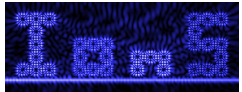


Electrodynamics

— Lecture A —

UNIVERSITY
OF TWENTE.

MESA+
INSTITUTE FOR NANOTECHNOLOGY



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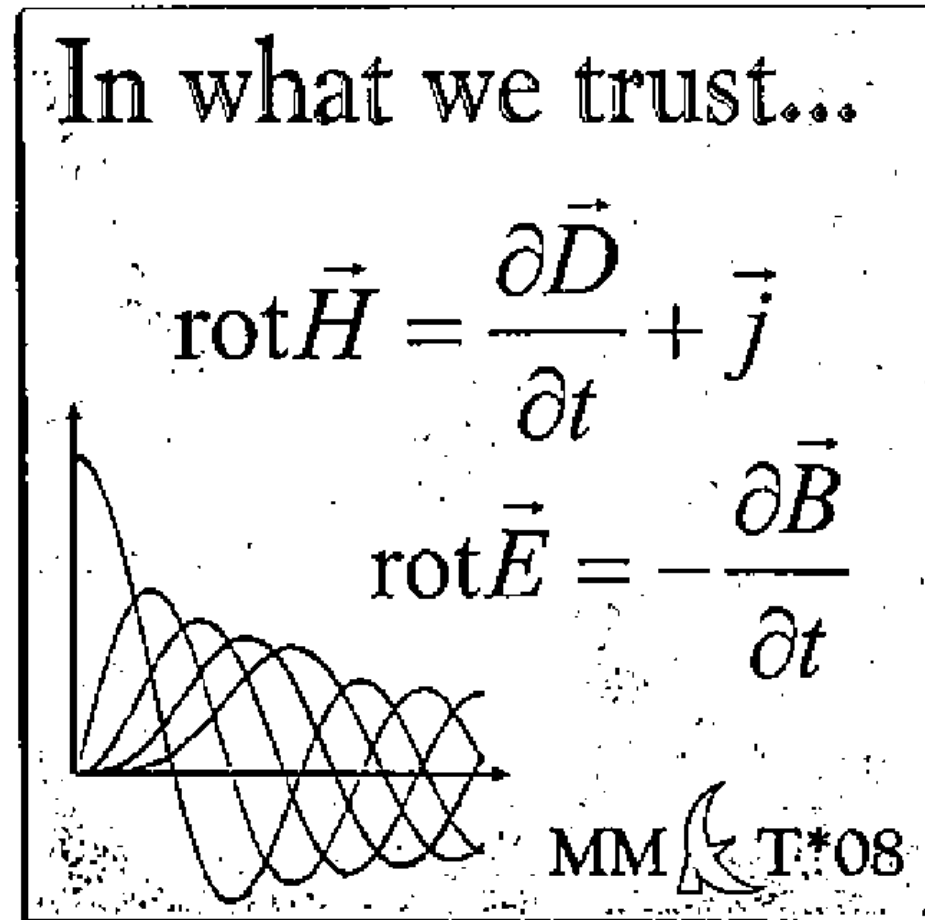
Integrated Optical MicroSystems
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Maxwell equations

SI, in matter, time domain, differential form:

$\nabla \cdot \mathbf{D} = \rho_f,$	$\mathbf{E}(\mathbf{r}, t)$: electric field,
$\nabla \times \mathbf{E} = -\dot{\mathbf{B}},$	$\mathbf{D}(\mathbf{r}, t)$: (di-)electric displacement,
$\nabla \cdot \mathbf{B} = 0,$	$\mathbf{B}(\mathbf{r}, t)$: magnetic induction (field, flux density),
$\nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}},$	$\mathbf{H}(\mathbf{r}, t)$: magnetic field (...),
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},$	$\rho_f(\mathbf{r}, t)$: density of free charges,
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}).$	$\mathbf{J}_f(\mathbf{r}, t)$: density of free currents,
	$\mathbf{P}(\mathbf{r}, t)$: polarization,
	$\mathbf{M}(\mathbf{r}, t)$: magnetization,
	ϵ_0 : free space permittivity,
(+ constitutive relations)	μ_0 : free space permeability.

Valid for more than a century, firm basis for further considerations.

Course overview

Electrodynamics

- A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,
- B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion
- C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields
- D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves
- E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity
- F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides
- G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation
- H Special relativity; transmission lines

Formalities

Organization of the course:

- Lectures ($9\times$)
- Homework ($7\times$)
- Tutorials ($8\times$)
- Intermediate tests ($2\times$)
- Final exam
(~~textbooks~~, ~~laptops~~, ~~calculators~~, (~~smart~~)phones, (~~...~~), 1 A4 sheet .)

Textbook:

Introduction to Electrodynamics, D.J. Griffiths, 3rd intl. ed., Prentice Hall, 2003

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Space and time coordinates: $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow (x, y, z), t.$
- Scalar and vector fields: $\phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t), \mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$
- Inner product: $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$
- Vector product: $\mathbf{A} \times \mathbf{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}.$
- Time derivatives: $\frac{\partial \phi}{\partial t}, \partial_t \phi, \dot{\phi}, \nabla_t \phi.$

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Del, nabla: $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$
- Gradient: $\text{grad}\phi = \nabla\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix}.$
- Divergence: $\text{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z.$
- Curl: $\text{curl}\mathbf{A} = \text{rot}\mathbf{A} = \nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}.$
- Laplacian: $\Delta = \nabla \cdot \nabla = \nabla^2, \Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi, \Delta\mathbf{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$

Integral theorems

- Gradient theorem:
$$\int_a^b \nabla \phi \cdot d\mathbf{s} = \phi(\mathbf{b}) - \phi(\mathbf{a}),$$

arbitrary paths $\mathbf{a} \rightarrow \mathbf{b}$.

