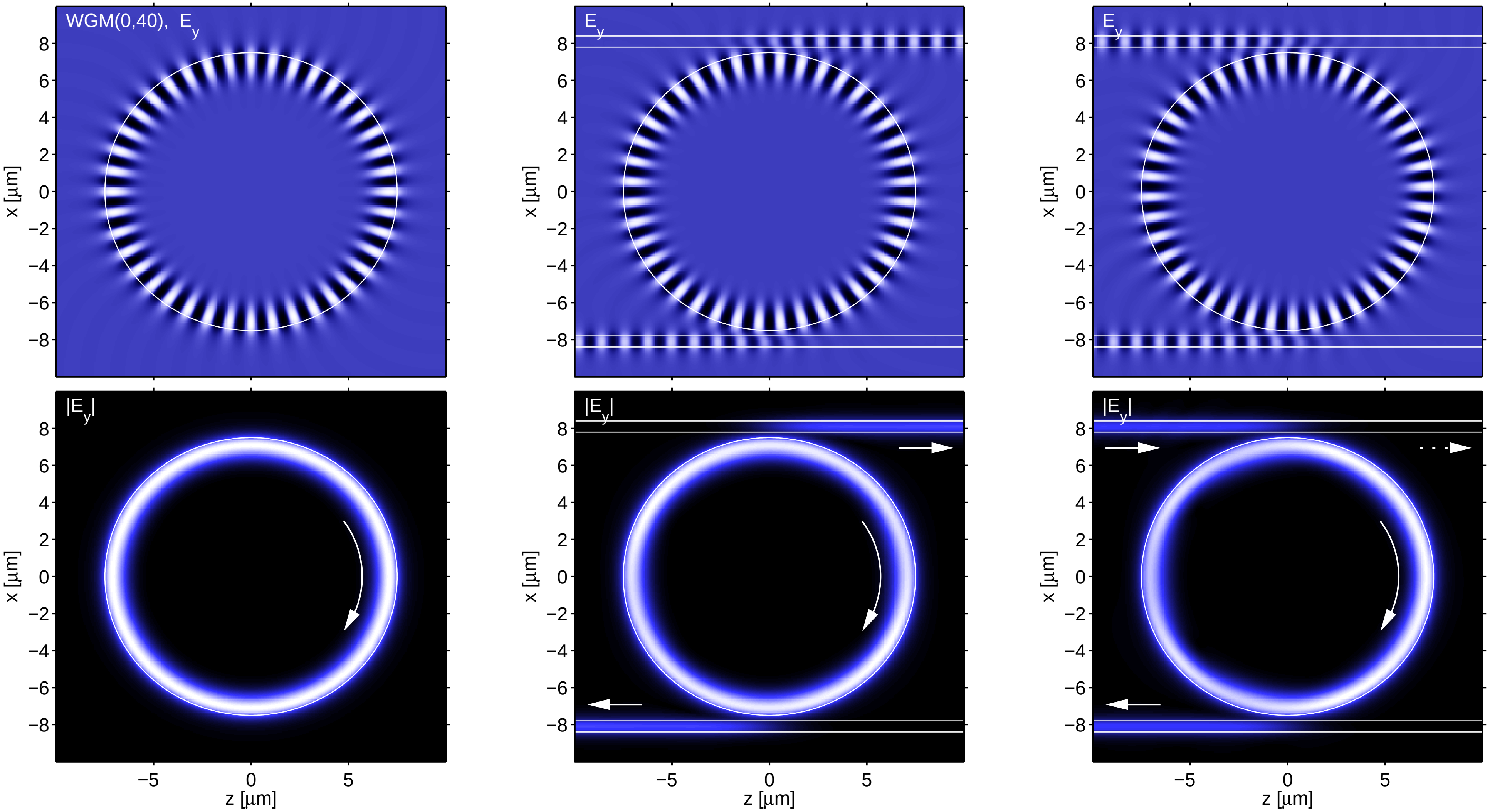
# *Micro-disk, resonant fields (0)*




$\lambda_r = 1.56514\,\mu$m,
$Q = 8.2 \cdot 10^5$,
$\Delta\lambda = 1.9 \cdot 10^{-6}\,\mu$m.

