# *Electrodynamics*

## *— Lecture E —*

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# *Course overview*

## Electrodynamics

A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,

B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion

C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields

D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves

E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity

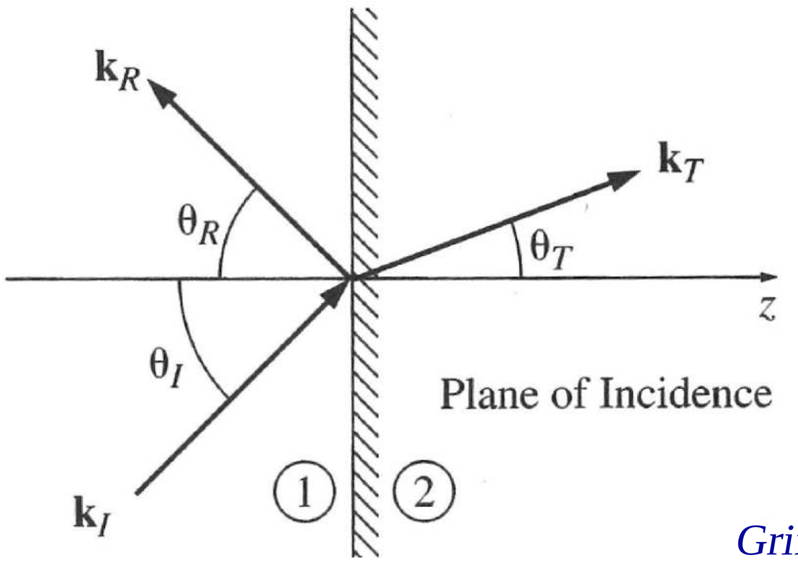
F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides

G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation

H Special relativity; transmission lines

## *Reflection and transmission*

. . . of plane waves
at (dielectric) interfaces.




*Griffiths, Fig. 9.14, page 387*

- $x$-$y$-plane: interface between regions 1, 2; linear homog. isotr. media; properties: $\epsilon_j,\ \mu_j,\ n_j = \sqrt{\epsilon_j \mu_j},\ \mathrm{c}_j = \mathrm{c}/n_j,\ j = 1, 2.$

## *Reflection and transmission*

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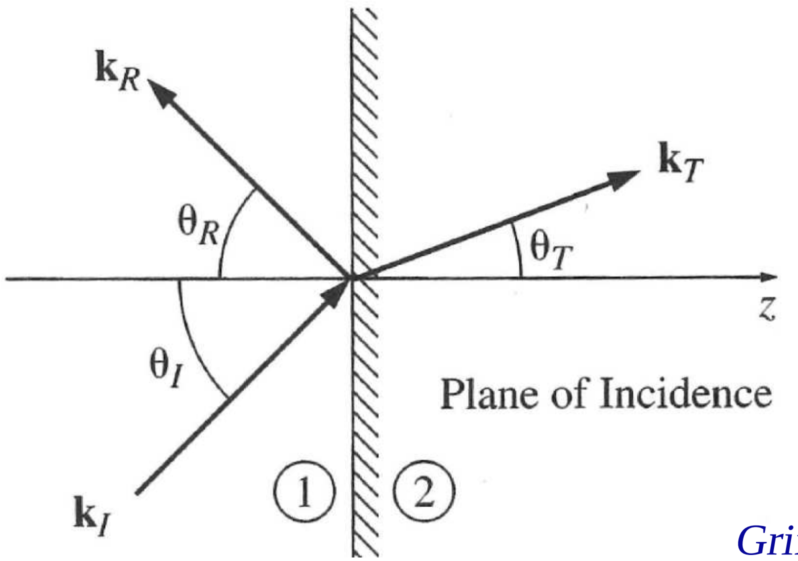


*Griffiths, Fig. 9.14, page 387*

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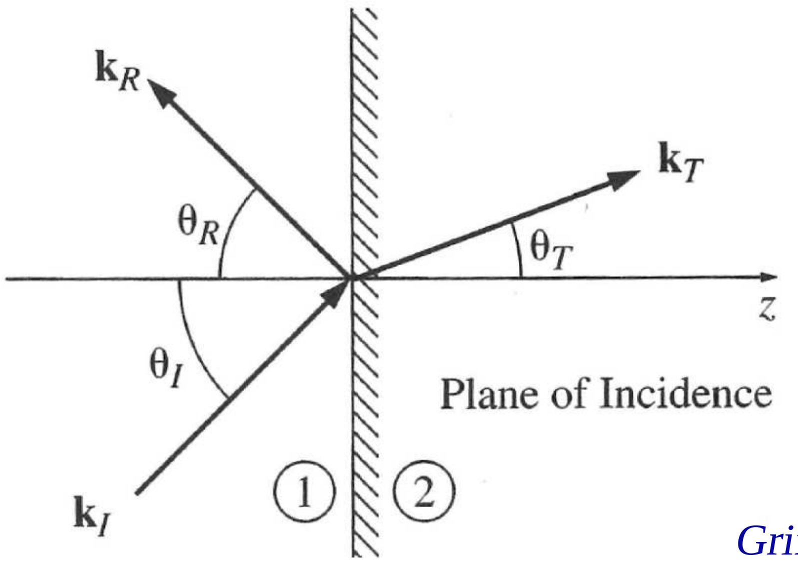
  properties: $\epsilon_j,\ \mu_j,\ n_j = \sqrt{\epsilon_j \mu_j},\ \mathrm{c}_j = \mathrm{c}/n_j,\ j = 1, 2.$
- Expectation:
  A given incoming wave in region 1 generates
  - a reflected wave in region 1,
  - a transmitted wave in region 2.

## *Reflection and transmission*

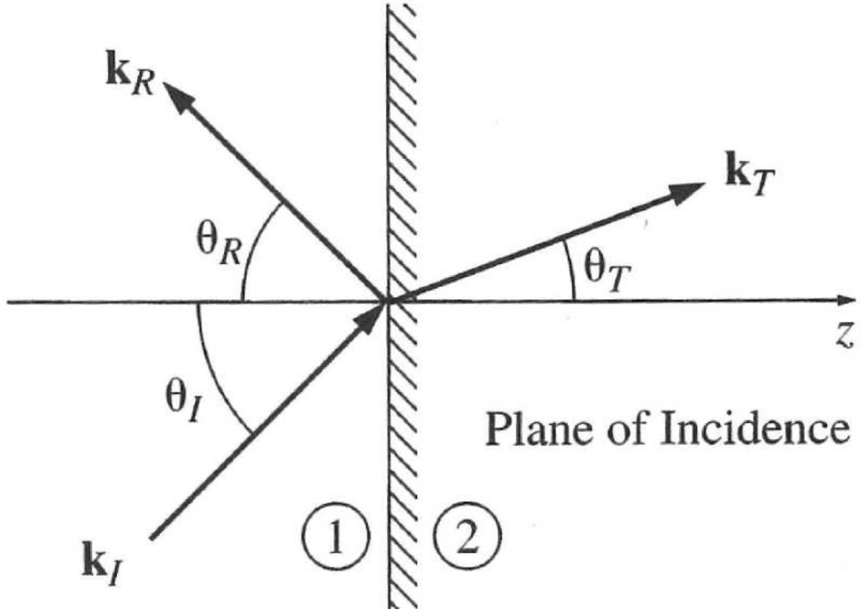
. . . of plane waves
at (dielectric) interfaces.




