1. *Plane harmonic electromagnetic waves*

   Consider an electromagnetic wave with an electric field of the form

   $$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},$$

   with frequency ω, wave vector $\boldsymbol{k}$, and amplitude vector $\boldsymbol{E}_0$ (remember the conventions for complex notation of time harmonic fields). The wave is assumed to propagate through a homogeneous linear medium without free charges or currents, characterized by the relative permittivity ϵ and relative permeability μ.

   (a) Write out the remaining parts $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ of the electromagnetic field in terms of the quantities introduced above. State conditions such that all Maxwell equations are satisfied.

   (b) Show that, for this wave, the time averaged energy density $\overline{u}$ and the time-averaged Poynting vector $\overline{\boldsymbol{S}}$ take the following alternative forms:

   $$\overline{u} = \frac{1}{2}\,\epsilon_0\epsilon\,|\boldsymbol{E}_0|^2 = \frac{1}{2}\,\mu_0\mu\,|\boldsymbol{H}_0|^2,$$

   $$\overline{\boldsymbol{S}} = \frac{1}{2}\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\,|\boldsymbol{E}_0|^2\,\frac{\boldsymbol{k}}{k} = \frac{1}{2}\sqrt{\frac{\mu_0\mu}{\epsilon_0\epsilon}}\,|\boldsymbol{H}_0|^2\,\frac{\boldsymbol{k}}{k} = \overline{u}\,\mathrm{c_m}\,\frac{\boldsymbol{k}}{k}.$$

   Here $\boldsymbol{H}_0$ is the amplitude vector associated with the magnetic part of the wave field.

2. *Polarization*

   Consider plane harmonic electromagnetic waves with frequency ω that propagate with wavenumber k along the Cartesian z-axis. Specifically, we look at waves with electric fields of the form

   $$\boldsymbol{E}(\boldsymbol{r},t) = E_{0x}\begin{pmatrix}1\\0\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{1}$$

   $$\boldsymbol{E}(\boldsymbol{r},t) = E_{0y}\begin{pmatrix}0\\1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{2}$$

   $$\boldsymbol{E}(\boldsymbol{r},t) = E_{\mathrm{l}}\begin{pmatrix}1\\\mathrm{i}\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{3}$$

   $$\boldsymbol{E}(\boldsymbol{r},t) = E_{\mathrm{r}}\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(kz-\omega t)}. \tag{4}$$

   Here E_{0x}, E_{0y}, E_{l}, and E_{r} are constant complex wave amplitudes.

   (a) Write out the *physical* electric field for each of the waves, using the conventions for the complex notation of time-harmonic fields. Where convenient, split the complex amplitudes $E_0 = |E_0|\exp(\mathrm{i}\varphi)$ into their absolute value $|E_0|$ and argument φ.

   (b) Draw rough schematics of these physical fields, for each of the three cases (1), (2), (3), (4):

      i. a snapshot at time t_0 of the electric components $E_x(z,t_0)$, $E_y(z,t_0)$ vs. z. Mark the wavelength $\lambda = 2\pi/k$ in your plots.

      ii. a plot of the electric components $E_x(z_0,t)$, $E_y(z_0,t)$ vs. time t, for fixed position z_0. Mark the period $T = 2\pi/\omega$ in your plots.

      iii. the trace of $\boldsymbol{E}(z_0,t)$ in the E_x-E_y-plane with time for fixed position z_0.

      iv. the trace of $\boldsymbol{E}(z,t_0)$ in the E_x-E_y-plane with position for fixed time t_0.

   Choose convenient values for t_0, z_0, and for the absolute values $|E_0|$ and arguments φ of the wave amplitudes. You might wish to combine different cases into single drawings.

(c) Show that a wave with general polarization

$$\boldsymbol{E}(\boldsymbol{r},t) = \begin{pmatrix} E_{0x} \\ E_{0y} \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{5}$$

can always be written

i. as the sum of two orthogonal linearly polarized waves, here an x-polarized wave (1) and a y-polarized wave (2),

ii. as the sum of a left circularly polarized wave (3) and a right circularly polarized wave (4).

3. *Spherical waves*

Show that the spherically symmetric field

$$\psi(\boldsymbol{r},t) = \frac{1}{r}\,a(kr-\omega t) + \frac{1}{r}\,b(kr+\omega t)$$

solves the scalar wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0$$

for arbitrary (smooth) functions a and b of one variable, provided that the relation $\omega^2 = k^2\mathrm{c}^2$ between frequency, wavenumber, and phase velocity is satisfied. Use the representation of the Laplace-operator in spherical coordinates.

Hand in your solutions until Wednesday, October 02, 10:45. Good luck!