Electrodynamics

— Lecture B —

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Integrated Optical MicroSystems
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Course overview

Electrodynamics

- A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,
- B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion
- C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields
- D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves
- E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity
- F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides
- G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation
- H Special relativity; transmission lines

Maxwell equations

SI, in matter, time domain, differential form:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{D} &= \rho_{\mathrm{f}}\,, \\
\boldsymbol{\nabla} \times \boldsymbol{E} &= -\dot{\boldsymbol{B}}, \\
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0, \\
\boldsymbol{\nabla} \times \boldsymbol{H} &= \boldsymbol{J}_{\mathrm{f}} + \dot{\boldsymbol{D}}, \\
\\
\boldsymbol{D} &= \epsilon_0 \boldsymbol{E} + \boldsymbol{P}, \\
\boldsymbol{B} &= \mu_0 (\boldsymbol{H} + \boldsymbol{M}).
\end{aligned}
$$

(+ constitutive relations)

$\boldsymbol{E}(\boldsymbol{r},t)$:	electric field,
$\boldsymbol{D}(\boldsymbol{r},t)$:	(di-)electric displacement,
$\boldsymbol{B}(\boldsymbol{r},t)$:	magnetic induction (field, flux density),
$\boldsymbol{H}(\boldsymbol{r},t)$:	magnetic field (. . .),
$\rho_{\mathrm{f}}(\boldsymbol{r},t)$:	density of free charges,
$\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r},t)$:	density of free currents,
$\boldsymbol{P}(\boldsymbol{r},t)$:	polarization,
$\boldsymbol{M}(\boldsymbol{r},t)$:	magnetization,
ϵ_0:	free space permittivity,
μ_0:	free space permeability.

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$\boldsymbol{P}(\boldsymbol{r}, t)$: polarization,
$\boldsymbol{M}(\boldsymbol{r}, t)$: magnetization,
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$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ (Lorentz-) force $\boldsymbol{F}$ on a charge q with velocity $\boldsymbol{v}$.

Polarization

$\boldsymbol{P}$: density of electric dipole moment (bound charges).

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}, \qquad [\boldsymbol{D}] = [\boldsymbol{P}] = \frac{\mathrm{As\,m}}{\mathrm{m}^3},\; [\boldsymbol{E}] = \frac{\mathrm{V}}{\mathrm{m}},$$

vacuum permittivity $\epsilon_0 = 8.854187817\ldots \cdot 10^{-12} \left[\frac{\mathrm{F}}{\mathrm{m}} = \frac{\mathrm{As}}{\mathrm{Vm}}\right].$

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- Simplest case: linear isotropic dielectrica,

$$\boldsymbol{P} = \epsilon_0 \chi_{\mathrm{e}} \boldsymbol{E}, \qquad \chi_{\mathrm{e}}\text{: dielectric susceptibility, } [\chi_{\mathrm{e}}] = 1.$$

$$\hookrightarrow \quad \boldsymbol{D} = \epsilon_0 (1 + \chi_{\mathrm{e}}) \boldsymbol{E} = \epsilon_0 \epsilon \boldsymbol{E}, \qquad \epsilon\text{: relative permittivity, } [\epsilon] = 1.$$

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- Complications: $\epsilon(\boldsymbol{r})$, $\epsilon(\omega)$, ϵ_{jk}, Imϵ, $\epsilon(T)$, $\epsilon(\boldsymbol{F})$, $\chi^{(2)}_{jkl} E_k E_l$, $\chi^{(3)}_{jklm} E_k E_l E_m$, ...

Magnetization

$\boldsymbol{M}$: density of magnetic dipole moments (bound currents).

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}, \qquad [\boldsymbol{H}] = [\boldsymbol{M}] = \frac{\mathrm{A\,m}^2}{\mathrm{m}^3},\ [\boldsymbol{B}] = \mathrm{T} = \frac{\mathrm{Vs}}{\mathrm{m}^2},$$

vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{\mathrm{N}}{\mathrm{A}^2} = \frac{\mathrm{Vs}}{\mathrm{Am}}\right].$

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- Complications: manifold.