$\lambda_r = 1.56454\,\mu$m,
$Q = 6.7 \cdot 10^2$,
$\Delta\lambda = 2.3 \cdot 10^{-3}\,\mu$m.

# *Micro-disk, resonant fields (0)*

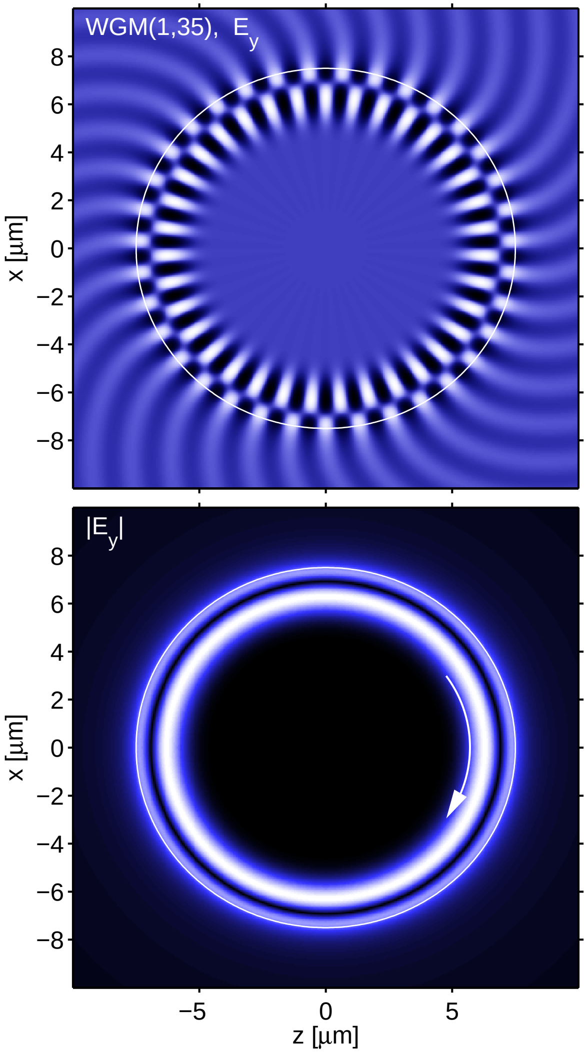



$\lambda_\mathrm{r} = 1.56514\,\mu$m,
$Q = 8.2 \cdot 10^5$,
$\Delta\lambda = 1.9 \cdot 10^{-6}\,\mu$m.

$\lambda_\mathrm{r} = 1.56454\,\mu$m,
$Q = 6.7 \cdot 10^2$,
$\Delta\lambda = 2.3 \cdot 10^{-3}\,\mu$m.

