A variational formulation of guided wave scattering problems

— hybrid analytic / numerical coupled mode modeling

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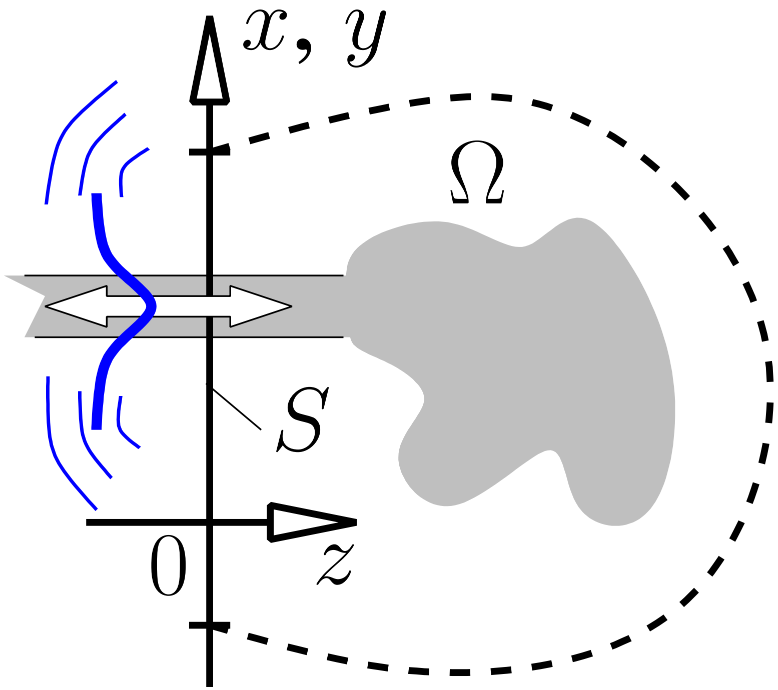
Abstract scattering problem



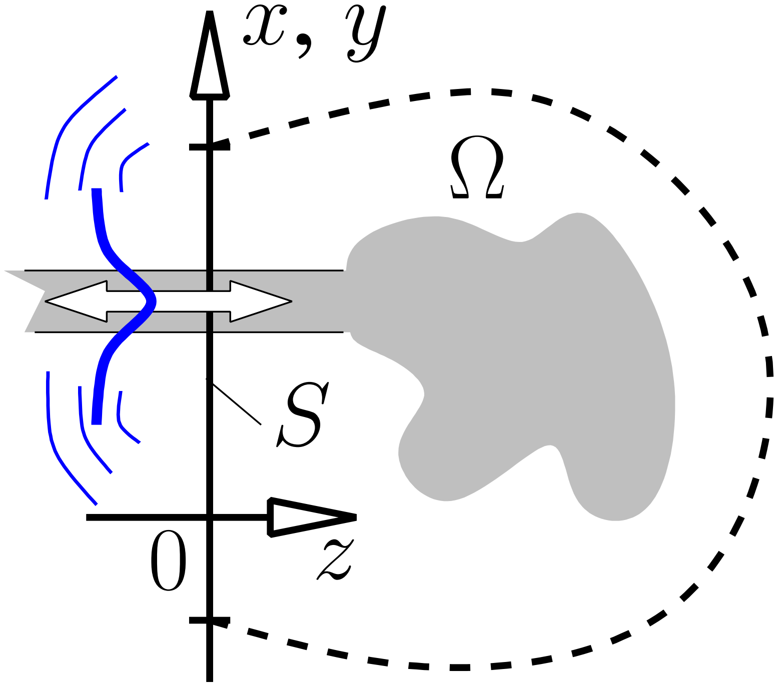
Ω: domain of interest,

$$\left.\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E} &= 0 \\ -\boldsymbol{\nabla}\times\boldsymbol{E} - \mathrm{i}\omega\mu_0\boldsymbol{H} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

for given frequency ω, permittivity $\epsilon = n^2$,

S: an exemplary port plane,
waveguides enter Ω through S.

Abstract scattering problem



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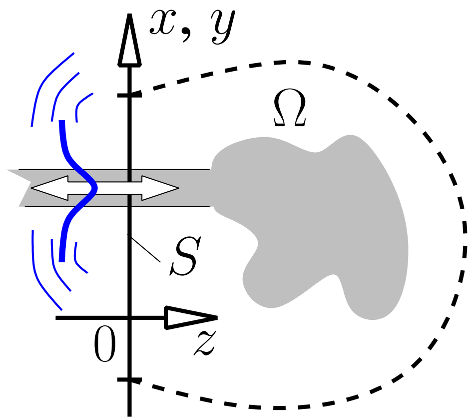
waveguides enter Ω through S.

Variational form including suitable boundary conditions ?

Outline

- Guided wave scattering problems
  - Transparent influx boundary conditions
  - Variational formulation
- Hybrid analytic / numerical coupled mode theory
  - Template, discretization
  - Variational scheme
  - Results, waveguide crossing
- Comments

Boundary conditions



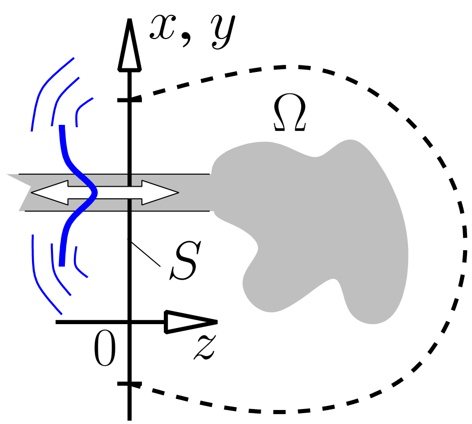
Ingredients:

- Complete set of normal modes on S,
  $(\tilde{\boldsymbol{E}}_m, \pm\tilde{\boldsymbol{H}}_m)(x, y)$ $\longleftrightarrow$ propagation along $\pm z$.
- Product on S: $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iint_S (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{e}_z \, \mathrm{d}x \, \mathrm{d}y$.
- Modal orthogonality properties $\langle \tilde{\boldsymbol{E}}_l, \tilde{\boldsymbol{H}}_k \rangle = \delta_{lk} N_k$, $N_k = \langle \tilde{\boldsymbol{E}}_k, \tilde{\boldsymbol{H}}_k \rangle$.