*Griffiths, Fig. 9.14, page 387*

- $x$-$y$-plane: interface between regions 1, 2; linear homog. isotr. media; properties: $\epsilon_j,\ \mu_j,\ n_j = \sqrt{\epsilon_j \mu_j},\ \mathrm{c}_j = \mathrm{c}/n_j,\ j = 1, 2.$
- Expectation:
  A given incoming wave in region 1 generates
  - a reflected wave in region 1,
  - a transmitted wave in region 2.
- Strategy:
  - write plane wave solutions of homogeneous ME for regions 1, 2,
  - piece these together at the $x$-$y$-plane by interface conditions.

## *Reflection and transmission, ansatz*

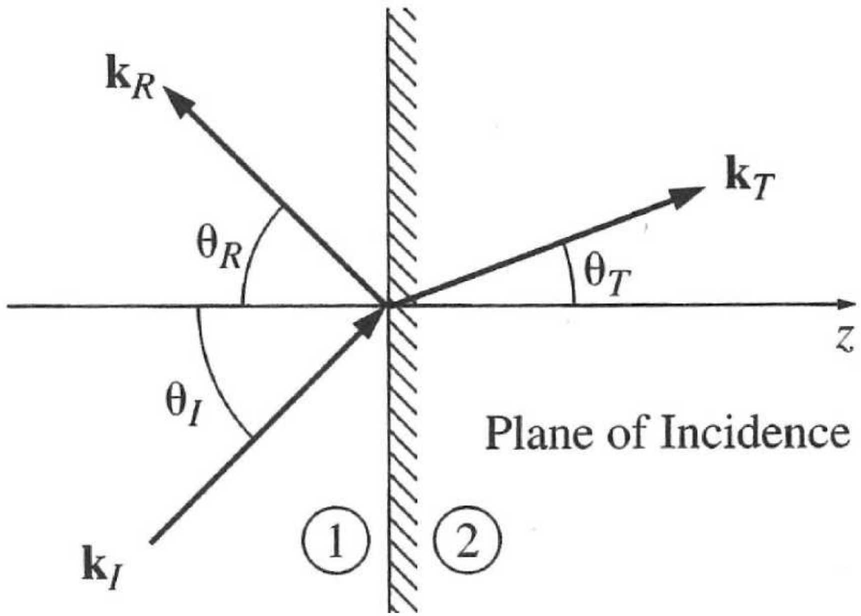



Ansatz:

- Incident: $\boldsymbol{E}_{\mathrm{I}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{I0}}\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{I}}\cdot\boldsymbol{r} - \omega_{\mathrm{I}} t)}$,
- Reflected: $\boldsymbol{E}_{\mathrm{R}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{R0}}\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{R}}\cdot\boldsymbol{r} - \omega_{\mathrm{R}} t)}$,
- Transmitted: $\boldsymbol{E}_{\mathrm{T}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{T0}}\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{T}}\cdot\boldsymbol{r} - \omega_{\mathrm{T}} t)}$,

$$\boldsymbol{k}_{\mathrm{I}} \perp \boldsymbol{E}_{\mathrm{I0}},\ \omega_{\mathrm{I}} = k_{\mathrm{I}} \mathrm{c}_1,\ \ \boldsymbol{k}_{\mathrm{R}} \perp \boldsymbol{E}_{\mathrm{R0}},\ \omega_{\mathrm{R}} = k_{\mathrm{R}} \mathrm{c}_1,\ \ \boldsymbol{k}_{\mathrm{T}} \perp \boldsymbol{E}_{\mathrm{T0}},\ \omega_{\mathrm{T}} = k_{\mathrm{T}} \mathrm{c}_2.$$

## *Reflection and transmission, ansatz*




Ansatz:

- Incident: $\boldsymbol{E}_{\mathrm{I}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{I0}}\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{I}} \cdot \boldsymbol{r} - \omega_{\mathrm{I}} t)}$,
- Reflected: $\boldsymbol{E}_{\mathrm{R}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{R0}}\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{R}} \cdot \boldsymbol{r} - \omega_{\mathrm{R}} t)}$,
- Transmitted: $\boldsymbol{E}_{\mathrm{T}}(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{T0}}\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{T}} \cdot \boldsymbol{r} - \omega_{\mathrm{T}} t)}$,

$$\boldsymbol{k}_{\mathrm{I}} \perp \boldsymbol{E}_{\mathrm{I0}},\ \omega_{\mathrm{I}} = k_{\mathrm{I}} \mathrm{c}_1,\ \ \boldsymbol{k}_{\mathrm{R}} \perp \boldsymbol{E}_{\mathrm{R0}},\ \omega_{\mathrm{R}} = k_{\mathrm{R}} \mathrm{c}_1,\ \ \boldsymbol{k}_{\mathrm{T}} \perp \boldsymbol{E}_{\mathrm{T0}},\ \omega_{\mathrm{T}} = k_{\mathrm{T}} \mathrm{c}_2.$$

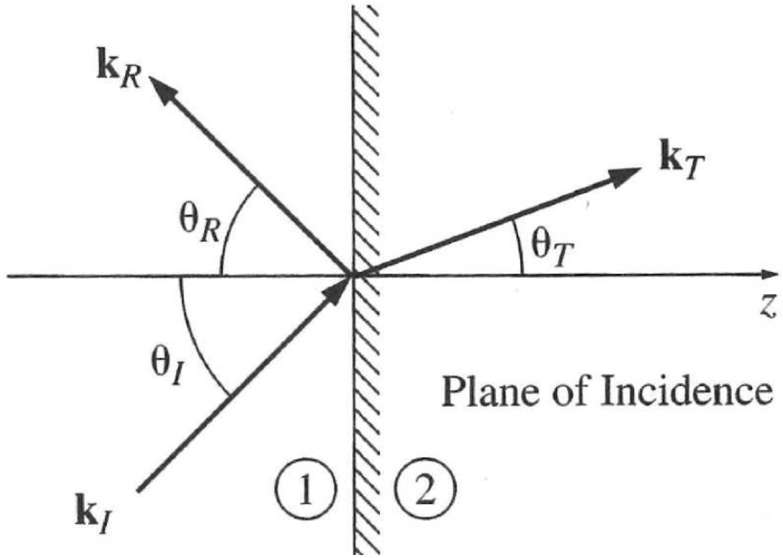
↪ Maxwell equations in regions 1, 2 are satisfied ( . . . ).

## *Plane of incidence*

Interface conditions, formal:

$$( \,.\, )\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{I}} \cdot \boldsymbol{r} - \omega_{\mathrm{I}} t)} + ( \,.\, )\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{R}} \cdot \boldsymbol{r} - \omega_{\mathrm{R}} t)} = ( \,.\, )\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}_{\mathrm{T}} \cdot \boldsymbol{r} - \omega_{\mathrm{T}} t)} \Big|_{\boldsymbol{r}=(x,y,0)}$$

$\forall\, x, y, t$ (!).



