1. *More about discrete Hamiltonian and Lagrangian systems*

   (a) For a discrete Hamiltonian system with position variables $\boldsymbol{q} = (q_1, \dots, q_N) \in \mathbb{R}^N$, momentum variables $\boldsymbol{p} = (p_1, \dots, p_N) \in \mathbb{R}^N$, and Hamiltonian (Hamilton function)

   $$H : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad (t, \boldsymbol{q}, \boldsymbol{p}) \to H(t, \boldsymbol{q}, \boldsymbol{p}),$$

   the canonical action functional $\mathcal{A}_c$ is defined as

   $$\mathcal{A}_c(\boldsymbol{q}, \boldsymbol{p}) = \int_{t_0}^{t_1} \left(\boldsymbol{p}(t) \cdot \partial_t \boldsymbol{q}(t) - H(t, \boldsymbol{q}(t), \boldsymbol{p}(t))\right) \mathrm{d}t\,,$$

   where $\boldsymbol{q}(t)$, $\boldsymbol{p}(t)$ are the candidates for the trajectory of the system in phase space. The canonical action principle states that the physical trajectories are critical points of the action functional. Find the differential equations that govern these trajectories, i.e. find the equations of motion (Hamilton equations) for the discrete Hamiltonian system.

   (b) For a single particle of mass m, described by its cartesian position $\boldsymbol{q}$ and momentum $\boldsymbol{p}$, the Hamiltonian (here: the total energy) for the motion of this particle in a potential $V(\boldsymbol{q})$ is

   $$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q})\,.$$

   Show that the Hamilton equations for this system lead to Newtons law, with $\boldsymbol{F} = -\boldsymbol{\nabla} V$ being the force acting on the particle.

   (c) i. Show that for an autonomous discrete Hamiltonian system with position variables $\boldsymbol{q}$, momentum variables $\boldsymbol{p}$, and a Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$ that does not explicitly depend on time, the Hamiltonian is conserved during the physical time evolution.

   ii. Show that for an autonomous discrete Lagrangian system described by position variables $\boldsymbol{q}$, velocities $\dot{\boldsymbol{q}} = \partial_t \boldsymbol{q}$, and a Lagrangian $L(\boldsymbol{q}, \boldsymbol{v})$ that does not explicitly depend on time, the quantity

   $$E = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

   is conserved during the physical time evolution.

2. *Harmonic oscillator in Lagrangian and Hamiltonian description*
   Page 48 of the lecture notes, exercise 42, parts 1 (a) and 2 (a).

3. *Nonlinear two-point boundary value problem*
   Page 51 of the lecture notes, section 3.5, exercise 2.

4. *Stationary states of a nonlinear diffusion equation*
   Page 54 of the lecture notes, section 3.5, exercise 6.

5. *Dido's problem: Find the surface of largest area, given the value of its perimeter.*
   Page 71 of the lecture notes, section 5.1, exercise 60, part 1.
   Program:

   (a) Specialize the problem in $\mathbb{R}^2$ to a part of the x-y-plane such that the boundary can be parametrized by polar coordinates. For a given trial curve $r(\phi)$ that defines the boundary, give expressions for the enclosed area $\mathcal{A}(r)$ and for the length of the boundary $\mathcal{L}(r)$.

   (b) State the constraint optimization problem for the area, given a fixed length L of the boundary, and derive the differential equation for the constraint critical point.

   (c) Observe that the equation is satisfied for the type of curves that you expect, and fix the value of the Lagrange multiplier by incorporating the constraint.

Solutions are to be handed in until Wednesday, June 01, 09:00.