Maxwell equations

- Linear dielectric & magnetic media:

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\mathrm{f}}, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \boldsymbol{D} = \epsilon_0 \epsilon \boldsymbol{E},$$
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- Static configurations, $\partial_t = 0$:

$$\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}, \quad \nabla \times \boldsymbol{E} = 0, \quad \boldsymbol{D} = \epsilon_0 \epsilon \boldsymbol{E} \quad \rightsquigarrow \quad \text{Electrostatics.}$$

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- Quasi-stationary configurations, $\partial_t\boldsymbol{D}=0$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{D}=\epsilon_0\epsilon\boldsymbol{E},$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\mathrm{f}},\quad \boldsymbol{B}=\mu_0\mu\boldsymbol{H} \quad \rightsquigarrow \quad \text{Induction, AC currents, \ldots}$$

Electrostatics

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=0,\quad \boldsymbol{D}=\epsilon_0\epsilon\boldsymbol{E}$$

$$\boldsymbol{\nabla}\cdot(\epsilon\boldsymbol{E})=\frac{\rho_{\mathrm{f}}}{\epsilon_0},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=0 \qquad \epsilon(\boldsymbol{r})\ !!!$$

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- Homogeneous media (piecewise ...), $\partial_r \epsilon = \epsilon \partial_r$:

  $$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_{\mathrm{f}}}{\epsilon_0 \epsilon}, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = 0.$$

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- Physical Gauss' law:

  $Q_{\mathrm{f},\mathcal{V}}$: free charges in $\mathcal{V}$

  $$\int_{\partial\mathcal{V}}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a}=\frac{Q_{\mathrm{f},\mathcal{V}}}{\epsilon\epsilon_0}.$$

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  $$\int_{\partial\mathcal{V}}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a}=\frac{Q_{\mathrm{f},\mathcal{V}}}{\epsilon\epsilon_0}.$$
- Scalar electrostatic potential $\phi(\boldsymbol{r})$: *(lecture A, potentials . . .)*
  $$\boldsymbol{E}=-\boldsymbol{\nabla}\phi,\quad \phi(\boldsymbol{r})-\phi(\boldsymbol{r}_0)=-\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\boldsymbol{E}(\boldsymbol{r}')\cdot\mathrm{d}\boldsymbol{r}',$$

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- Scalar electrostatic potential $\phi(\boldsymbol{r})$:

(lecture A, potentials ...)

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi, \qquad \phi(\boldsymbol{r}) - \phi(\boldsymbol{r}_0) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\boldsymbol{E}(\boldsymbol{r}')\cdot\mathrm{d}\boldsymbol{r}',$$

- Poisson equation: $\quad \Delta\phi = -\dfrac{\rho_{\mathrm{f}}}{\epsilon_0\epsilon}.$

Electrostatics, solutions

- Poisson equation: $\Delta\phi = -\dfrac{\rho_\mathrm{f}}{\epsilon_0\epsilon}.$

- Solutions ($\mathcal{V} \to \infty$):

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\int \frac{\rho_\mathrm{f}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\mathcal{V}', \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\int \rho_\mathrm{f}(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\mathcal{V}'.$$

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- Point charge q at $\boldsymbol{r}_0$, $\rho_\mathrm{f}(\boldsymbol{r}) = q\,\delta(\boldsymbol{r}-\boldsymbol{r}_0)$:

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0\epsilon}\,\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}, \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0\epsilon}\,\frac{(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}.$$

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- Coulomb's law, force on another charge $\tilde{q}$ at $\boldsymbol{r}$:

$$\boldsymbol{F}_{q\to\tilde{q}} = \frac{\tilde{q}\,q}{4\pi\epsilon_0\epsilon}\,\frac{(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}.$$

Electrostatics, charge densities

Variants of charge densities:

- $\rho(\boldsymbol{r})$, volume, $Q = \int_{\mathcal{V}} \rho \, \mathrm{d}\mathcal{V}$, $[\rho] = \mathrm{As}/m^3$,
- $\sigma(\boldsymbol{r})$, surface, $Q = \int_{\mathcal{S}} \sigma \, \mathrm{d}a$, $[\sigma] = \mathrm{As}/m^2$,
- $\lambda(\boldsymbol{r})$, line, $Q = \int_{\mathcal{P}} \lambda \, \mathrm{d}s$, $[\lambda] = \mathrm{As}/m$,
- Q, point at $\boldsymbol{r}_0$, $\rho(\boldsymbol{r}) = Q\,\delta(\boldsymbol{r} - \boldsymbol{r}_0)$, $[Q] = \mathrm{As} = \mathrm{C}$.