- Divergence theorem
(Gauss', Green's):
$$\int_{\mathcal{V}} \nabla \cdot \mathbf{A} d\mathcal{V} = \int_{\partial\mathcal{V}} \mathbf{A} \cdot d\mathbf{a},$$

$d\mathbf{a}/|d\mathbf{a}|$ an outward oriented normal vector on the surface $\partial\mathcal{V}$ of \mathcal{V} .

- Curl theorem
(Stokes'):
$$\int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\partial\mathcal{S}} \mathbf{A} \cdot d\mathbf{s},$$

sign consistency required for $d\mathbf{a}$ and $d\mathbf{s}$ (right hand rule).

- Variants & corollaries.

Dirac delta

A linear functional that extracts the value of a function at one point (...):

$$\text{1-D: } \int_a^b f(x) \delta(x - x_0) \, dx = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x - x_0) = 0, \text{ if } x \neq x_0.$$

$$\text{3-D: } \int_{\mathcal{V}} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) \, d\mathcal{V} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(\mathbf{r} - \mathbf{r}_0) = 0, \text{ if } \mathbf{r} \neq \mathbf{r}_0.$$


Implications: manifold.

Laplace of $1/r$

Example:

- $\Delta \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = 0$, if $\mathbf{r} \neq \mathbf{r}_0$,
- $\int_{\mathcal{V}} \Delta \frac{1}{|\mathbf{r} - \mathbf{r}_0|} d\mathcal{V} = \begin{cases} -4\pi, & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$

(exercise)

 $-\frac{1}{4\pi} \int_{\mathcal{V}} f(\mathbf{r}) \Delta \frac{1}{|\mathbf{r} - \mathbf{r}_0|} d\mathcal{V} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$

$$-\frac{1}{4\pi} \Delta \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = \delta(\mathbf{r} - \mathbf{r}_0).$$

Potentials

$\mathbf{F}(\mathbf{r})$: a well-behaved vector field, $\mathbf{F}(\mathbf{r} \rightarrow \infty) \rightarrow 0$.



Then \mathbf{F} can be written

$$\mathbf{F} = \nabla\phi + \nabla \times \mathbf{A},$$

with (e.g.)

$$\phi(\mathbf{r}) = -\frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathcal{V}', \quad \text{and} \quad \mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\nabla \times \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathcal{V}'.$$

(proof)

- \mathbf{F} is fully determined by $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.
- $\nabla \times \mathbf{F} = 0$  $\mathbf{F} = \nabla\phi, \quad \Delta\phi = \nabla \cdot \mathbf{F}.$
- $\nabla \cdot \mathbf{F} = 0$  $\mathbf{F} = \nabla \times \mathbf{A}, \quad \Delta\mathbf{A} = -\nabla \times \mathbf{F}.$



Potentials, sources

$\mathbf{F}(\mathbf{r})$: a well-behaved vector field, $\mathbf{F}(\mathbf{r} \rightarrow \infty) \rightarrow 0$,
with given $s = \nabla \cdot \mathbf{F}$ and $\mathbf{c} = \nabla \times \mathbf{F}$. Then \mathbf{F} can be written

$$\mathbf{F} = \nabla \phi + \nabla \times \mathbf{A},$$

with (e.g.)

$$\phi(\mathbf{r}) = -\frac{1}{4\pi} \int \frac{s(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathcal{V}', \quad \text{and} \quad \mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\mathbf{c}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathcal{V}'.$$

- \mathbf{F} is fully determined by s and \mathbf{c} .
- $\nabla \times \mathbf{F} = 0, \nabla \cdot \mathbf{F} = s$  $\mathbf{F} = \nabla \phi, \quad \Delta \phi = s.$
- $\nabla \cdot \mathbf{F} = 0, \nabla \times \mathbf{F} = \mathbf{c}$  $\mathbf{F} = \nabla \times \mathbf{A}, \quad \Delta \mathbf{A} = -\mathbf{c}.$

Taylor expansion

1-D: A function $f(x)$ of one variable, expansion around $x = x_0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \rightsquigarrow \quad f(x_0 + \Delta x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) \Delta x^n.$$

3-D: Scalar field $\phi(\mathbf{r})$, expansion around $\mathbf{r} = \mathbf{r}_0$:

$$\phi(\mathbf{r}_0 + \Delta \mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\Delta \mathbf{r} \cdot \nabla)^n \phi(\mathbf{r}_0) = [\exp(\Delta \mathbf{r} \cdot \nabla)] \phi(\mathbf{r}_0).$$

(derivation)

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2$.
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k)$.
- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k)$.
- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$.
- $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$.

Fourier transform

3-D: A field $\phi(\mathbf{r})$:

$$\phi(\mathbf{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3k, \quad \tilde{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3r.$$

4-D: A field $\phi(\mathbf{r}, t)$:

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}^4} \int \tilde{\phi}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3k d\omega,$$

$$\tilde{\phi}(\mathbf{k}, \omega) = \frac{1}{\sqrt{2\pi}^4} \int \phi(\mathbf{r}, t) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3r dt.$$

Upcoming

Next lecture:

- Maxwell equations
- Brush up on electro- and magnetostatics
- Multipole expansion