"Any" electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ on S can be expanded as

$$\boldsymbol{E} = \sum_m e_m \tilde{\boldsymbol{E}}_m, \quad e_m = \frac{1}{N_m} \langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m \rangle, \quad \boldsymbol{H} = \sum_m h_m \tilde{\boldsymbol{H}}_m, \quad h_m = \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle,$$

Boundary conditions



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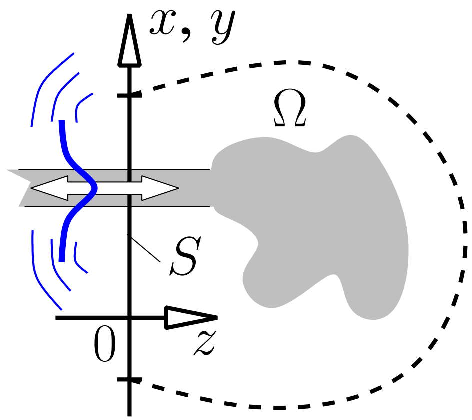
or $$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_m f_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix} + \sum_m b_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ -\tilde{\boldsymbol{H}}_m \end{pmatrix}, \qquad \begin{aligned} f_m &= (e_m + h_m)/2, \\ b_m &= (e_m - h_m)/2 \end{aligned}$$

(transverse components only).

Transparent influx boundary conditions (TIBCs)



. . . on S for inhomogeneous exterior, incoming waveguides:

$$\boldsymbol{E} = \sum_m 2F_m \tilde{\boldsymbol{E}}_m - \sum_m \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle \tilde{\boldsymbol{E}}_m \,,$$

$$\boldsymbol{H} = \sum_m 2F_m \tilde{\boldsymbol{H}}_m - \sum_m \frac{1}{N_m} \langle \boldsymbol{E}, \tilde{\boldsymbol{H}}_m \rangle \tilde{\boldsymbol{H}}_m \,;$$

F_m: influx, given coefficients of incoming waves; $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}_{\mathrm{inc}} = \sum_m F_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix}.$

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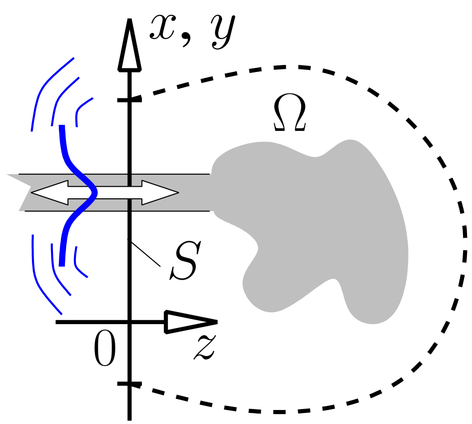

... on S for inhomogeneous exterior, incoming waveguides:

$$\boldsymbol{E} = \sum_m 2F_m \tilde{\boldsymbol{E}}_m - \sum_m \frac{1}{N_m} \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H} \rangle \tilde{\boldsymbol{E}}_m \,,$$

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For a general field of the form $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_m f_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ \tilde{\boldsymbol{H}}_m \end{pmatrix} + \sum_m b_m \begin{pmatrix} \tilde{\boldsymbol{E}}_m \\ -\tilde{\boldsymbol{H}}_m \end{pmatrix}$

the TIBCs require $f_m = F_m$, while b_m can be arbitrary.

Frequency domain Maxwell equations, variational form

Consider the functional

$$\mathcal{L}(\boldsymbol{E},\boldsymbol{H}) = \iiint_\Omega \left\{\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) + \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^2 + \mathrm{i}\omega\mu_0\boldsymbol{H}^2\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(C. Vassallo. *Optical Waveguide Concepts*. Elsevier, Amsterdam, 1991).

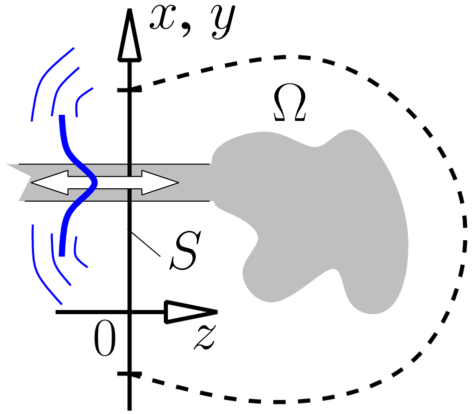
First variation:

$$\begin{aligned}\delta\mathcal{L}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) &= \iiint_\Omega \big\{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H} - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}) \\ &\qquad + 2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E} + \mathrm{i}\omega\mu_0\boldsymbol{H})\big\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &- \iint_{\partial\Omega}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H} + (\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.\end{aligned}$$

Stationarity $\delta\mathcal{L}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) = 0$ for arbitrary $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ implies

- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the Maxwell equations in Ω
- and that transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish on $\partial\Omega$.