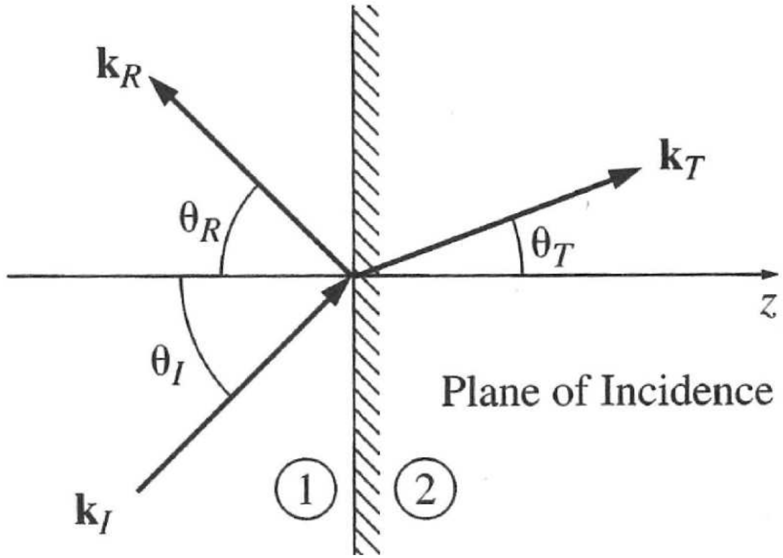
## *Plane of incidence*

Interface conditions, formal:

$$( \, . \, )\, e^{i(\boldsymbol{k}_\mathrm{I}\cdot\boldsymbol{r} - \omega_\mathrm{I} t)} + ( \, . \, )\, e^{i(\boldsymbol{k}_\mathrm{R}\cdot\boldsymbol{r} - \omega_\mathrm{R} t)} = ( \, . \, )\, e^{i(\boldsymbol{k}_\mathrm{T}\cdot\boldsymbol{r} - \omega_\mathrm{T} t)}\Big|_{\boldsymbol{r}=(x,y,0)}$$

$\forall\, x, y, t$ (!).

$$\omega_\mathrm{I} = \omega_\mathrm{R} = \omega_\mathrm{T} =: \omega,$$
$$k_{\mathrm{I}x} = k_{\mathrm{R}x} = k_{\mathrm{T}x} =: k_x,$$
$$k_{\mathrm{I}y} = k_{\mathrm{R}y} = k_{\mathrm{T}y} =: k_y.$$



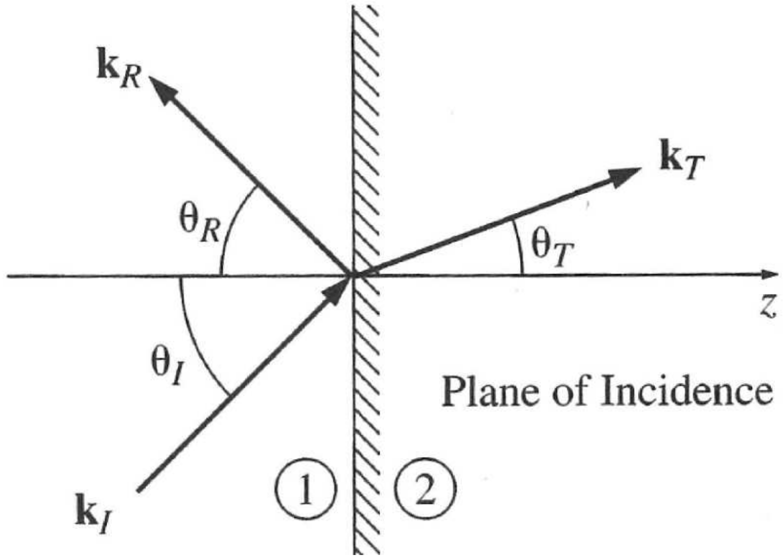
## *Plane of incidence*

Interface conditions, formal:

$$( \, . \, ) \, \mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}_\mathrm{I} \cdot \boldsymbol{r} - \omega_\mathrm{I} t\right)} + ( \, . \, ) \, \mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}_\mathrm{R} \cdot \boldsymbol{r} - \omega_\mathrm{R} t\right)} = ( \, . \, ) \, \mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{r} - \omega_\mathrm{T} t\right)} \Big|_{\boldsymbol{r}=(x,y,0)}$$

$\forall \, x, y, t$ (!).

$$\omega_\mathrm{I} = \omega_\mathrm{R} = \omega_\mathrm{T} =: \omega,$$
$$k_{\mathrm{I}x} = k_{\mathrm{R}x} = k_{\mathrm{T}x} =: k_x,$$
$$k_{\mathrm{I}y} = k_{\mathrm{R}y} = k_{\mathrm{T}y} =: k_y.$$




Orient the coordinates such that $k_{\mathrm{I}y} = k_y = 0$:

$$\boldsymbol{k}_\mathrm{I} = (k_x, 0, k_{\mathrm{I}z}),$$
$$\boldsymbol{k}_\mathrm{R} = (k_x, 0, k_{\mathrm{R}z}),$$
$$\boldsymbol{k}_\mathrm{T} = (k_x, 0, k_{\mathrm{T}z}).$$

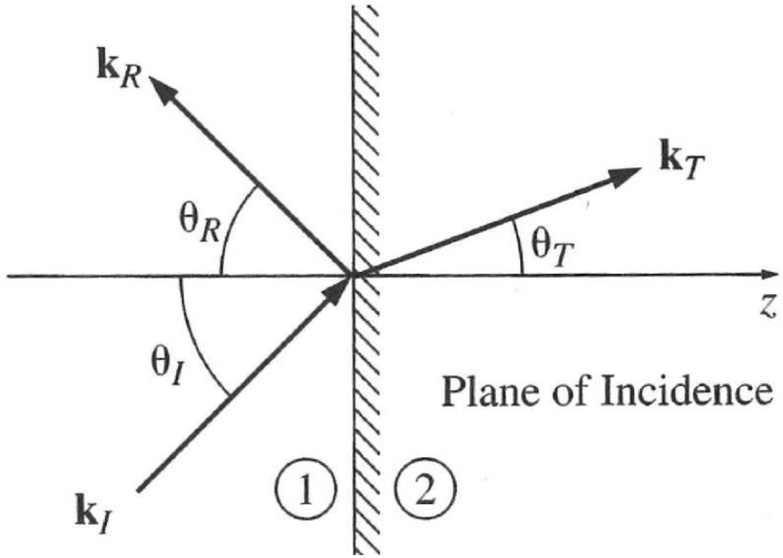
# *Plane of incidence*

Interface conditions, formal:

$$( \,.\, )\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\mathrm{I} \cdot \boldsymbol{r} - \omega_\mathrm{I} t)} + ( \,.\, )\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\mathrm{R} \cdot \boldsymbol{r} - \omega_\mathrm{R} t)} = ( \,.\, )\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{r} - \omega_\mathrm{T} t)} \Big|_{\boldsymbol{r}=(x,y,0)}$$

$\forall\, x, y, t$ (!).