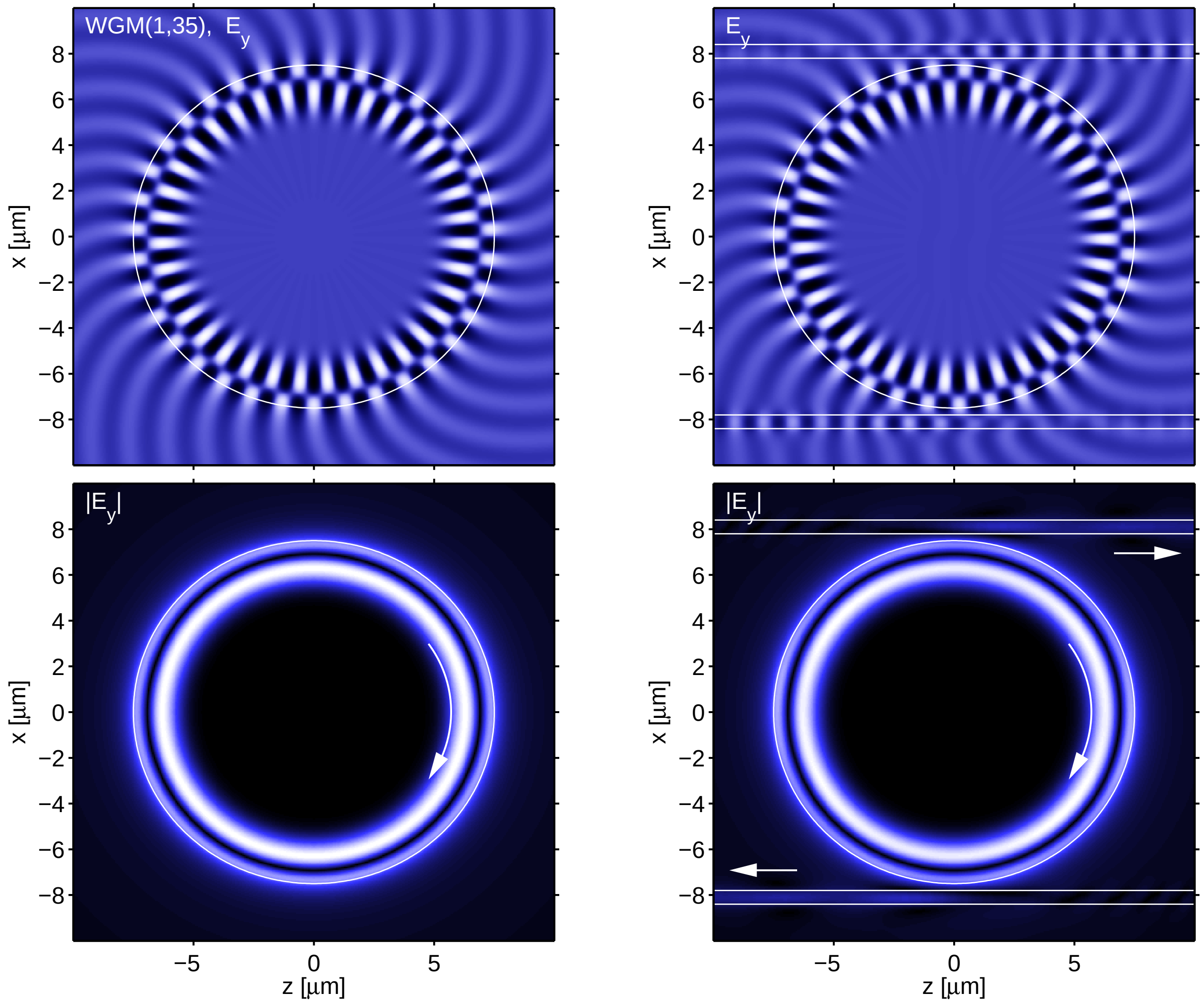
$\lambda_\mathrm{r} = 1.56456\,\mu$m.

# *Micro-disk, resonant fields (1)*




$\lambda_{\mathrm{r}} = 1.57444\,\mu\mathrm{m}$,
$Q = 1.6 \cdot 10^3$,
$\Delta\lambda = 9.2 \cdot 10^{-4}\,\mu\mathrm{m}$.

