1. *Gauss' theorem, alternative forms*

   Verify that Gauss' theorem can be given the following alternative forms

   (a) $\displaystyle\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{B}\, \mathrm{d}\mathcal{V} = \oint_{\partial\mathcal{V}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}\,,$

   (b) $\displaystyle\int_{\mathcal{V}} \boldsymbol{\nabla} \times \boldsymbol{B}\, \mathrm{d}\mathcal{V} = -\oint_{\partial\mathcal{V}} \boldsymbol{B} \times \mathrm{d}\boldsymbol{a}\,,$

   (c) $\displaystyle\int_{\mathcal{V}} \boldsymbol{\nabla}\phi\, \mathrm{d}\mathcal{V} = \oint_{\partial\mathcal{V}} \phi\, \mathrm{d}\boldsymbol{a}\,.$

   for scalar fields $\phi(\boldsymbol{r})$, vector fields $\boldsymbol{B}(\boldsymbol{r})$, and integrals that extend over a domain $\mathcal{V}$ and its boundary $\partial\mathcal{V}$. Start by applying the original identity (a) to products $\boldsymbol{A} \times \boldsymbol{B}$ and $\boldsymbol{A}\phi$, where $\boldsymbol{A}$ is an arbitrary *constant* vector.

2. *Multipole moments*

   (a) Compute the first two moments Q, $\boldsymbol{p}$ of the charge densities

      i. $\rho(\boldsymbol{r}) = q\,\delta(\boldsymbol{r})$,

      ii. $\rho(\boldsymbol{r}) = q\,\delta(\boldsymbol{r} - \boldsymbol{a})$,

      iii. $\rho(\boldsymbol{r}) = -q\,\delta(\boldsymbol{r} + \boldsymbol{a}/2) + q\,\delta(\boldsymbol{r} - \boldsymbol{a}/2)$, and

      iv. $\rho(\boldsymbol{r}) = -q\,\delta(\boldsymbol{r}) + q\,\delta(\boldsymbol{r} - \boldsymbol{a})$,

      for charges $\pm q$ at "small" distances $\boldsymbol{a}$.

   (b) Calculate the first three moments Q, $\boldsymbol{p}$, and Q_{jk} of the charge density $\rho(\boldsymbol{r}) = -q\,\delta(\boldsymbol{r} + \boldsymbol{a}/2) + 2q\,\delta(\boldsymbol{r}) - q\,\delta(\boldsymbol{r} - \boldsymbol{a}/2)$.
      Specify the result for Q_{jk} to $\boldsymbol{a} = (0, 0, a)$.

   (c) Determine the magnetic dipole moment $\boldsymbol{m}$ of a current I, running through a plane loop $\mathcal{C}$ of "small" extensions. Does the particular shape of $\mathcal{C}$ matter?
      Program: Replace $\int \sqcup\, \boldsymbol{J}\, \mathrm{d}^3 r \to I \oint \sqcup\, \mathrm{d}\boldsymbol{s}$. Verify that $\boldsymbol{\nabla} \times (\boldsymbol{b} \times \boldsymbol{r}) = 2\,\boldsymbol{b}$ for an arbitrary *constant* vector $\boldsymbol{b}$. Use a cyclic shift of the triple product $\sqcup \cdot (\sqcup \times \sqcup)$, and apply Stokes' theorem. Alternative: Recall the geometrical properties of the vector product.

3. *Dipole fields*

   (a) Show that the electric dipole potential $\displaystyle\phi_{\mathrm{D}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3}$

      relates to the electric field $\displaystyle\boldsymbol{E}_{\mathrm{D}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\boldsymbol{p} \cdot \boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3}\right).$

   (b) Show that the magnetic dipole potential $\displaystyle\boldsymbol{A}_{\mathrm{D}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\,\frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}$

      relates to the magnetic induction $\displaystyle\boldsymbol{B}_{\mathrm{D}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3}\right).$

   (c) Write the potential ϕ_{D} in spherical coordinates.

   (d) Write the field $\boldsymbol{E}_{\mathrm{D}}$ in spherical coordinates.

   (e) Verify that you receive the result of (d) when you derive the field in spherical coordinates, starting from the result of (c).

   You might wish to apply the rules for the evaluation of $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b})$ and $\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b})$. Keep in mind that $\boldsymbol{p}$ and $\boldsymbol{m}$ are constant vectors. In previous exercises you already computed $\boldsymbol{\nabla}(r^{\alpha})$. Use the direction of $\boldsymbol{p}$ as the polar axis for the evaluation in spherical coordinates.

Hand in your solutions until Wednesday, September 18, 10:45. Good luck!