Variational form of the scattering problem

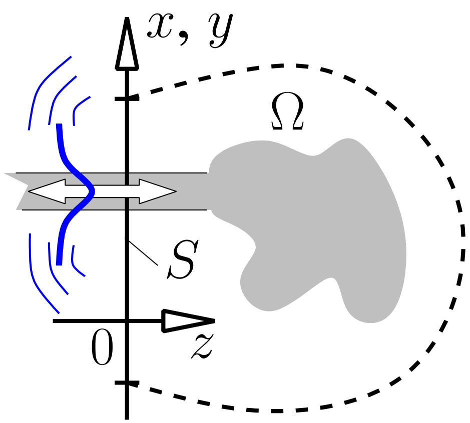


... based on the functional:

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) = \iiint_{\Omega} \{\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) + \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^2 + \mathrm{i}\omega\mu_0\boldsymbol{H}^2\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$- \sum_m 2F_m \left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle - \langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\right\}$$
$$+ \sum_m \frac{1}{2N_m}\left\{\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle^2 - \langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle^2\right\}$$

Variational form of the scattering problem, first variation

$$
\begin{aligned}
\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H}) \quad = \quad & \iiint_\Omega \Big\{2\delta\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}-\mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}) \\
& \qquad +2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}+\mathrm{i}\omega\mu_0\boldsymbol{H})\Big\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
+ \quad & \Big\langle \boldsymbol{E}-\sum_m 2F_m\tilde{\boldsymbol{E}}_m+\sum_m\frac{1}{N_m}\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle\tilde{\boldsymbol{E}}_m,\,\delta\boldsymbol{H}\Big\rangle \\
- \quad & \Big\langle \delta\boldsymbol{E},\,\boldsymbol{H}-\sum_m 2F_m\tilde{\boldsymbol{H}}_m+\sum_m\frac{1}{N_m}\langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\tilde{\boldsymbol{H}}_m\Big\rangle \\
- \quad & \iint_{\partial\Omega\setminus S}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H}+(\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.
\end{aligned}
$$



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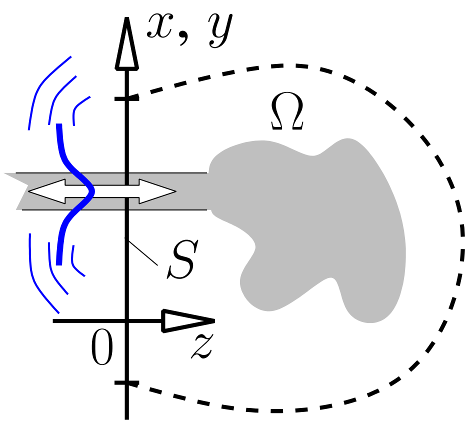
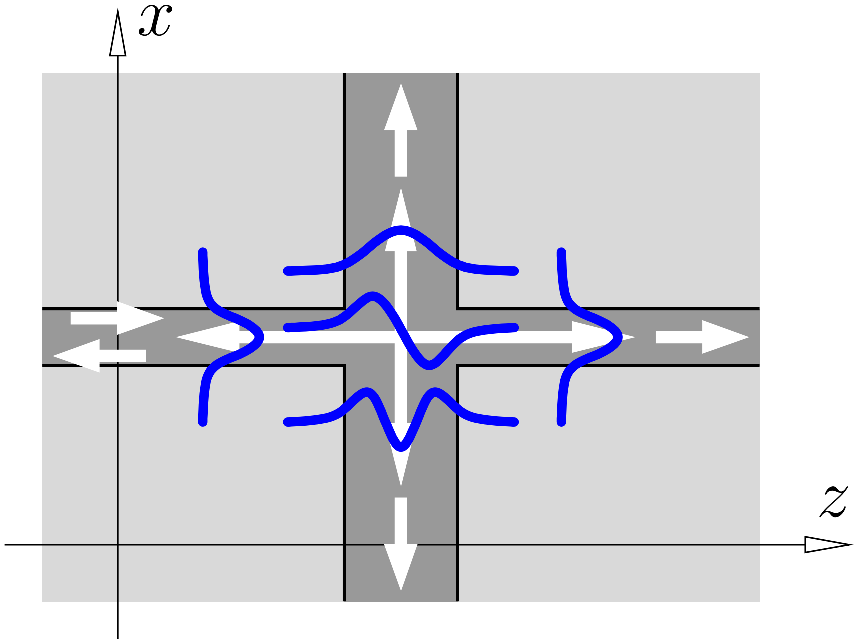
& \qquad + 2\delta\boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}+\mathrm{i}\omega\mu_0\boldsymbol{H})\big\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
& + \Big\langle\boldsymbol{E}-\sum_m 2F_m\tilde{\boldsymbol{E}}_m+\sum_m\frac{1}{N_m}\langle\tilde{\boldsymbol{E}}_m,\boldsymbol{H}\rangle\tilde{\boldsymbol{E}}_m,\,\delta\boldsymbol{H}\Big\rangle \\
& - \Big\langle\delta\boldsymbol{E},\,\boldsymbol{H}-\sum_m 2F_m\tilde{\boldsymbol{H}}_m+\sum_m\frac{1}{N_m}\langle\boldsymbol{E},\tilde{\boldsymbol{H}}_m\rangle\tilde{\boldsymbol{H}}_m\Big\rangle \\
& - \iint_{\partial\Omega\backslash S}\{(\boldsymbol{n}\times\boldsymbol{E})\cdot\delta\boldsymbol{H}+(\boldsymbol{n}\times\boldsymbol{H})\cdot\delta\boldsymbol{E}\}\,\mathrm{d}A\,.
\end{aligned}
$$



Stationarity $\delta\mathcal{F}(\boldsymbol{E},\boldsymbol{H};\delta\boldsymbol{E},\delta\boldsymbol{H})=0$ for arbitrary $\delta\boldsymbol{E}$, $\delta\boldsymbol{H}$ implies

- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the Maxwell equations in Ω,
- that $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy TIBCs on S,
- and that transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ vanish on $\partial\Omega\backslash S$.