↪
$$\omega_\mathrm{I} = \omega_\mathrm{R} = \omega_\mathrm{T} =: \omega,$$
$$k_{\mathrm{I}x} = k_{\mathrm{R}x} = k_{\mathrm{T}x} =: k_x,$$
$$k_{\mathrm{I}y} = k_{\mathrm{R}y} = k_{\mathrm{T}y} =: k_y.$$




Orient the coordinates such that $k_{\mathrm{I}y} = k_y = 0$:

↪
$$\boldsymbol{k}_\mathrm{I} = (k_x, 0, k_{\mathrm{I}z}),$$
$$\boldsymbol{k}_\mathrm{R} = (k_x, 0, k_{\mathrm{R}z}),$$
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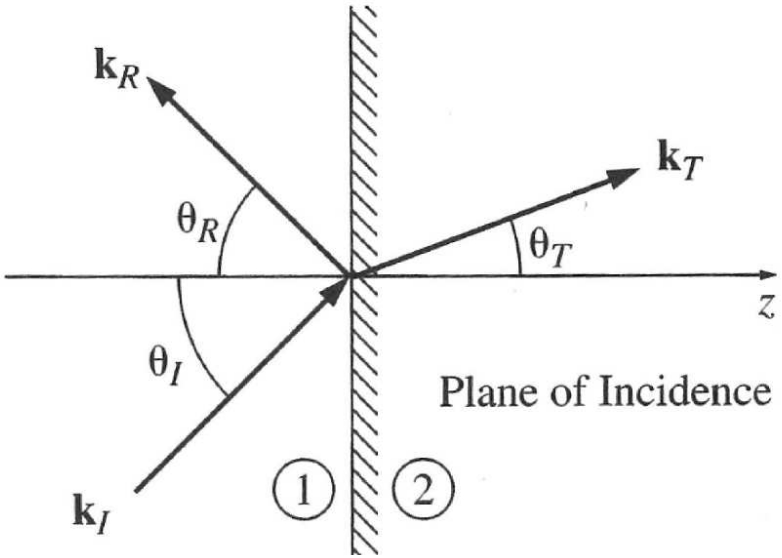
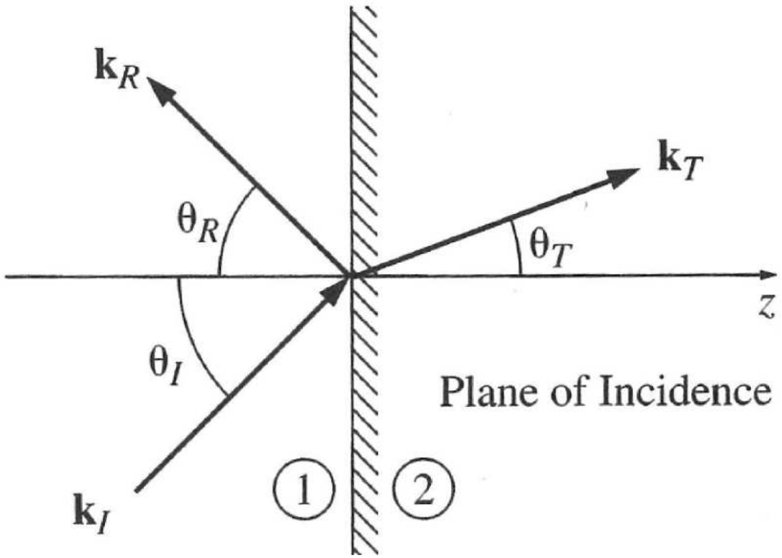
⇝ Plane of incidence spanned by $\boldsymbol{k}_\mathrm{I}$ and the interface normal $\boldsymbol{e}_z$.

# *Reflection and refraction*




$$k_\mathrm{I} = \frac{\omega}{c_1} = k_0 n_1 = k_\mathrm{R}, \quad k_\mathrm{T} = \frac{\omega}{c_2} = k_0 n_2$$

## *Reflection and refraction*



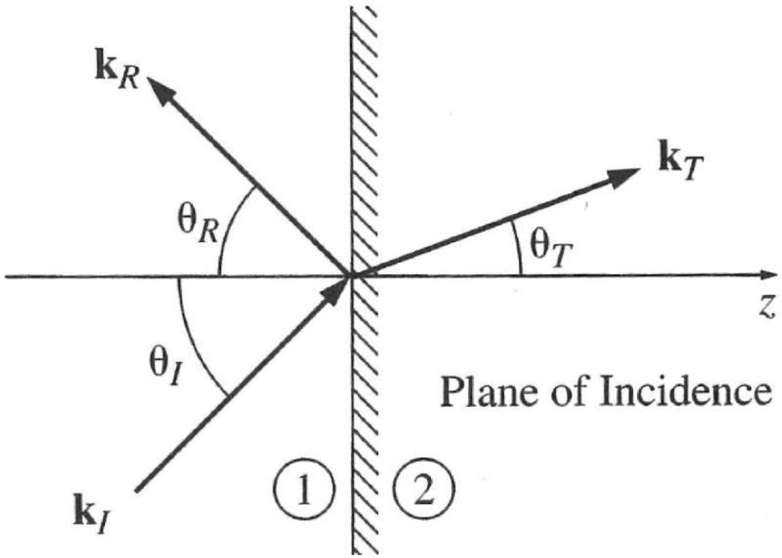

$$k_{\rm I} = \frac{\omega}{c_1} = k_0 n_1 = k_{\rm R}, \quad k_{\rm T} = \frac{\omega}{c_2} = k_0 n_2$$

$$\boldsymbol{k}_{\rm I} = k_0 n_1 (\sin\theta_1, 0, \cos\theta_1),$$

$$\boldsymbol{k}_{\rm R} = k_0 n_1 (\sin\theta_1, 0, -\cos\theta_1),$$

$$\boldsymbol{k}_{\rm T} = k_0 n_2 (\sin\theta_2, 0, \cos\theta_2).$$

## *Reflection and refraction*

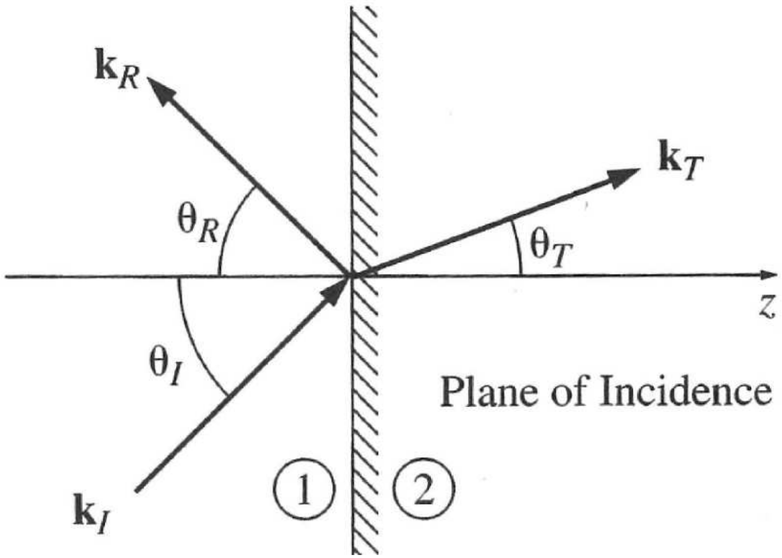



$$k_{\mathrm{I}} = \frac{\omega}{c_1} = k_0 n_1 = k_{\mathrm{R}}, \quad k_{\mathrm{T}} = \frac{\omega}{c_2} = k_0 n_2$$

$$\boldsymbol{k}_{\mathrm{I}} = k_0 n_1 (\sin\theta_1, 0, \cos\theta_1),$$
$$\boldsymbol{k}_{\mathrm{R}} = k_0 n_1 (\sin\theta_1, 0, -\cos\theta_1),$$
$$\boldsymbol{k}_{\mathrm{T}} = k_0 n_2 (\sin\theta_2, 0, \cos\theta_2).$$

- $\theta_{\mathrm{I}} = \theta_{\mathrm{R}} = \theta_1$, angle of incidence $=$ angle of reflection.

