Electrodynamics

— Lecture C —

UNIVERSITY OF TWENTE.

MESA+ INSTITUTE FOR NANOTECHNOLOGY

Manfred Hammer*, Hugo Hoekstra,
Lantian Chang, Mustafa Sefünç, Yean-Sheng Yong

Integrated Optical MicroSystems
MESA$^+$ Institute for Nanotechnology
University of Twente, The Netherlands

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* *Department of Electrical Engineering, University of Twente* *P.O. Box 217, 7500 AE Enschede, The Netherlands*
Phone: +31/53/489-3448 *Fax: +31/53/489-3996* *E-mail: m.hammer@utwente.nl*

Course overview

Electrodynamics

- A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,
- B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion
- C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields
- D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves
- E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity
- F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides
- G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation
- H Special relativity; transmission lines

Maxwell equations

SI, in matter, time domain, differential form:

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho_{\mathrm{f}}, \\ \nabla \times \boldsymbol{E} &= -\dot{\boldsymbol{B}}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J}_{\mathrm{f}} + \dot{\boldsymbol{D}}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{D} &= \epsilon_0 \boldsymbol{E} + \boldsymbol{P}, \\ \boldsymbol{B} &= \mu_0 (\boldsymbol{H} + \boldsymbol{M}). \end{aligned}$$

(+ constitutive relations)

- $\boldsymbol{E}(\boldsymbol{r}, t)$: electric field,
- $\boldsymbol{D}(\boldsymbol{r}, t)$: (di-)electric displacement,
- $\boldsymbol{B}(\boldsymbol{r}, t)$: magnetic induction (field, flux density),
- $\boldsymbol{H}(\boldsymbol{r}, t)$: magnetic field (. . .),
- $\rho_{\mathrm{f}}(\boldsymbol{r}, t)$: density of free charges,
- $\boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r}, t)$: density of free currents,
- $\boldsymbol{P}(\boldsymbol{r}, t)$: polarization,
- $\boldsymbol{M}(\boldsymbol{r}, t)$: magnetization,
- ϵ_0: free space permittivity,
- μ_0: free space permeability.

$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ (Lorentz-) force $\boldsymbol{F}$ on a charge q with velocity $\boldsymbol{v}$.

Maxwell equations

- Macroscopic equations, linear isotropic media:
  $$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\rm f},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\rm f}+\dot{\boldsymbol{D}},$$
  $$\boldsymbol{D}=\epsilon_0\epsilon\boldsymbol{E},\quad \boldsymbol{B}=\mu_0\mu\boldsymbol{H}.$$
- Microscopic equations: $(\epsilon=1,\ \mu=1,\ \boldsymbol{D}=\epsilon_0\boldsymbol{E},\ \boldsymbol{B}=\mu_0\boldsymbol{H})$
  $$\epsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\dot{\boldsymbol{E}},$$
- . . . alternatively:
  $$\epsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\mu_0\dot{\boldsymbol{H}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{H}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}+\epsilon_0\dot{\boldsymbol{E}}.$$
- Homogeneous macroscopic equations: $(\rho_{\rm f}=0,\ \boldsymbol{J}_{\rm f}=0)$
  $$\boldsymbol{\nabla}\cdot\boldsymbol{D}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\dot{\boldsymbol{D}},$$
  $$\boldsymbol{D}=\epsilon_0\epsilon\boldsymbol{E},\quad \boldsymbol{B}=\mu_0\mu\boldsymbol{H}.$$
- Homogeneous microscopic equations: $(\rho=0,\ \boldsymbol{J}=0)$
  $$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\epsilon_0\dot{\boldsymbol{E}},$$
- . . . alternatively:
  $$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\mu_0\dot{\boldsymbol{H}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{H}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\epsilon_0\dot{\boldsymbol{E}}.$$