Hybrid coupled mode theory: waveguide crossing



Basis elements:

- guided modes of the horizontal WG

$$\psi_m^{\mathrm{f,b}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_m^{\mathrm{f,b}}(x)\, \mathrm{e}^{\mp \mathrm{i} \beta_m^{\mathrm{f,b}} z},$$

- guided modes of the vertical WG

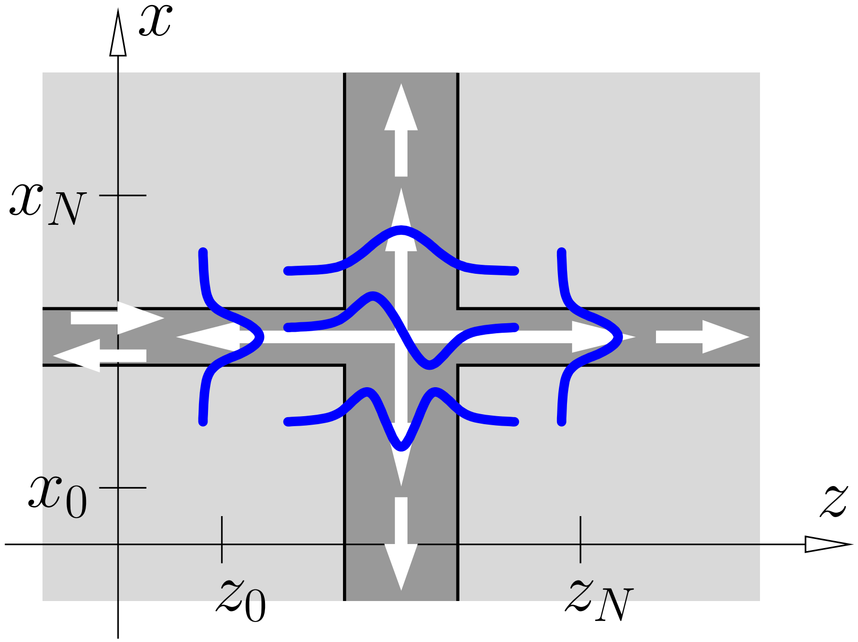
$$\psi_m^{\mathrm{u,d}}(x,z) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}_m^{\mathrm{u,d}}(z)\, \mathrm{e}^{\mp \mathrm{i} \beta_m^{\mathrm{u,d}} x}$$

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \sum_m f_m(z) \boldsymbol{\psi}_m^{\mathrm{f}}(x,z) + \sum_m b_m(z) \boldsymbol{\psi}_m^{\mathrm{b}}(x,z)$$

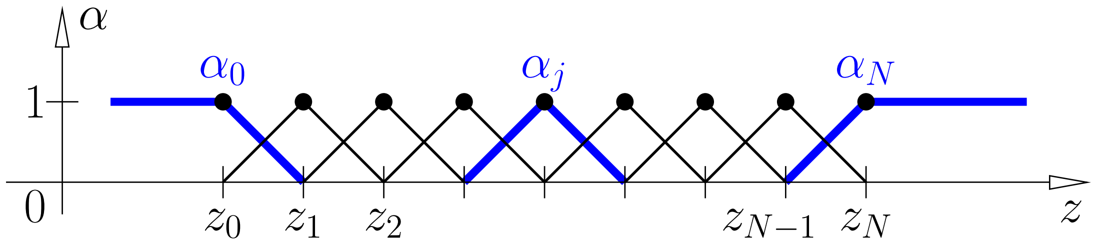
$$+ \sum_m u_m(x) \boldsymbol{\psi}_m^{\mathrm{u}}(x,z) + \sum_m d_m(x) \boldsymbol{\psi}_m^{\mathrm{d}}(x,z)$$

$f_m,\, b_m,\, u_m,\, d_m$: ?

Amplitude functions, discretization



1-D linear finite elements



$$f_m(z) = \sum_{j=0}^{N} f_{m,j}\, \alpha_j(z)\,,$$

$b_m(z)$, $u_m(x)$, $d_m(x)$ analogous.

$$\hookrightarrow \qquad \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \sum_k a_k \Big(\alpha_\cdot(\cdot)\boldsymbol{\psi}_\cdot^\cdot(x,z) \Big) =: \sum_k a_k \begin{pmatrix} \boldsymbol{E}_k \\ \boldsymbol{H}_k \end{pmatrix}(x,z),$$

$k \in \{$waveguides, modes, elements$\}$, $\quad a_k \in \{f_{m,j}, b_{m,j}, u_{m,j}, d_{m,j}\}$.

Variational HCMT scheme

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{H}) \xrightarrow{\;(\boldsymbol{E}, \boldsymbol{H}) = \sum_k a_k (\boldsymbol{E}_k, \boldsymbol{H}_k)\;} \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Variational HCMT scheme

$$(\boldsymbol{E}, \boldsymbol{H}) = \sum_k a_k (\boldsymbol{E}_k, \boldsymbol{H}_k)$$

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{H}) \longrightarrow \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Restricted functional:

$$\mathcal{F}_{\mathrm{r}}(\boldsymbol{a}) = \sum_{l,k} a_l F_{lk} a_k + \sum_l R_l a_l + \sum_{l,k} a_l B_{lk} a_k \,,$$

$$F_{lk} = \iiint_\Omega \Big\{ \boldsymbol{E}_l \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}_k) + \boldsymbol{H}_l \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}_k) - \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_l \cdot \boldsymbol{E}_k + \mathrm{i}\omega\mu_0 \boldsymbol{H}_l \cdot \boldsymbol{H}_k \Big\} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,,$$

$$R_l = -\sum_m 2F_m \left\{ \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}_l \rangle - \langle \boldsymbol{E}_l, \tilde{\boldsymbol{H}}_m \rangle \right\},$$

$$B_{lk} = \sum_m \frac{1}{2N_m} \left\{ \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}_l \rangle \langle \tilde{\boldsymbol{E}}_m, \boldsymbol{H}_k \rangle - \langle \boldsymbol{E}_l, \tilde{\boldsymbol{H}}_m \rangle \langle \boldsymbol{E}_k, \tilde{\boldsymbol{H}}_m \rangle \right\},$$

+ contributions R, B from other port planes.