# ***Reflection and refraction***

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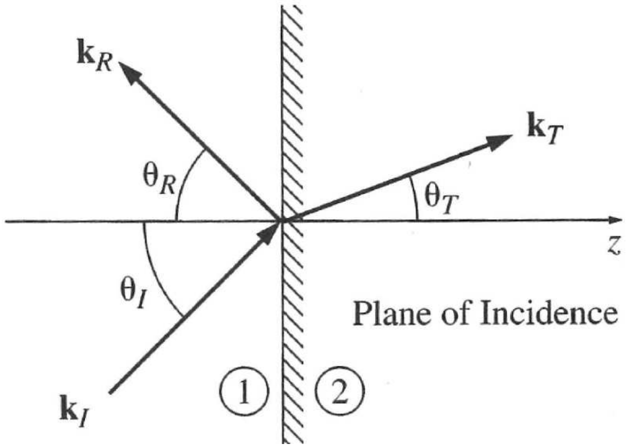
$$\boldsymbol{k}_{\mathrm{R}} = k_0 n_1 (\sin\theta_1, 0, -\cos\theta_1),$$

$$\boldsymbol{k}_{\mathrm{T}} = k_0 n_2 (\sin\theta_2, 0, \cos\theta_2).$$

- $\theta_{\mathrm{I}} = \theta_{\mathrm{R}} = \theta_1$, angle of incidence $=$ angle of reflection.
- $n_1 \sin\theta_1 = n_2 \sin\theta_2$, Snell's law, law of refraction.

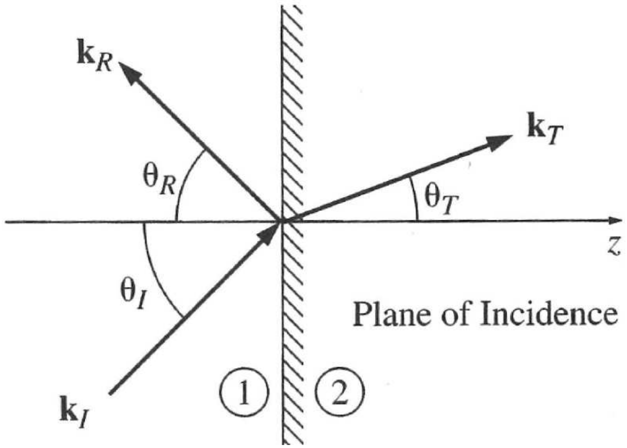
# *Reflection and transmission*




$$\boldsymbol{E}_1(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{I0}}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_1(\sin\theta_1\, x + \cos\theta_1\, z) - \omega t\right)} + \boldsymbol{E}_{\mathrm{R0}}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_1(\sin\theta_1\, x - \cos\theta_1\, z) - \omega t\right)},$$

$$\boldsymbol{E}_2(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{T0}}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_2(\sin\theta_2\, x + \cos\theta_2\, z) - \omega t\right)}.$$

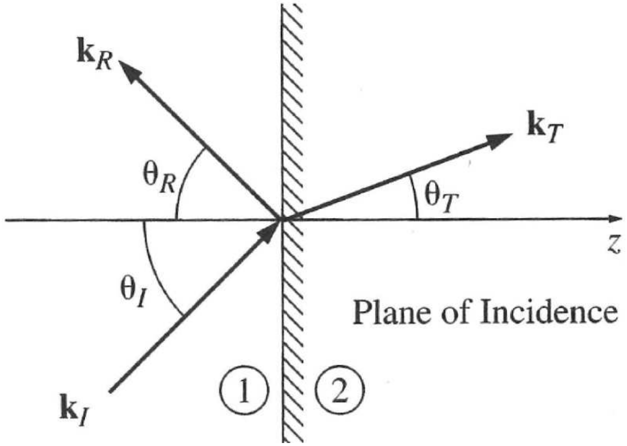
## *Reflection and transmission*




$$\begin{aligned} \boldsymbol{E}_1(\boldsymbol{r},t) &= \boldsymbol{E}_{\mathrm{I0}}\,\mathrm{e}^{\mathrm{i}\,(k_0 n_1(\sin\theta_1\,x+\cos\theta_1\,z)-\omega t)} \\ &+ \boldsymbol{E}_{\mathrm{R0}}\,\mathrm{e}^{\mathrm{i}\,(k_0 n_1(\sin\theta_1\,x-\cos\theta_1\,z)-\omega t)}, \\ \boldsymbol{E}_2(\boldsymbol{r},t) &= \boldsymbol{E}_{\mathrm{T0}}\,\mathrm{e}^{\mathrm{i}\,(k_0 n_2(\sin\theta_2\,x+\cos\theta_2\,z)-\omega t)}. \end{aligned}$$

- $\boldsymbol{\nabla}\times\boldsymbol{E} = -\mu_0\mu\dot{\boldsymbol{H}}$ ⤳ $\boldsymbol{H}_{j0} = \dfrac{1}{\omega\mu_0\mu_{1,2}}\boldsymbol{k}_j\times\boldsymbol{E}_{j0},\quad j=\mathrm{I},\mathrm{R},\mathrm{T}.$

# *Reflection and transmission*



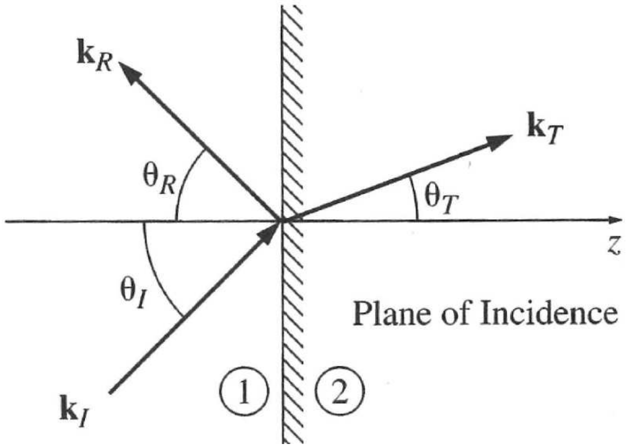

$$\boldsymbol{E}_1(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{I0}}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_1 (\sin\theta_1\, x + \cos\theta_1\, z) - \omega t\right)} + \boldsymbol{E}_{\mathrm{R0}}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_1 (\sin\theta_1\, x - \cos\theta_1\, z) - \omega t\right)},$$

$$\boldsymbol{E}_2(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{T0}}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_2 (\sin\theta_2\, x + \cos\theta_2\, z) - \omega t\right)}.$$

- $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}$ ⟿ $\boldsymbol{H}_{j0} = \dfrac{1}{\omega \mu_0 \mu_{1,2}} \boldsymbol{k}_j \times \boldsymbol{E}_{j0}, \quad j = \mathrm{I, R, T}.$

- Assume $\boldsymbol{E}_{\mathrm{I0}} = E_{\mathrm{I0}} \boldsymbol{e}_y$, $\boldsymbol{E}_{\mathrm{R0}} = E_{\mathrm{R0}} \boldsymbol{e}_y$, $\boldsymbol{E}_{\mathrm{T0}} = E_{\mathrm{T0}} \boldsymbol{e}_y$ *(Lecture F, 2-D TE)*
  s-polarization, polarization of $\boldsymbol{E}_{\mathrm{I}} \perp$ plane of incidence.