Conductors, electrostatic configurations, free mobility of charges:

- External field induces charges σ_{f} on the surface of the conductor.
- These cancel the external field in the interior
  $\rightsquigarrow$ $\boldsymbol{E} = 0$ inside the conductor.
- $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \dfrac{\rho_{\mathrm{f}}}{\epsilon_0 \epsilon}$ $\rightsquigarrow$ $\rho_{\mathrm{f}} = 0$ inside the conductor.
- $\phi(\boldsymbol{r}) = \phi(\boldsymbol{r}_0) - \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s}$ $\rightsquigarrow$ ϕ constant over the conductor.
- $\boldsymbol{E}_{\mathrm{outside}} \perp$ conductor surface.
- (Faraday cage).

Magnetostatics

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_\mathrm{f},\quad \boldsymbol{B}=\mu_0\mu\boldsymbol{H}$$

$$\hookrightarrow\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \frac{1}{\mu_0}\boldsymbol{\nabla}\times\frac{1}{\mu}\boldsymbol{B}=\boldsymbol{J}_\mathrm{f}$$

$\mu(\boldsymbol{r})$!!!

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- Ampère's circuital law:

  $I_{\mathrm{f},\mathcal{S}}$: free current through $\mathcal{S}$

  $$\oint_{\partial \mathcal{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu \mu_0 \, I_{\mathrm{f},\mathcal{S}}.$$

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- Vector potential $\boldsymbol{A}(\boldsymbol{r})$: *(lecture A, potentials ...)*
  $\boldsymbol{B} = \nabla \times \boldsymbol{A}.$

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- Poisson equation, vectorial: $\Delta \boldsymbol{A} = -\mu_0 \mu \boldsymbol{J}_{\mathrm{f}}.$

Magnetostatics, solutions

- Poisson equation, vectorial: $\Delta \boldsymbol{A} = -\mu_0 \mu \boldsymbol{J}_{\mathrm{f}}$.

- Solutions ($\mathcal{V} \to \infty$):

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \frac{\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}\mathcal{V}', \quad \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}') \times \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}\mathcal{V}'.$$

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$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \frac{\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}\mathcal{V}', \quad \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}') \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \mathrm{d}\mathcal{V}'.$$

- Current I along $\mathcal{C}$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} I \oint_{\mathcal{C}} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}\boldsymbol{s}', \quad \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} I \oint_{\mathcal{C}} \mathrm{d}\boldsymbol{s}' \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

Magnetostatics, solutions

- Poisson equation, vectorial: $\Delta \boldsymbol{A} = -\mu_0 \mu \boldsymbol{J}_{\mathrm{f}}.$

- Solutions ($\mathcal{V} \to \infty$):

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \frac{\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}\mathcal{V}', \quad \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}') \times \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}\mathcal{V}'.$$

- Current I along $\mathcal{C}$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} I \oint_{\mathcal{C}} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}\boldsymbol{s}', \quad \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} I \oint_{\mathcal{C}} \mathrm{d}\boldsymbol{s}' \times \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}.$$

- Ampère's force law, force on another current $\tilde{I}$ along $\tilde{\mathcal{C}}$:

$$\boldsymbol{F}_{\mathcal{C} \to \tilde{\mathcal{C}}} = \frac{\mu_0 \mu}{4\pi} \tilde{I} I \oint_{\tilde{\mathcal{C}}} \oint_{\mathcal{C}} \frac{\mathrm{d}\boldsymbol{s} \times (\mathrm{d}\boldsymbol{s}' \times (\boldsymbol{r} - \boldsymbol{r}'))}{|\boldsymbol{r} - \boldsymbol{r}'|^3}.$$

Magnetostatics, current densities

Variants of current densities:

- $\boldsymbol{J}(\boldsymbol{r})$, volume, $I = \int_{\mathcal{S}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}$, $\qquad [\boldsymbol{J}] = \mathrm{A}/m^2$,
- $\boldsymbol{K}(\boldsymbol{r})$, surface, $I = \int_{\mathcal{P}} \boldsymbol{K} \cdot \boldsymbol{t}\, \mathrm{d}s$, $\quad |\boldsymbol{t}| = 1,\ \boldsymbol{t} \parallel \text{surface},\ \boldsymbol{t} \perp \mathrm{d}\boldsymbol{s}$, $\qquad [\boldsymbol{K}] = \mathrm{A}/m$,
- $\boldsymbol{I}(\boldsymbol{r})$, line, $I = |\boldsymbol{I}|$, $\qquad [\boldsymbol{I}] = \mathrm{A}$,

- Current density $\leftrightsquigarrow$ moving charge density:

  $\boldsymbol{J} = \rho\, \boldsymbol{v}, \quad \boldsymbol{K} = \sigma\, \boldsymbol{v}, \quad \boldsymbol{I} = \lambda\, \boldsymbol{v}.$

Multipole expansion

Electrostatics:

localized charge density $\rho_{\mathrm{f}}(\boldsymbol{r})$, $\quad \Delta\phi = -\dfrac{\rho_{\mathrm{f}}}{\epsilon_0\epsilon}$,

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\int \frac{\rho_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\mathcal{V}'.$$

Magnetostatics:

localized current density $\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r})$, $\quad \Delta\boldsymbol{A} = -\mu_0\mu\boldsymbol{J}_{\mathrm{f}}$,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0\mu}{4\pi}\int \frac{\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\mathcal{V}'.$$

Consider $\phi(\boldsymbol{r})$, $\boldsymbol{A}(\boldsymbol{r})$, for $|\boldsymbol{r}| \gg |\boldsymbol{r}'|_{\rho(r')\neq 0,\ J(r')\neq 0}$: $\ \cdots$

Multipole expansion, 1/r

Constant $\boldsymbol{r}_s$; second order expansion of $\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|}$ around $\boldsymbol{r}=\boldsymbol{0}$:

$$\frac{1}{|\Delta\boldsymbol{r}-\boldsymbol{r}_s|} \approx \frac{1}{|\boldsymbol{r}_s|} + \frac{\boldsymbol{r}_s\cdot\Delta\boldsymbol{r}}{|\boldsymbol{r}_s|^3} + \frac{1}{2}\left\{\frac{3(\Delta\boldsymbol{r}\cdot\boldsymbol{r}_s)^2 - \boldsymbol{r}_s^2\Delta\boldsymbol{r}^2}{|\boldsymbol{r}_s|^5}\right\} \pm \ \ldots .$$

(exercise)

Multipole expansion, 1/r

Constant $\boldsymbol{r}_s$; second order expansion of $\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|}$ around $\boldsymbol{r}=\mathbf{0}$:

$$\frac{1}{|\Delta\boldsymbol{r}-\boldsymbol{r}_s|} \approx \frac{1}{|\boldsymbol{r}_s|} + \frac{\boldsymbol{r}_s\cdot\Delta\boldsymbol{r}}{|\boldsymbol{r}_s|^3} + \frac{1}{2}\left\{\frac{3(\Delta\boldsymbol{r}\cdot\boldsymbol{r}_s)^2 - \boldsymbol{r}_s^2\Delta\boldsymbol{r}^2}{|\boldsymbol{r}_s|^5}\right\} \pm \ \ldots\ .$$

(exercise)

Expansion required for $|\boldsymbol{r}| \gg |\boldsymbol{r}'|$; substitute $\Delta\boldsymbol{r}\to\boldsymbol{r}',\ \ r'=|\boldsymbol{r}'|,$

$\boldsymbol{r}_s\to\boldsymbol{r},\ \ r=|\boldsymbol{r}|\ :$

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} + \frac{1}{2}\left\{\frac{3(\boldsymbol{r}'\cdot\boldsymbol{r})^2 - \boldsymbol{r}^2(\boldsymbol{r}')^2}{r^5}\right\} \pm \ \ldots\ .$$

Multipole expansion, ϕ

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\int \frac{\rho_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\mathcal{V}'$$

&

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} + \frac{1}{2}\left(\frac{3(\boldsymbol{r}'\cdot\boldsymbol{r})^2 - \boldsymbol{r}^2(\boldsymbol{r}')^2}{r^5}\right) \pm \ldots$$

Multipole expansion, ϕ

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\int\frac{\rho_\mathrm{f}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\mathcal{V}'$$

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$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} + \frac{1}{2}\left(\frac{3(\boldsymbol{r}'\cdot\boldsymbol{r})^2 - \boldsymbol{r}^2(\boldsymbol{r}')^2}{r^5}\right) \pm \ldots$$

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\left\{ \frac{1}{r}\int\rho_\mathrm{f}(\boldsymbol{r}')\,\mathrm{d}^3r' + \frac{1}{r^3}\,\boldsymbol{r}\cdot\int\rho_\mathrm{f}(\boldsymbol{r}')\,\boldsymbol{r}'\,\mathrm{d}^3r' + \frac{1}{2\,r^5}\int\rho_\mathrm{f}(\boldsymbol{r}')\left(3(\boldsymbol{r}'\cdot\boldsymbol{r})^2 - \boldsymbol{r}^2(\boldsymbol{r}')^2\right)\,\mathrm{d}^3r' \pm \ldots \right\}$$

