Electrodynamics

— Lecture A —

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Integrated Optical MicroSystems
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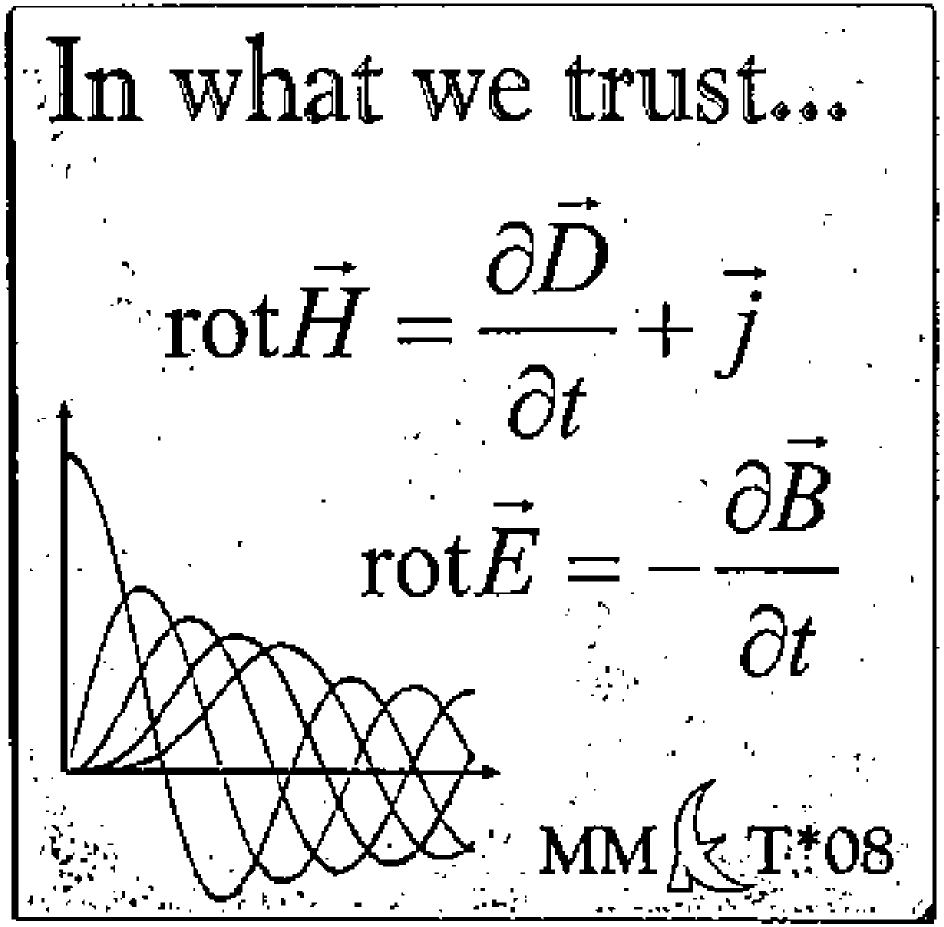
Motto



MMET'08, Mathematical Methods in Electromagnetic Theory, Odesa, Ukraine, June 29 - July 2, 2008

Maxwell equations

SI, in matter, time domain, differential form:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho_{\mathrm{f}},\\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\dot{\boldsymbol{B}},\\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\ \boldsymbol{\nabla}\times\boldsymbol{H} &= \boldsymbol{J}_{\mathrm{f}}+\dot{\boldsymbol{D}},\end{aligned}$$

$$\begin{aligned}\boldsymbol{D} &= \epsilon_0\boldsymbol{E}+\boldsymbol{P},\\ \boldsymbol{B} &= \mu_0(\boldsymbol{H}+\boldsymbol{M}).\end{aligned}$$

(+ constitutive relations)

$\boldsymbol{E}(\boldsymbol{r},t)$: electric field,
$\boldsymbol{D}(\boldsymbol{r},t)$: (di-)electric displacement,
$\boldsymbol{B}(\boldsymbol{r},t)$: magnetic induction (field, flux density),
$\boldsymbol{H}(\boldsymbol{r},t)$: magnetic field (...),
$\rho_{\mathrm{f}}(\boldsymbol{r},t)$: density of free charges,
$\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r},t)$: density of free currents,
$\boldsymbol{P}(\boldsymbol{r},t)$: polarization,
$\boldsymbol{M}(\boldsymbol{r},t)$: magnetization,
ϵ_0: free space permittivity,
μ_0: free space permeability.

Valid for more than a century, firm basis for further considerations.

Course overview

Electrodynamics

A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,

B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion

C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields

D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves

E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity

F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides

G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation

H Special relativity; transmission lines

Formalities

Organization of the course:

- Lectures ($9\times$)
- Homework ($7\times$)
- Tutorials ($8\times$)
- Intermediate tests ($2\times$)
- Final exam
  (~~textbooks~~, ~~laptops~~, ~~calculators~~, ~~(smart-)phones~~, ~~(...)~~, 1 A4 sheet .)

Textbook:
Introduction to Electrodynamics, D.J. Griffiths, 3rd intl. ed., Prentice Hall, 2003

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z),\ t.$

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- Vector product: $\boldsymbol{A} \times \boldsymbol{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}.$

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- Time derivatives: $\dfrac{\partial \phi}{\partial t},\ \partial_t \phi,\ \dot{\phi},\ \cancel{\nabla_t \phi}.$

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- Curl: $\mathrm{curl}\boldsymbol{A} = \mathrm{rot}\boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}.$

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- Laplacian: $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2$, $\Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_x^2\phi$, $\Delta\boldsymbol{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}$.

