1. *Waves*

Assume that the following expressions represent the electric part of electromagnetic fields. Here, for simplicity, $\boldsymbol{E}_0 \in \mathbb{R}^3$ is a real-valued amplitude vector, $\boldsymbol{k} \in \mathbb{R}^3$ and $\omega > 0$ are the wave-vector and frequency associated with the waves:

(a) $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$,

(b) $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$,

(c) $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)}$,

(d) $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + \boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)}$,

(e) $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + \frac{1}{2}\,\boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)}$.

Using the conventions for complex notation of time-harmonic fields, evaluate the physical electric field, simplify as far as reasonable. Clarify the notions of a *forward traveling wave*, *backward traveling wave*, *standing wave*, and *partly traveling, partly standing wave*.

2. *Derivatives of harmonic plane waves*

Given time-harmonic scalar and vector fields of the form

$$\psi(\boldsymbol{r},t) = \psi_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \qquad \boldsymbol{F}(\boldsymbol{r},t) = \boldsymbol{F}_0\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

with constant ψ_0 and $\boldsymbol{F}_0$, make the substitution rules "$\partial_t \to -\mathrm{i}\omega$", "$\boldsymbol{\nabla} \to \mathrm{i}\boldsymbol{k}$" precise by evaluating the following expressions:

$$\dot{\psi},\ \dot{\boldsymbol{F}},\ \ddot{\psi},\ \ddot{\boldsymbol{F}},\ \boldsymbol{\nabla}\psi,\ \boldsymbol{\nabla}\cdot\boldsymbol{F},\ \boldsymbol{\nabla}\times\boldsymbol{F},\ \Delta\psi,\ \Delta\boldsymbol{F}.$$

3. *Maxwell stress tensor and force*

Consider a *large* parallel-plate capacitor with plates at a distance d, filled with a dielectric material with relative permittivity ϵ, connected to a voltage U.

(a) Introduce a convenient coordinate system (draw a schematic), and state the electromagnetic field.

(b) Evaluate the components T_{jk} of the Maxwell stress tensor for this configuration.

(c) Calculate the force per unit area on each of the plates.

4. *Time-averaged Poynting vector and energy density*

(a) Suppose that $a(\boldsymbol{r},t)$ and $b(\boldsymbol{r},t)$ are scalar fields (or components of vector fields) of the form $a(\boldsymbol{r},t) = \tilde{a}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$, $b(\boldsymbol{r},t) = \tilde{b}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$. Using the conventions for the complex notation of time-harmonic fields, here for angular frequency $\omega = 2\pi/T$, with time period T, write out the (real, physical) product $a\,b$, and show that the time-average

$$\bar{f}(t) = \frac{1}{T}\int_t^{t+T} f(t')\,\mathrm{d}t'$$

of that product evaluates to $\overline{a\,b} = \frac{1}{2}\mathrm{Re}(\tilde{a}^*\,\tilde{b})$, where * denotes complex conjugation.

(b) Show that, for time-harmonic electromagnetic fields of the form $\boldsymbol{E}(\boldsymbol{r},t) = \tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ ($\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ analogously), the time-averaged Poynting vector $\overline{\boldsymbol{S}}$ and the time averaged energy density $\overline{u}$ take the forms

$$\overline{\boldsymbol{S}} = \frac{1}{2}\,\mathrm{Re}\left(\tilde{\boldsymbol{E}}^* \times \tilde{\boldsymbol{H}}\right), \quad \text{and} \quad \overline{u} = \frac{1}{4}\,\mathrm{Re}\left(\tilde{\boldsymbol{E}}^* \cdot \tilde{\boldsymbol{D}} + \tilde{\boldsymbol{H}}^* \cdot \tilde{\boldsymbol{B}}\right).$$

5. *d'Alembert's solution of the wave equation*
Consider the 1-D wave equation with initial conditions

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(x,t) = 0\,, \quad \psi(x,0) = \psi_0(x)\,, \quad \dot{\psi}(x,0) = \dot{\psi}_0(x)\,,$$

for $t \geq 0$, $-\infty < x < \infty$, where c is a positive constant and ψ_0, $\dot{\psi}_0$ are given (smooth) functions.

(a) Show that, for two arbitrary (smooth) functions f and b of one variable, the expression

$$\psi(x,t) = f(x-ct) + b(x+ct)$$

solves the wave equation.

(b) Verify that the function

$$\psi(x,t) = \frac{1}{2}\psi_0(x+ct) + \frac{1}{2}\psi_0(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{\psi}_0(s)\,\mathrm{d}s$$

satisfies the initial conditions.

(c) Use these results to show that the function defined in 5b satisfies the initial value problem.

Hand in your solutions until Wednesday, September 25, 10:45. Good luck!