## *Reflection and transmission*




$$\boldsymbol{E}_1(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{I}0}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_1 (\sin\theta_1\, x + \cos\theta_1\, z) - \omega t\right)} + \boldsymbol{E}_{\mathrm{R}0}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_1 (\sin\theta_1\, x - \cos\theta_1\, z) - \omega t\right)},$$

$$\boldsymbol{E}_2(\boldsymbol{r},t) = \boldsymbol{E}_{\mathrm{T}0}\, \mathrm{e}^{\mathrm{i}\left(k_0 n_2 (\sin\theta_2\, x + \cos\theta_2\, z) - \omega t\right)}.$$

- $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}$ $\rightsquigarrow$ $\boldsymbol{H}_{j0} = \frac{1}{\omega \mu_0 \mu_{1,2}} \boldsymbol{k}_j \times \boldsymbol{E}_{j0}, \quad j = \mathrm{I}, \mathrm{R}, \mathrm{T}.$

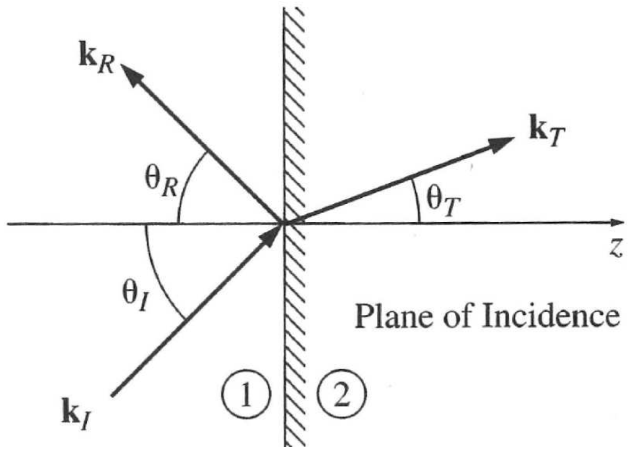
- Assume $\boldsymbol{E}_{\mathrm{I}0} = E_{\mathrm{I}0}\boldsymbol{e}_y,\ \boldsymbol{E}_{\mathrm{R}0} = E_{\mathrm{R}0}\boldsymbol{e}_y,\ \boldsymbol{E}_{\mathrm{T}0} = E_{\mathrm{T}0}\boldsymbol{e}_y$ *(Lecture F, 2-D TE)*
  s-polarization, polarization of $\boldsymbol{E}_{\mathrm{I}} \perp$ plane of incidence.

$\hookrightarrow$

$$\boldsymbol{H}_{\mathrm{I}0} = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_1}{\mu_1} E_{\mathrm{I}0}(-\cos\theta_1, 0, \sin\theta_1),\ \ \boldsymbol{H}_{\mathrm{R}0} = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_1}{\mu_1} E_{\mathrm{R}0}(\cos\theta_1, 0, \sin\theta_1),$$

$$\boldsymbol{H}_{\mathrm{T}0} = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_2}{\mu_2} E_{\mathrm{T}0}(-\cos\theta_2, 0, \sin\theta_2).$$

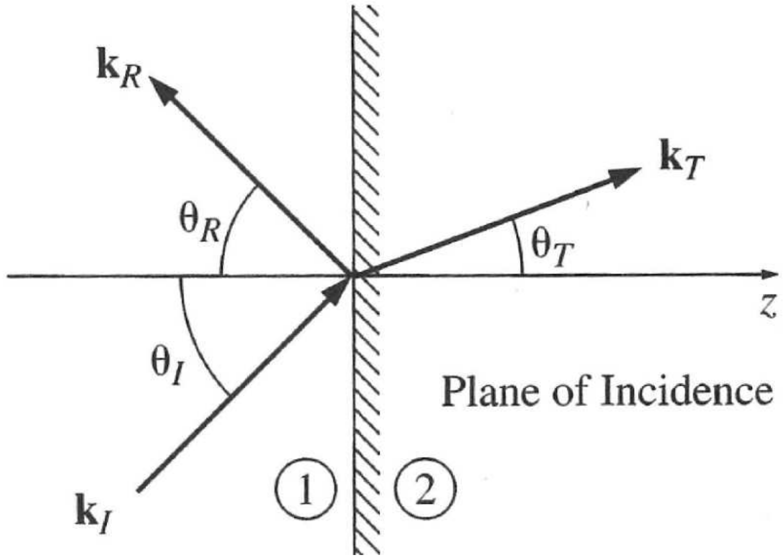
# Reflection and transmission, fields




$$
\begin{aligned}
\boldsymbol{E}_1(\boldsymbol{r},t) &= E_{\mathrm{I0}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\,(k_0 n_1 (\sin\theta_1\, x + \cos\theta_1\, z) - \omega t)} \\
&+ E_{\mathrm{R0}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\,(k_0 n_1 (\sin\theta_1\, x - \cos\theta_1\, z) - \omega t)}, \\
\boldsymbol{E}_2(\boldsymbol{r},t) &= E_{\mathrm{T0}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\,(k_0 n_2 (\sin\theta_2\, x + \cos\theta_2\, z) - \omega t)},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{H}_1(\boldsymbol{r},t) &= \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{n_1}{\mu_1}\, E_{\mathrm{I0}} \begin{pmatrix} -\cos\theta_1 \\ 0 \\ \sin\theta_1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\,(k_0 n_1 (\sin\theta_1\, x + \cos\theta_1\, z) - \omega t)} \\
&+ \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{n_1}{\mu_1}\, E_{\mathrm{R0}} \begin{pmatrix} \cos\theta_1 \\ 0 \\ \sin\theta_1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\,(k_0 n_1 (\sin\theta_1\, x - \cos\theta_1\, z) - \omega t)}, \\
\boldsymbol{H}_2(\boldsymbol{r},t) &= \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{n_2}{\mu_2}\, E_{\mathrm{T0}} \begin{pmatrix} -\cos\theta_2 \\ 0 \\ \sin\theta_2 \end{pmatrix} \mathrm{e}^{\mathrm{i}\,(k_0 n_2 (\sin\theta_2\, x + \cos\theta_2\, z) - \omega t)}.
\end{aligned}
$$

## *Reflection and transmission, interface conditions*




Interface conditions at $z = 0$ $(\sigma_{\rm f} = 0, \boldsymbol{K}_{\rm f} = 0)$:

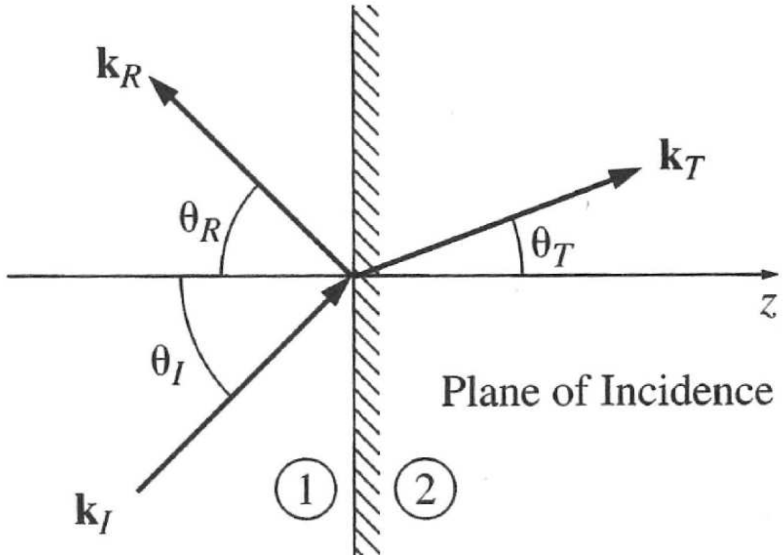
1.) $H^{\parallel}$ continuous, $H_{1x} = H_{2x}, \quad H_{1y} = H_{2y},$