Integral theorems

- Gradient theorem: $$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\nabla}\phi \cdot \mathrm{d}\boldsymbol{s} = \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}),$$

arbitrary paths $\boldsymbol{a} \to \boldsymbol{b}$.

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- Divergence theorem (Gauss', Green's):
$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{A}\, \mathrm{d}\mathcal{V} = \int_{\partial\mathcal{V}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{a},$$
$\mathrm{d}\boldsymbol{a}/|\mathrm{d}\boldsymbol{a}|$ an outward oriented normal vector on the surface $\partial\mathcal{V}$ of $\mathcal{V}$.

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- Curl theorem (Stokes'):

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{a} = \oint_{\partial\mathcal{S}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s},$$

sign consistency required for $\mathrm{d}\boldsymbol{a}$ and $\mathrm{d}\boldsymbol{s}$ (right hand rule).

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- Variants & corollaries.

Dirac delta

A linear functional that extracts the value of a function at one point (...):

1-D: $$\int_a^b f(x)\,\delta(x-x_0)\,\mathrm{d}x = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x-x_0) = 0, \text{ if } x \neq x_0.$$

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3-D: $$\int_{\mathcal{V}} f(\boldsymbol{r})\,\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}\mathcal{V} = \begin{cases} f(\boldsymbol{r}_0), & \text{if } \boldsymbol{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

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Implications: manifold.

Laplace of 1/r

Example:

- $\Delta \dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} = 0, \text{ if } \boldsymbol{r} \neq \boldsymbol{r}_0,$
- $\displaystyle\int_{\mathcal{V}} \Delta \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\, \mathrm{d}\mathcal{V} = \begin{cases} -4\pi, & \text{if } \boldsymbol{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$

(exercise)

$$\hookrightarrow \quad -\frac{1}{4\pi}\int_{\mathcal{V}} f(\boldsymbol{r}) \Delta \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\, \mathrm{d}\mathcal{V} = \begin{cases} f(\boldsymbol{r}_0), & \text{if } \boldsymbol{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$-\frac{1}{4\pi}\Delta \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} = \delta(\boldsymbol{r}-\boldsymbol{r}_0).$$

Potentials

$\boldsymbol{F}(\boldsymbol{r})$: a well-behaved vector field, $\boldsymbol{F}(r \to \infty) \to 0$.
Then $\boldsymbol{F}$ can be written

$$\boldsymbol{F} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{A},$$

with (e.g.)

$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int \frac{\boldsymbol{\nabla}\cdot\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\mathcal{V}', \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{\boldsymbol{\nabla}\times\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\mathcal{V}'.$$

~~*(proof)*~~

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- $\boldsymbol{\nabla}\times\boldsymbol{F} = 0 \quad \rightsquigarrow \quad \boldsymbol{F} = \boldsymbol{\nabla}\phi, \quad \Delta\phi = \boldsymbol{\nabla}\cdot\boldsymbol{F}.$

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Potentials, sources

$\boldsymbol{F}(\boldsymbol{r})$: a well-behaved vector field, $\boldsymbol{F}(r \to \infty) \to 0$,
with given $s = \boldsymbol{\nabla} \cdot \boldsymbol{F}$ and $\boldsymbol{c} = \boldsymbol{\nabla} \times \boldsymbol{F}$. Then $\boldsymbol{F}$ can be written

$$\boldsymbol{F} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{A},$$

with (e.g.)

$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi} \int \frac{s(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}\mathcal{V}', \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi} \int \frac{\boldsymbol{c}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}\mathcal{V}'.$$

- $\boldsymbol{F}$ is fully determined by s and $\boldsymbol{c}$.
- $\boldsymbol{\nabla} \times \boldsymbol{F} = 0, \ \boldsymbol{\nabla} \cdot \boldsymbol{F} = s \quad \rightsquigarrow \quad \boldsymbol{F} = \boldsymbol{\nabla}\phi, \quad \Delta\phi = s.$
- $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0, \ \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{c} \quad \rightsquigarrow \quad \boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{A}, \quad \Delta\boldsymbol{A} = -\boldsymbol{c}.$

Taylor expansion

1-D: A function $f(x)$ of one variable, expansion around $x = x_0$:

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$$\phi(\boldsymbol{r}_0 + \Delta \boldsymbol{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\Delta \boldsymbol{r} \cdot \boldsymbol{\nabla})^n \phi(\boldsymbol{r}_0) \quad = \quad [\exp(\Delta \boldsymbol{r} \cdot \boldsymbol{\nabla})] \phi(\boldsymbol{r}_0).$$

(derivation)

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\ \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

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- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k)$.
- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$.
- $\delta(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k$.

Fourier transform

3-D: A field $\phi(\boldsymbol{r})$:

$$\phi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\boldsymbol{k})\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k, \qquad \tilde{\phi}(\boldsymbol{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r.$$

Fourier transform

3-D: A field $\phi(\boldsymbol{r})$:

$$\phi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\boldsymbol{k})\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k, \qquad \tilde{\phi}(\boldsymbol{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r.$$

4-D: A field $\phi(\boldsymbol{r}, t)$:

$$\phi(\boldsymbol{r}, t) = \frac{1}{\sqrt{2\pi}^4} \int \tilde{\phi}(\boldsymbol{k}, \omega)\ \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 k\, \mathrm{d}\omega,$$

$$\tilde{\phi}(\boldsymbol{k}, \omega) = \frac{1}{\sqrt{2\pi}^4} \int \phi(\boldsymbol{r}, t)\ \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 r\, \mathrm{d}t.$$

Upcoming

Next lecture:

- Maxwell equations
- Brush up on electro- and magnetostatics
- Multipole expansion