Variational HCMT scheme

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) \xrightarrow{\displaystyle (\boldsymbol{E},\boldsymbol{H}) = \sum_k a_k(\boldsymbol{E}_k,\boldsymbol{H}_k)} \mathcal{F}_{\mathrm{r}}(\boldsymbol{a})$$

Restricted functional:

$\mathcal{F}_{\mathrm{r}}(\boldsymbol{a}) = \boldsymbol{a}\cdot \mathsf{M}\boldsymbol{a} + \boldsymbol{R}\cdot\boldsymbol{a}.$

Variational HCMT scheme

$$(\boldsymbol{E}, \boldsymbol{H}) = \sum_k a_k (\boldsymbol{E}_k, \boldsymbol{H}_k)$$

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{H}) \longrightarrow \mathcal{F}_\mathrm{r}(\boldsymbol{a})$$

Restricted functional:

$\mathcal{F}_\mathrm{r}(\boldsymbol{a}) = \boldsymbol{a} \cdot \mathsf{M}\boldsymbol{a} + \boldsymbol{R} \cdot \boldsymbol{a}.$

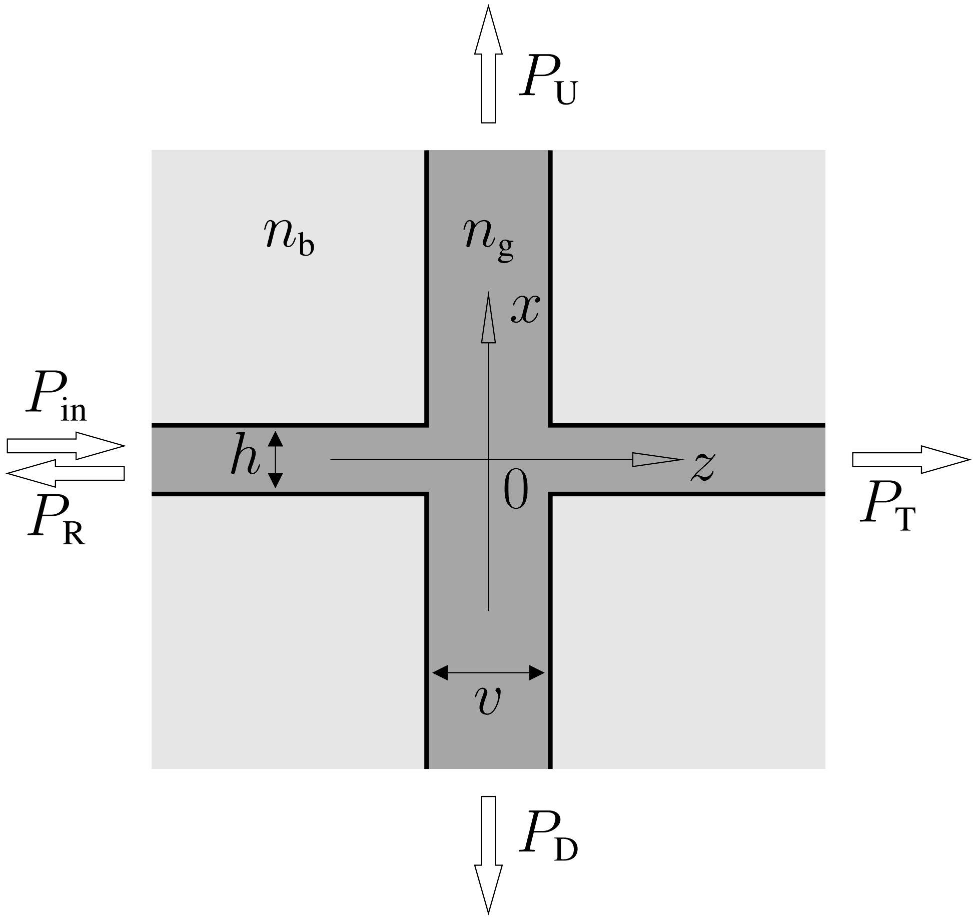
Require $\delta\mathcal{F}_\mathrm{r} = \delta\boldsymbol{a} \cdot \left(\left(\mathsf{M} + \mathsf{M}^\mathsf{T}\right) \boldsymbol{a} + \boldsymbol{R}\right) = 0$ for all $\delta\boldsymbol{a}$,

$\left(\mathsf{M} + \mathsf{M}^\mathsf{T}\right) \boldsymbol{a} + \boldsymbol{R} = 0,$

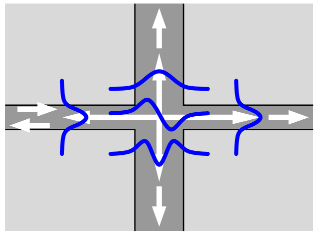
$\boldsymbol{a},$ $(\ldots)$

$f_m,\ b_m,\ u_m,\ d_m,\ \ \boldsymbol{E},\ \boldsymbol{H}.$

Waveguide crossing, results



$n_g = 3.4,\ n_b = 1.45,\ \lambda = 1.55\,\mu$m,
$h = 0.2\,\mu$m, v variable, TE polarization.



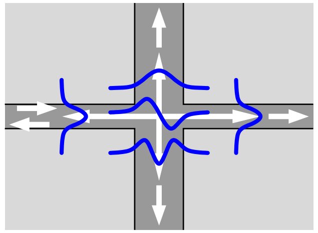
Basis elements:
guided modes of the horizontal
and vertical cores
(directional variants).

FEM discretization:
$z \in [v/2 - 1.5\,\mu\text{m}, v/2 + 1.5\,\mu\text{m}]$, $\Delta x = 0.025\,\mu$m,
$x \in [w/2 - 1.5\,\mu\text{m}, w/2 + 1.5\,\mu\text{m}]$, $\Delta z = 0.025\,\mu$m.

