1. *Bilinear functionals and quadratic forms*

   Exercise 10, pages 17 & 18 of the lecture notes; look through section 2.1.7 first.

2. *Restriction of functionals* (≈ Exercise 12, pages 19, 20)

   Consider the functional

   $$\mathcal{L}(u) = \int_{-1}^{1} (u(x) - \sin(x))^2 \, \mathrm{d}x$$

   for functions in $\{u : [-1, 1] \to \mathbb{R}\}$ where it is well defined. Minimizing this functional is an example of the least squares method to approximate the function $\sin$ by a function u from a given subset. Clearly, if any function u would be allowed, the solution of the minimization problem would be $u_{\text{min}} = \sin$.

   (a) Let P_2 be the set of polynomials of degree at most two:

   $$P_2 = \left\{u : \mathbb{R} \to \mathbb{R},\, u(x) = a_0 + a_1 x + a_2 x^2 \,|\, a_0, a_1, a_2 \in \mathbb{R}\right\}.$$

   This is a three dimensional space. A set of polynomials P_3 of degree three is defined analogously. Restrict the functional to the spaces P_2 and P_3 and find the corresponding functions L_{P_2} and L_{P_3} (these are functions of three or four variables, respectively; evaluate them as far as possible).

   (b) Find the minimizers of these functions and write down the minimizing polynomials. Illustrate the approximations of the $\sin$ function by means of suitable plots.

   (c) Interpret the minimizing polynomials as the projection of the function $\sin$ on the sets of polynomials. To make that statement explicit, show that a function u_P minimizes the functional $\mathcal{L}$ on a given set P, $u_P \in \text{Min}_{u\in P}\{\mathcal{L}(u)\}$, if it is given by $u_P = \rho_P\, v_P$ with $\rho_P = \langle \sin, v_P\rangle$, where v_P is the solution of the maximization problem $v_P \in \text{Max}_{v\in P}\left\{\langle v, \sin\rangle \,|\, ||v||^2 = 1\right\}$. Here $||\ \ ||$ and $\langle\ ,\ \rangle$ denote the usual L_2-norm and inner product. Start with the identity

   $$||u - \sin||^2 = ||u||^2 - 2\,\langle u, \sin\rangle + ||\sin||^2\,,$$

   and replace u by its 'length' $\rho = ||u||$ and its 'orientation' $v = u/||u||$ (Draw the corresponding situation for the projection of a given vector on a subspace).

   (d) Find the projection of $\sin$ (the least squares approximation) on the polynomial sets P_2 and P_3 for the norm $||\ \ ||_1$ given by

   $$||u||_1^2 = \int_{-1}^{1} u_x^2(x)\, \mathrm{d}x\,.$$

   Wherever possible, use Maple to do the manipulations.

3. *Lagranges Lemma* (Exercise 24)

   Give a *sketch* of a proof of the Lagrange Lemma as stated on page 24 of the lecture notes. Remarks:

   - For the definition of the space of test functions $C_0^\infty(\Omega)$ you might wish to consider section 2.1.2.
   - In case you need a test function with specific properties that are compatible with the definition of $C_0^\infty(\Omega)$: State the properties explicitly, and assume that such a function exists; do not try to construct it explicitly. Note that in general $\Omega \subset \mathbb{R}^n$.
   - The Lemma addresses functions that map to real numbers $f : \Omega \to \mathbb{R}$ as well as vector functions $f : \Omega \to \mathbb{R}^m$ (with the dot product $f \cdot \eta$). It may be convenient to consider these cases separately.

Solutions are to be handed in until Wednesday, May 04, 09:00.