2.) $B^{\perp}$ continuous, $\mu_0\mu_1 H_{1z} = \mu_0\mu_2 H_{2z},$

3.) $E^{\parallel}$ continuous, $E_{1x} = E_{2x}, \quad E_{1y} = E_{2y},$

4.) $D^{\perp}$ continuous, $\epsilon_0\epsilon_1 E_{1z} = \epsilon_0\epsilon_2 E_{2z}.$

## *Reflection and transmission, interface conditions*

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1.) $H^{\parallel}$ continuous, $H_{1x} = H_{2x}, \quad H_{1y} = H_{2y},$

2.) $B^{\perp}$ continuous, $\mu_0\mu_1 H_{1z} = \mu_0\mu_2 H_{2z},$

3.) $E^{\parallel}$ continuous, $E_{1x} = E_{2x}, \quad E_{1y} = E_{2y},$

4.) $D^{\perp}$ continuous, $\epsilon_0\epsilon_1 E_{1z} = \epsilon_0\epsilon_2 E_{2z}.$

$$\hookrightarrow \quad E_{\mathrm{I}0} + E_{\mathrm{R}0} = E_{\mathrm{T}0}, \quad \frac{n_1}{\mu_1}(E_{\mathrm{I}0} - E_{\mathrm{R}0})\cos\theta_1 = \frac{n_2}{\mu_2} E_{\mathrm{T}0}\cos\theta_2.$$

# ***Reflection and transmission, amplitudes***

s-polarization, $\boldsymbol{E} \perp$ plane of incidence:

$$E_{\mathrm{T0}}^{(\mathrm{s})} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + \dfrac{\mu_1}{\mu_2} n_2 \cos\theta_2} E_{\mathrm{I0}}^{(\mathrm{s})}, \qquad E_{\mathrm{R0}}^{(\mathrm{s})} = \frac{n_1 \cos\theta_1 - \dfrac{\mu_1}{\mu_2} n_2 \cos\theta_2}{n_1 \cos\theta_1 + \dfrac{\mu_1}{\mu_2} n_2 \cos\theta_2} E_{\mathrm{I0}}^{(\mathrm{s})}.$$

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p-polarization, $\boldsymbol{E} \parallel$ plane of incidence: *(exercise)*

$$E_{\mathrm{T0}}^{(\mathrm{p})} = \frac{2n_1 \cos\theta_1}{\dfrac{\mu_1}{\mu_2} n_2 \cos\theta_1 + n_1 \cos\theta_2} E_{\mathrm{I0}}^{(\mathrm{p})}, \quad E_{\mathrm{R0}}^{(\mathrm{p})} = \frac{\dfrac{\mu_1}{\mu_2} n_2 \cos\theta_1 - n_1 \cos\theta_2}{\dfrac{\mu_1}{\mu_2} n_2 \cos\theta_1 + n_1 \cos\theta_2} E_{\mathrm{I0}}^{(\mathrm{p})}.$$

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- $n_1\sin\theta_1 = n_2\sin\theta_2 \quad \rightsquigarrow \quad \cos\theta_2 = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_1}.$

## *Fresnel equations*

For $\mu_1 = \mu_2$, specifically for $\omega \in$ optical regime, $\mu_1 = \mu_2 = 1$:

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Alternative form: eliminate $n_1, n_2$ using Snell's law.

At normal incidence $\theta_1 = \theta_2 = 0$, (s) $\tilde{=}$ (p):

$$\frac{E_{\mathrm{T0}}}{E_{\mathrm{I0}}} = \frac{2n_1}{n_1 + n_2}, \qquad \frac{E_{\mathrm{R0}}}{E_{\mathrm{I0}}} = \pm\frac{n_1 - n_2}{n_1 + n_2}.$$

## ***Transmittance and reflectance***

- Power flow density $\overline{\boldsymbol{S}}$ for the plane waves: $\overline{\boldsymbol{S}} = \frac{1}{2}\sqrt{\frac{\epsilon_0 \epsilon}{\mu_0 \mu}}|E_0|^2 \frac{\boldsymbol{k}}{k}.$

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- Transmittance $T$, transmission coefficient: $T = \frac{I_\mathrm{T}}{I_\mathrm{I}} = \sqrt{\frac{\epsilon_2\mu_1}{\mu_2\epsilon_1}}\left|\frac{E_\mathrm{T0}}{E_\mathrm{I0}}\right|^2\frac{\cos\theta_2}{\cos\theta_1},$

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- Power balance: $I_{\mathrm{I}} = I_{\mathrm{R}} + I_{\mathrm{T}}, \quad R + T = 1.$ *(exercise)*

## ***Brewster's angle***

- Reflectance for (p)-polarization:

$$R^{(\mathrm{p})} = \frac{I_\mathrm{R}}{I_\mathrm{I}} = \left|\frac{E_{\mathrm{R}0}}{E_{\mathrm{I}0}}\right|^2 \quad \text{with} \quad \frac{E^{(\mathrm{p})}_{\mathrm{R}0}}{E^{(\mathrm{p})}_{\mathrm{I}0}} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}.$$

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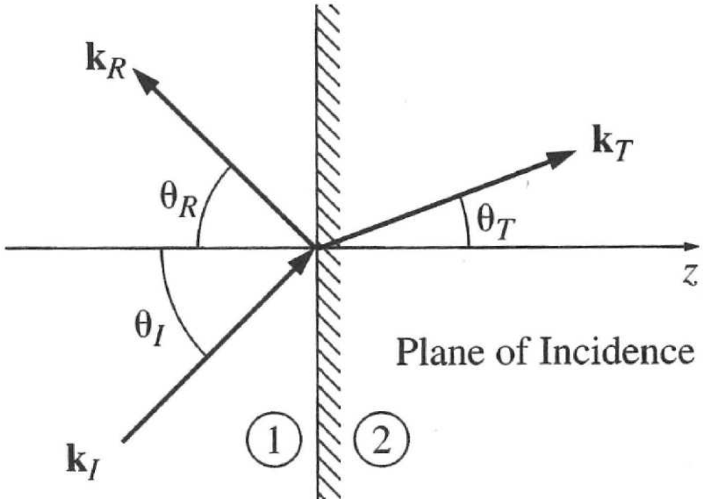
Unpolarized incidence at $\theta_1 = \theta_\mathrm{B}$:

⇝ The reflected waves are purely s-polarized,
p-polarized waves are fully transmitted.