Multipole moments, ϕ

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\left\{\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{j,k=1}^{3} Q_{j,k}\frac{r_j r_k}{r^5} \;\; \pm \;\; \dots \quad \right\},$$

$$Q = \int \rho_{\mathrm{f}}(\boldsymbol{r}')\,\mathrm{d}^3 r', \quad \text{total charge, monopole moment,}$$

$$\boldsymbol{p} = \int \rho_{\mathrm{f}}(\boldsymbol{r}')\,\boldsymbol{r}'\,\mathrm{d}^3 r', \quad \text{electric dipole moment,}$$

$$Q_{jk} = \int \rho_{\mathrm{f}}(\boldsymbol{r}')\left(3r'_j r'_k - (\boldsymbol{r}')^2\delta_{jk}\right)\,\mathrm{d}^3 r', \quad \text{electric quadrupole moment.}$$

Multipole moments, ϕ

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\left\{\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{j,k=1}^{3} Q_{j,k}\frac{r_j r_k}{r^5} \quad \pm \quad \ldots \quad \right\},$$

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$|\boldsymbol{r}| \gg |\boldsymbol{r}'|_{\rho(r')\neq 0}$:

- $Q \neq 0$: monopole dominant; $\phi(\boldsymbol{r}) \approx \dfrac{1}{4\pi\epsilon_0\epsilon}\dfrac{q}{r}$,
- $Q = 0,\ \boldsymbol{p} \neq 0$: dipole dominant; $\phi(\boldsymbol{r}) \approx \dfrac{1}{4\pi\epsilon_0\epsilon}\dfrac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$,
- $Q = 0,\ \boldsymbol{p} = 0$: quadru- $(\ldots)$-pole dominant.

Multipole moments, ϕ

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0\epsilon}\left\{\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{j,k=1}^{3} Q_{j,k}\frac{r_j r_k}{r^5} \ \pm \ \ldots \ \right\},$$

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- $Q = 0,\ \boldsymbol{p} = 0$: quadru- (…)-pole dominant.

$\boldsymbol{E} = -\boldsymbol{\nabla}\phi \rightsquigarrow$ corresponding electric fields.

Multipole expansion, A

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \int \frac{\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}\mathcal{V}'$$

&

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \approx \frac{1}{r} + \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r^3} \pm \ldots$$

Multipole expansion, $\boldsymbol{A}$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0\mu}{4\pi}\int \frac{\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\mathcal{V}'$$

&

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{1}{r} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^3} \pm \ldots$$

$$\hookrightarrow \quad \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0\mu}{4\pi}\left\{ \quad \frac{1}{r}\int \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')\,\mathrm{d}^3r' + \frac{1}{r^3}\int \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')\,(\boldsymbol{r}\cdot\boldsymbol{r}')\,\mathrm{d}^3r' \pm \ldots \right\}$$

Multipole moments, $\boldsymbol{A}$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 \mu}{4\pi} \left\{ 0 \; + \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \;\; \pm \;\; \ldots \qquad\qquad\qquad \right\},$$

no magnetic monopoles,

$$\boldsymbol{m} = \frac{1}{2} \int \boldsymbol{r}' \times \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}') \, \mathrm{d}^3 r',$$ magnetic dipole moment.

Multipole moments, $\boldsymbol{A}$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0\mu}{4\pi}\left\{ 0 \; + \frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3} \;\; \pm \;\; \ldots \qquad\qquad\qquad \right\},$$

no magnetic monopoles,

$$\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{r}' \times \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}')\,\mathrm{d}^3r',$$ magnetic dipole moment.

$|\boldsymbol{r}| \gg |\boldsymbol{r}'|_{\rho(r')\neq 0}$:

- $\boldsymbol{m} \neq 0$: dipole term dominant; $\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0\mu}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$,
- $\boldsymbol{m} = 0$: higher order moments become relevant.

Multipole moments, A

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- $\boldsymbol{m}\neq 0$: dipole term dominant; $\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0\mu}{4\pi}\,\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$,
- $\boldsymbol{m} = 0$: higher order moments become relevant.

$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ $\rightsquigarrow$ corresponding magnetic fields.

Upcoming

Next lecture:

- Maxwell equations: free space and in media, time- and frequency domain, differential and integral form, interface conditions, continuity equation
- Energy and momentum of electromagnetic fields, Poynting theorem