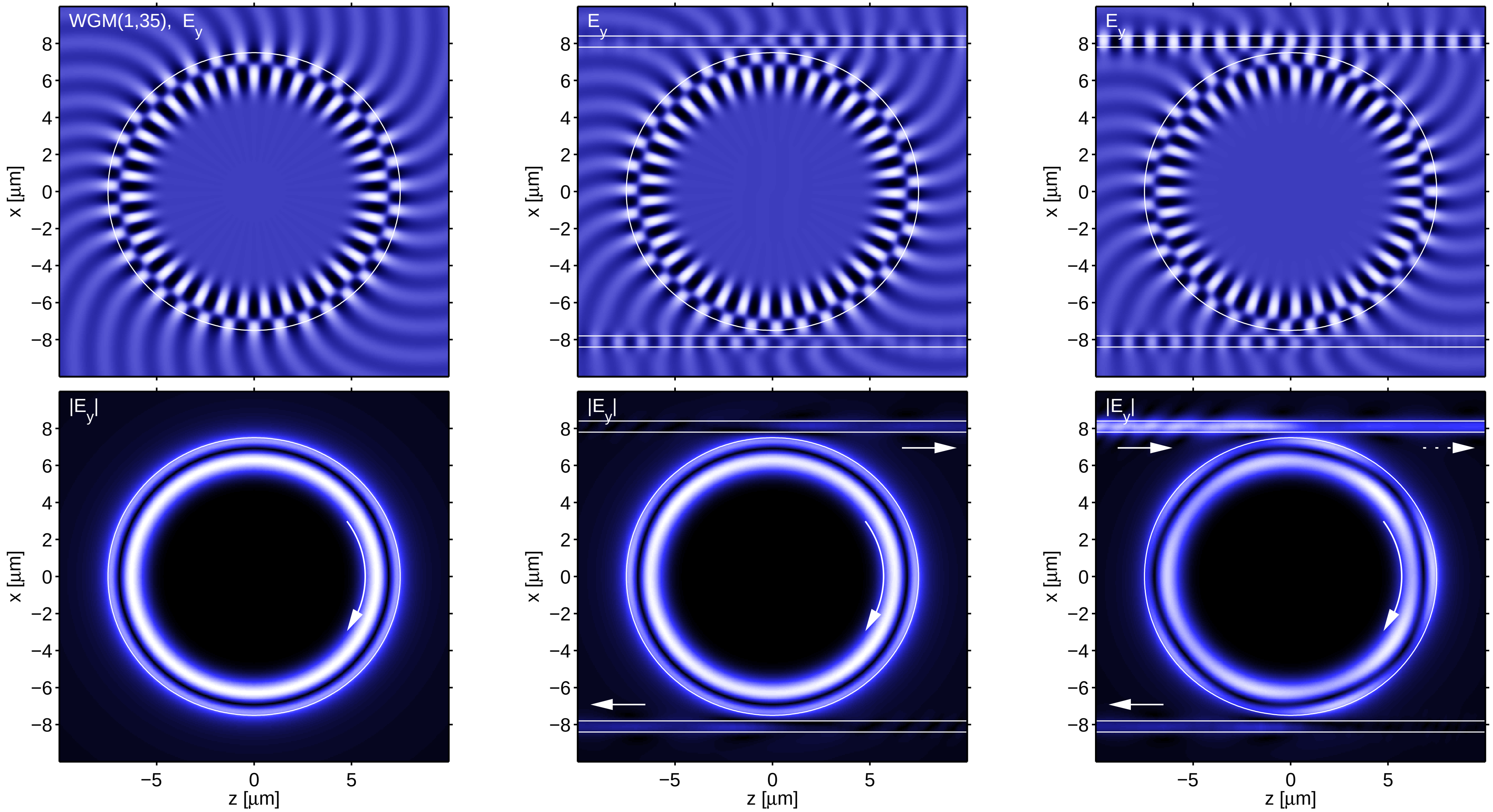
# *Micro-disk, resonant fields (1)*




$\lambda_r = 1.57444\,\mu$m,
$Q = 1.6 \cdot 10^3$,
$\Delta\lambda = 9.2 \cdot 10^{-4}\,\mu$m.

$\lambda_r = 1.57320\,\mu$m,
$Q = 1.1 \cdot 10^3$,
$\Delta\lambda = 1.4 \cdot 10^{-3}\,\mu$m.