# *Total internal reflection*

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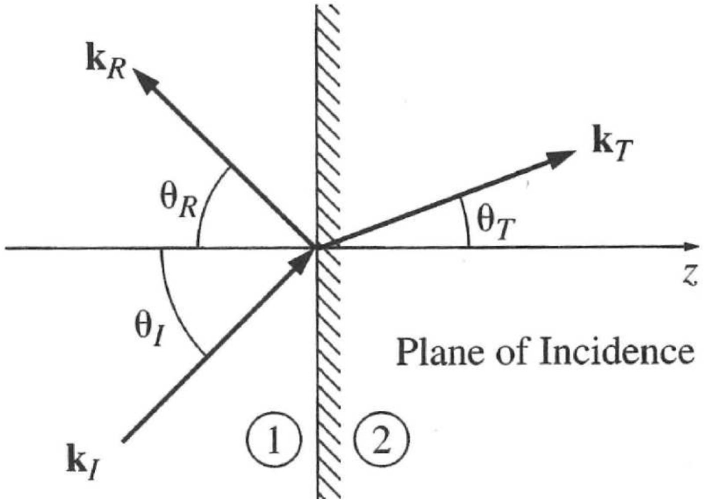


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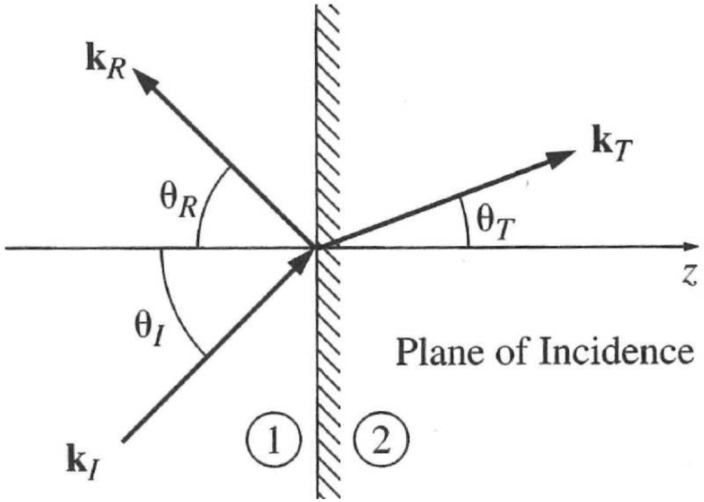


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Incidence at $\theta_1 > \theta_c$: *(exercise)*

- no transmitted propagating wave, evanescent fields in region 2,
- the incident power is fully reflected back into region 1.

## *Wave propagation in conductors*

Specifically: homogeneous isotropic conductors, linear media.

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$$\hookrightarrow \quad \dot{\rho}_{\mathrm{f}} = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_{\mathrm{f}}, \quad \rho_{\mathrm{f}}(\boldsymbol{r}, t) = \rho_{\mathrm{f}}(\boldsymbol{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0)\right),$$

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# *Telegrapher equation*

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$$\Delta \boldsymbol{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\boldsymbol{E}} - \mu_0 \mu \sigma \dot{\boldsymbol{E}} = 0, \quad \Delta \boldsymbol{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\boldsymbol{H}} - \mu_0 \mu \sigma \dot{\boldsymbol{H}} = 0,$$

Telegrapher equation.

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Telegrapher equation.

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Frequency domain: $\boldsymbol{E}(\boldsymbol{r},t)=\tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$,

$$\Delta\tilde{\boldsymbol{E}}+\left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu+\mathrm{i}\omega\mu_0\mu\sigma\right)\tilde{\boldsymbol{E}}=0.$$

Nonconducting media $\sigma=0,\quad \Delta\tilde{\boldsymbol{E}}+\left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\right)\tilde{\boldsymbol{E}}=0.$

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Define $\bar{\epsilon}$ such that $\frac{\omega^2}{\mathrm{c}^2}\bar{\epsilon}\mu=\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu+\mathrm{i}\omega\mu_0\mu\sigma$, i.e. $\bar{\epsilon}=\epsilon+\mathrm{i}\,\frac{\sigma}{\epsilon_0\omega}$

$k_0=\frac{\omega}{\mathrm{c}}$

↪ $\Delta\tilde{\boldsymbol{E}}+k_0^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}}=0,$ Helmholtz equation, $\bar{\epsilon}\in\mathbb{C}$.

## *Wave attenuation*

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↪ solutions $\sim$ e$^{\mathrm{i}\,(k_0 \bar{n} z - \omega t)}$

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$$\mathrm{e}^{\mathrm{i}(k_0\bar{n}z - \omega t)} = \mathrm{e}^{\mathrm{i}(k_0 n' z - \omega t)}\, \mathrm{e}^{-k_0 n'' z},$$

damped plane wave solutions.

Issues:

- penetration depth,
- $\boldsymbol{S}$ and $u$ decay with $z$,
- still transverse waves,
- $\boldsymbol{E}$, $\boldsymbol{H}$ no longer in phase,
- notions of wavenumber $\in \mathbb{C}$, wavelength, phase velocity.

## ***Localized waves***

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- $\omega = \frac{k\mathrm{c}}{n(\omega)}$, solve for $\omega(k)$: dispersion relation.

---

$$\Psi(z,t) = \int f(k)\, \mathrm{e}^{\mathrm{i}(kz - \omega(k)\, t)} \mathrm{d}k.$$

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- Assume $f(k)$ is sharply peaked around $k_\mathrm{p}$, (⟵⟿ weak localization of $\Psi$ vs. $z$) such that only $\omega(k)$ for $k$ close to $k_\mathrm{p}$ contribute; expand $\omega(k)$ around $k_\mathrm{p}$:

$$\omega(k) \approx \omega(k_\mathrm{p}) + (k - k_\mathrm{p}) \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_\mathrm{p}}, \qquad \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_\mathrm{p}} = v_\mathrm{g}, \text{ group velocity}.$$

# *Wave packets*

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$\hookrightarrow$

$$\Psi(z,t) = \mathrm{e}^{\mathrm{i}\,(k_\mathrm{p} z - \omega(k_\mathrm{p})\,t)}\,\hat{\Psi}(z - v_\mathrm{g} t),$$

$$\hat{\Psi}(z - v_\mathrm{g} t) = \int f(k_\mathrm{p} + q)\,\mathrm{e}^{\mathrm{i}\,q(z - v_\mathrm{g} t)}\mathrm{d}q.$$

## *Wave packets*

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Wave packet $\Psi(z,t)$: here a plane wave $(k_\mathrm{p}, \omega(k_\mathrm{p}))$, modulated by an envelope $\hat{\Psi}$ that moves with velocity $v_\mathrm{g}$ in direction $z$.

## *Wave packets*

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- Simplest case: *no dispersion*, $\omega(k) = \dfrac{k\mathrm{c}}{n}$, $n$ const.

↪ $\dfrac{\mathrm{d}\omega}{\mathrm{d}k} = \dfrac{\mathrm{c}}{n} = \mathrm{c_m} = \dfrac{\omega(k)}{k}$, phase velocity $\mathrm{c_m}$ = group velocity $v_\mathrm{g}$,

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## *Wave packets*

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## *Wave packets*

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- Dispersive case: $v_\mathrm{g} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k} \neq \dfrac{\omega(k)}{k} = \mathrm{c}_\mathrm{m}.$

- Group velocity $v_\mathrm{g}$ :
  - velocity for energy transport of the wave packet,
  - velocity for signal / information transport,
  - $v_\mathrm{g} \leq \mathrm{c}.$

## *Upcoming*

Next lecture:

- Maxwell equations, vectorial and scalar Helmholtz equation,
- 2D configurations (examples),
- mode problems, metallic and dielectric waveguides (examples).