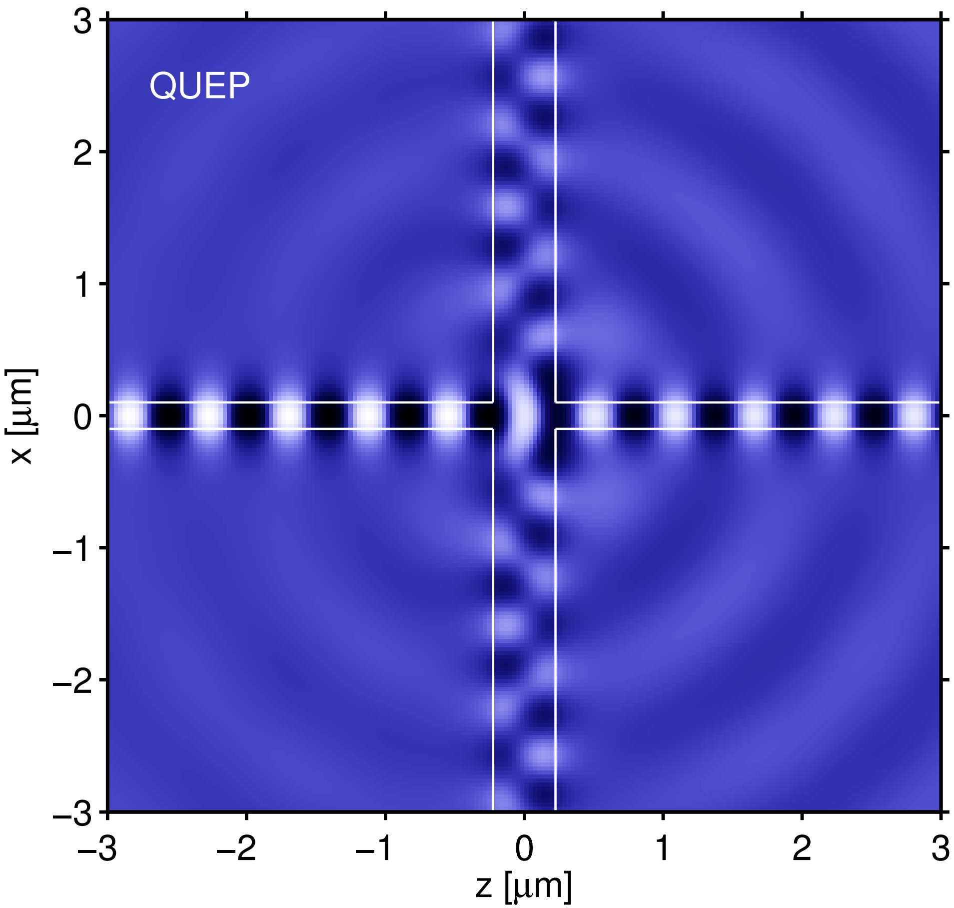
Computational window:
$z \in [-4\,\mu\text{m}, 4\,\mu\text{m}]$, $x \in [-4\,\mu\text{m}, 4\,\mu\text{m}]$.

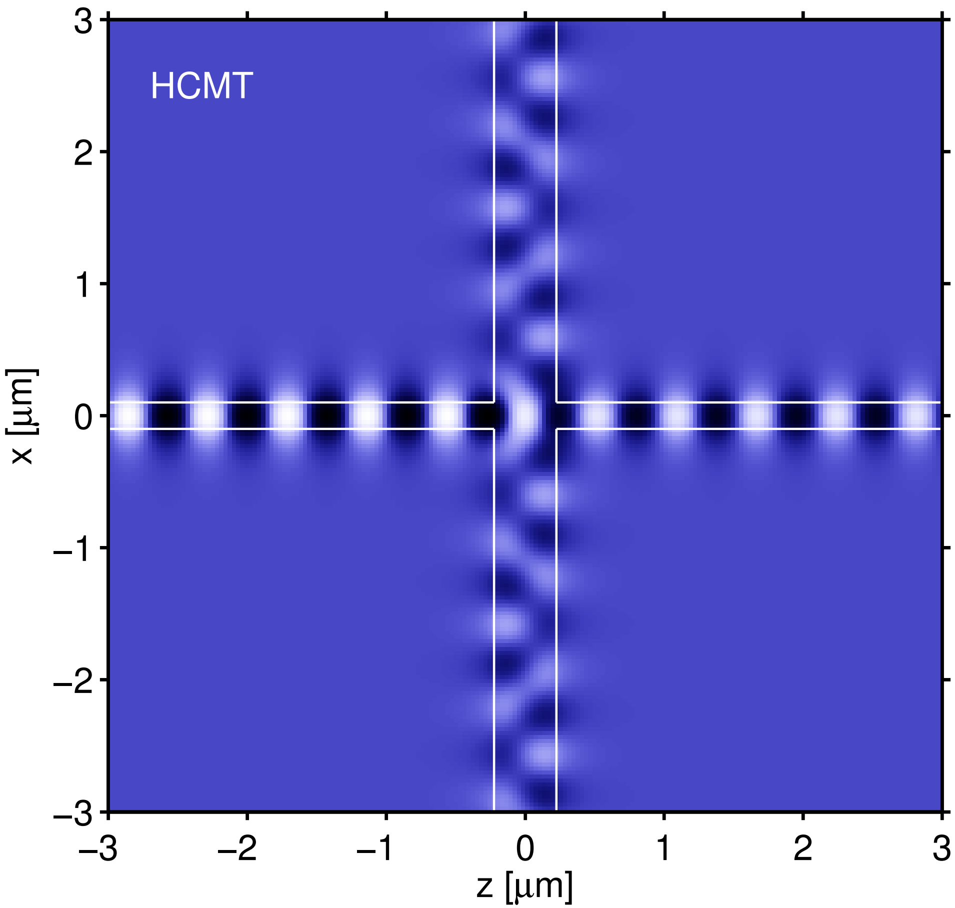
Waveguide crossing, fields

$v = 0.45\,\mu$m:

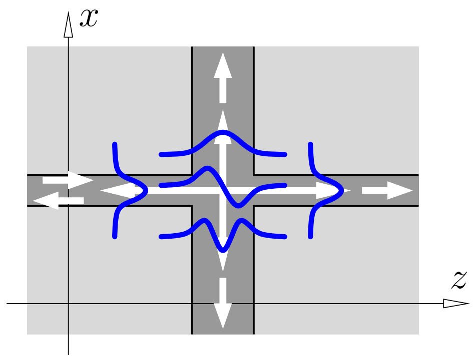


reference

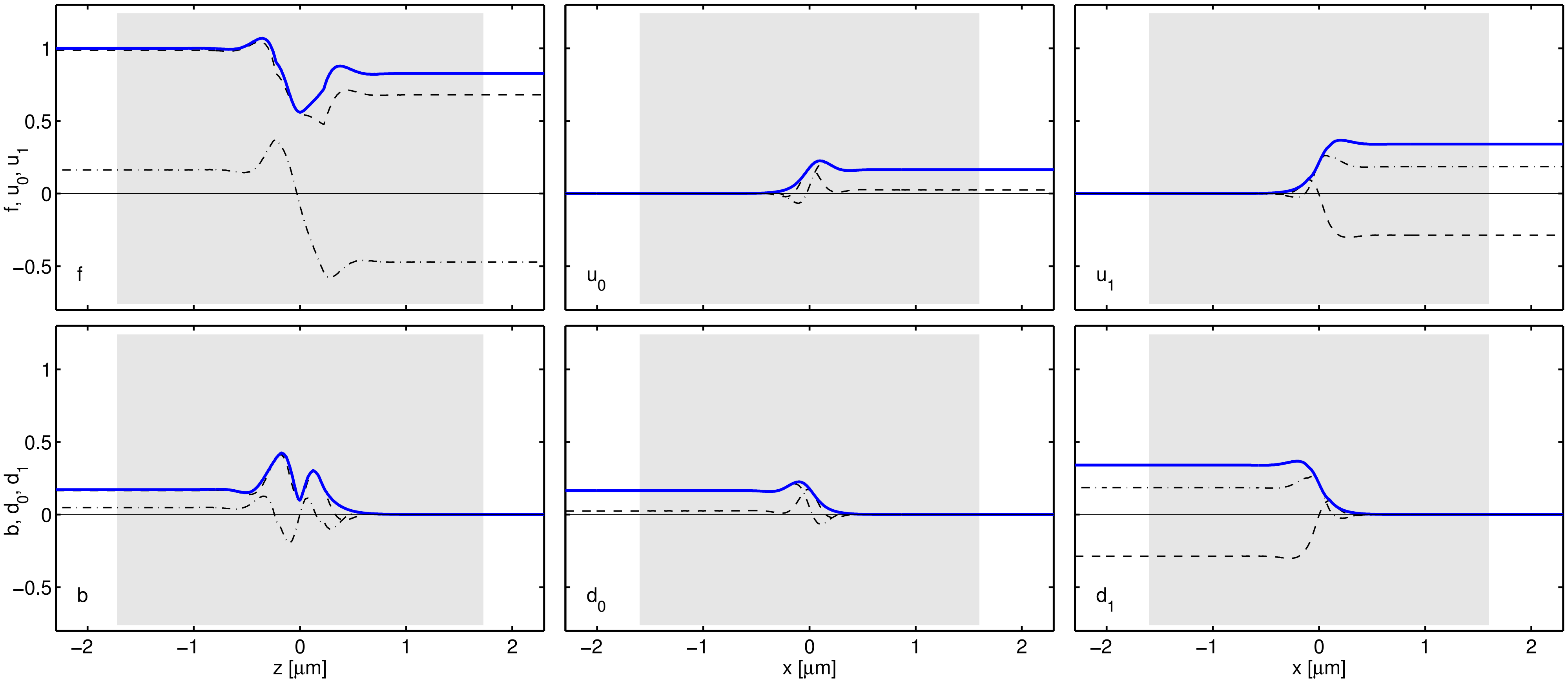




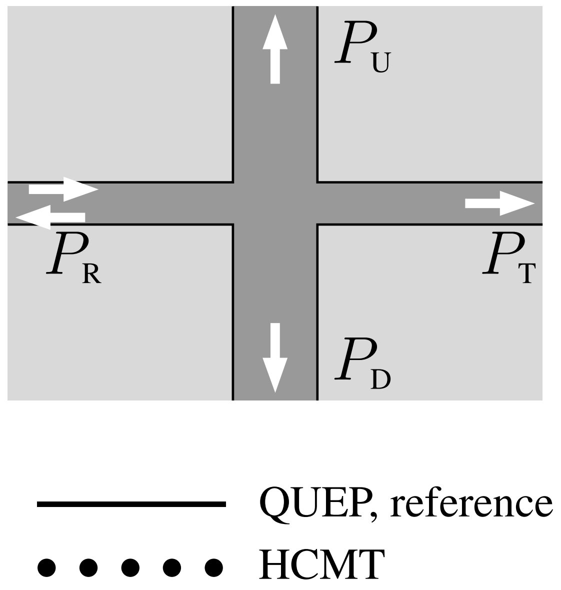
Waveguide crossing, amplitude functions

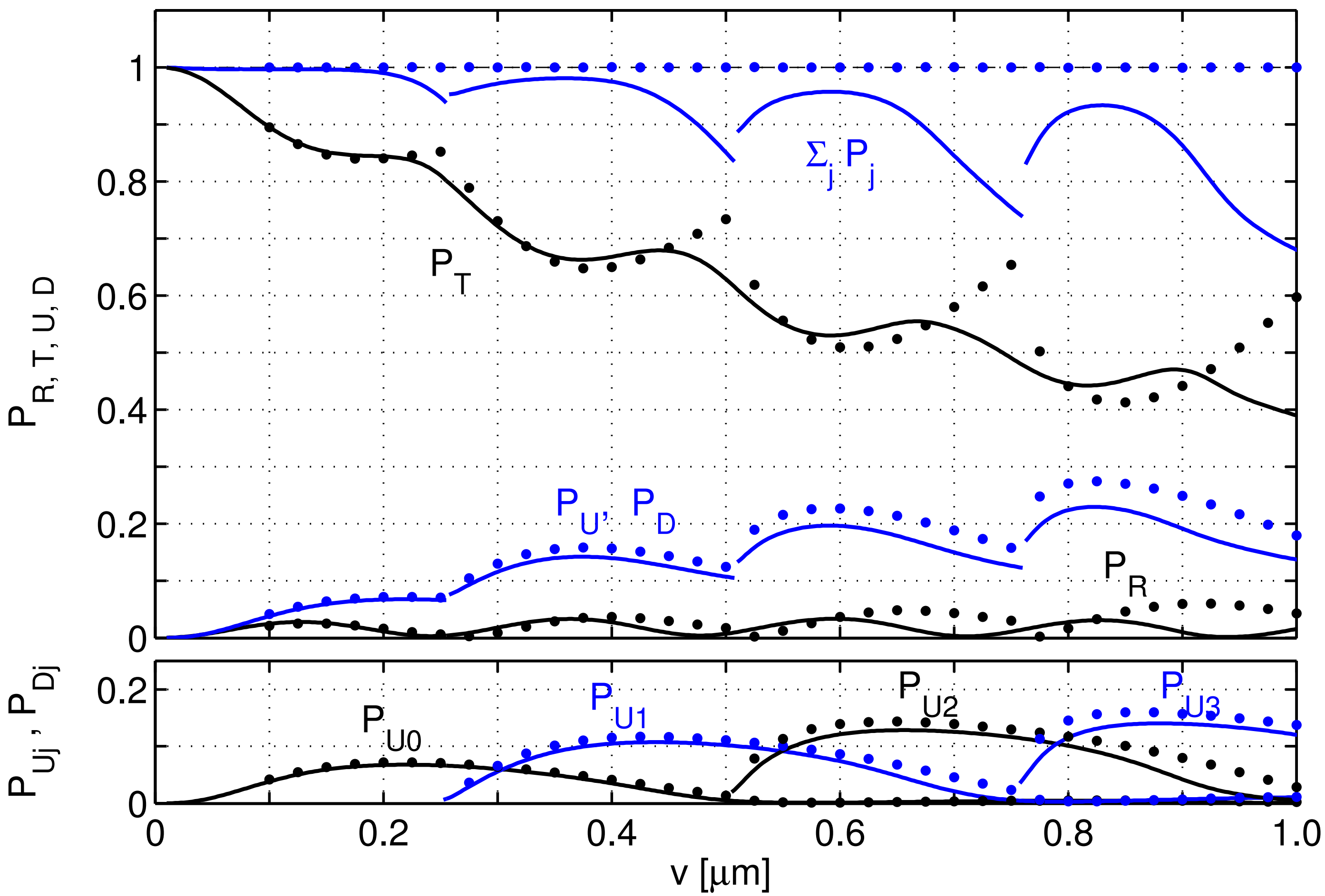


$v = 0.45\,\mu$m:

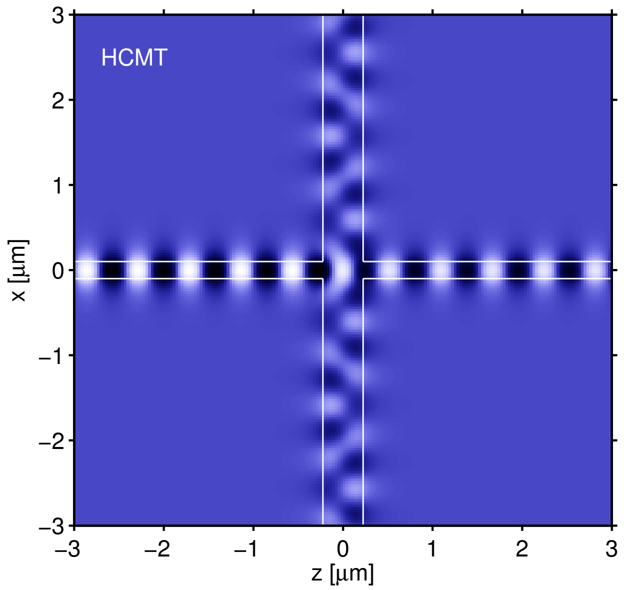


Waveguide crossing, power transfer





Comments



HCMT scheme based on a variational form of the guided wave scattering problem:

- Expansions at the TIBC ports reduce to single terms due to modal orthogonality.
- Bidirectional basis fields are required for all channels in the field templates.

Alternative functional:

$$\mathcal{C}(\boldsymbol{E},\boldsymbol{H}) = \iiint_\Omega \Big\{ \boldsymbol{E}^* \cdot (\boldsymbol{\nabla}\times\boldsymbol{H}) - \boldsymbol{H}^* \cdot (\boldsymbol{\nabla}\times\boldsymbol{E}) - \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}^*\cdot\boldsymbol{E} + \mathrm{i}\omega\mu_0\boldsymbol{H}^*\cdot\boldsymbol{H}\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

?

Extend $\mathcal{C}$ by boundary integrals such that

- the boundary terms in $\delta\mathcal{C}$ cancel ⟷ the Galerkin scheme (OWTNM'06) could be viewed as a variational restriction of $\mathcal{C}$.
- TIBCs are satisfied as natural boundary conditions if $\mathcal{C}$ becomes stationary ⟷ variational scheme with complex conjugate fields.