# *Micro-disk, resonant fields (1)*

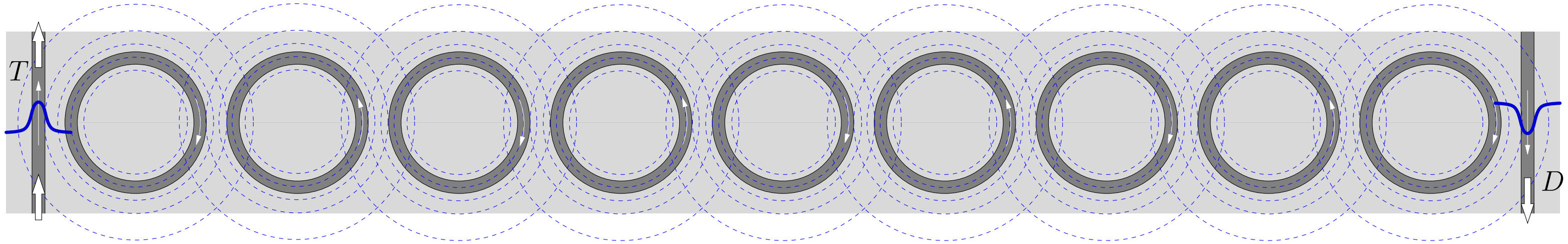


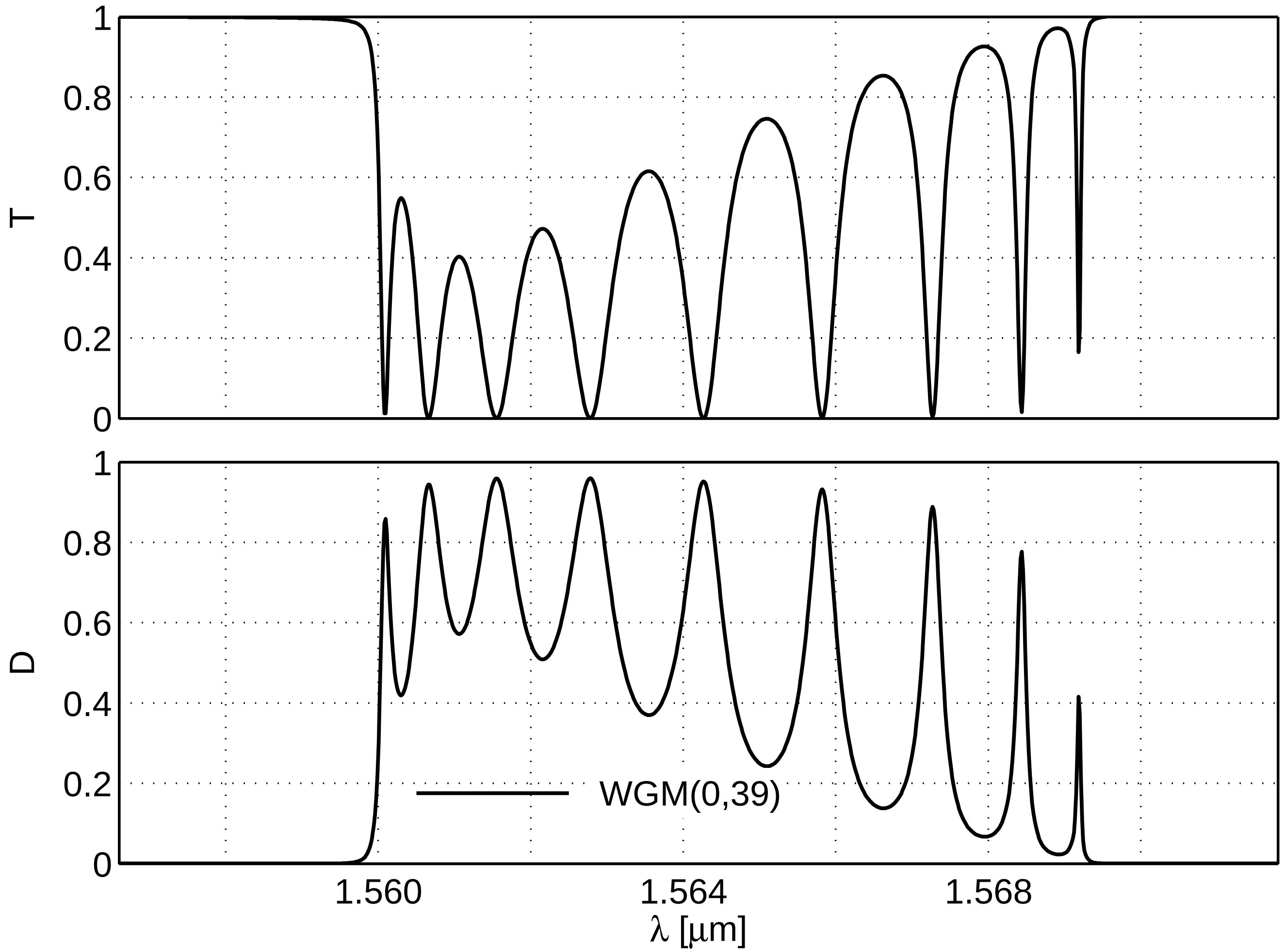

$\lambda_r = 1.57444\,\mu$m,
$Q = 1.6 \cdot 10^3$,
$\Delta\lambda = 9.2 \cdot 10^{-4}\,\mu$m.

$\lambda_r = 1.57320\,\mu$m,
$Q = 1.1 \cdot 10^3$,
$\Delta\lambda = 1.4 \cdot 10^{-3}\,\mu$m.

$\lambda_r = 1.57306\,\mu$m.

# *CROW, spectral response II*



# *CROW, spectral response II*