Maxwell equations, integral form

- $$\nabla \cdot \boldsymbol{D} = \rho_\mathrm{f}$$

  $$\rightsquigarrow \quad \int_{\partial\mathcal{V}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} = Q_{\mathrm{f},\mathcal{V}}, \qquad Q_{\mathrm{f},\mathcal{V}} = \int_{\mathcal{V}} \rho_\mathrm{f} \, \mathrm{d}\mathcal{V}.$$

- $$\nabla \cdot \boldsymbol{B} = 0$$

  $$\rightsquigarrow \quad \int_{\partial\mathcal{V}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = 0.$$

- $$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$$

  $$\rightsquigarrow \quad \oint_{\partial\mathcal{S}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\dot{\Phi}_{\mathrm{B},\mathcal{S}}, \qquad \Phi_{\mathrm{B},\mathcal{S}} = \int_{\mathcal{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \quad (\leftrightarrow \partial_t \mathcal{S} \ldots).$$

- $$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}$$

  $$\rightsquigarrow \quad \oint_{\partial\mathcal{S}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = I_{\mathrm{f},\mathcal{S}} + \dot{\Phi}_{\mathrm{D},\mathcal{S}}, \quad I_{\mathrm{f},\mathcal{S}} = \int_{\mathcal{S}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}, \quad \Phi_{\mathrm{D},\mathcal{S}} = \int_{\mathcal{S}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a}.$$

- . . . variants.

Maxwell equations, Fourier transform

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\mathrm{f}}+\dot{\boldsymbol{D}},$$

$$\boldsymbol{D}=\epsilon_0\epsilon\boldsymbol{E},\quad \boldsymbol{B}=\mu_0\mu\boldsymbol{H}$$

& $$\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)\ \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega,\quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\frac{1}{\sqrt{2\pi}}\int\boldsymbol{F}(\boldsymbol{r},t)\ \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t$$

↪

$$\boldsymbol{E}(\boldsymbol{r},t),\ \boldsymbol{D}(\boldsymbol{r},t),\ \boldsymbol{B}(\boldsymbol{r},t),\ \boldsymbol{H}(\boldsymbol{r},t),\ \rho_{\mathrm{f}}(\boldsymbol{r},t),\ \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r},t)$$

$$\leftrightsquigarrow\quad \tilde{\boldsymbol{E}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{D}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{H}}(\boldsymbol{r},\omega),\ \tilde{\rho}_{\mathrm{f}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{J}}_{\mathrm{f}}(\boldsymbol{r},\omega),$$

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}}=\tilde{\rho}_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}=\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}}=0,\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm{f}}-\mathrm{i}\omega\tilde{\boldsymbol{D}},$$

$$\tilde{\boldsymbol{D}}=\epsilon_0\epsilon\tilde{\boldsymbol{E}},\quad \tilde{\boldsymbol{B}}=\mu_0\mu\tilde{\boldsymbol{H}}\quad \text{(Caution !)}.$$

- . . . variants.

Maxwell equations, frequency domain

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}}=\tilde{\rho}_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}=\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}}=0,\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm{f}}-\mathrm{i}\omega\tilde{\boldsymbol{D}},$$

$$\tilde{\boldsymbol{D}}=\epsilon_0\epsilon\tilde{\boldsymbol{E}},\quad \tilde{\boldsymbol{B}}=\mu_0\mu\tilde{\boldsymbol{H}}.$$

- $\boldsymbol{F}\in\mathbb{R} \rightsquigarrow \quad \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega)=(\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^*$
- "at frequency ω_0": $\quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\delta(\omega-\omega_0)+\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\delta(\omega+\omega_0)$

↪ $$\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{2}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}+\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}\right\},$$

$$\boldsymbol{F}(\boldsymbol{r},t)=\mathrm{Re}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\right\},$$

"$\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{2}\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}+\mathrm{c.c.}$".

↪ $$\bar{\boldsymbol{E}}(\boldsymbol{r}),\ \bar{\boldsymbol{D}}(\boldsymbol{r}),\ \bar{\boldsymbol{B}}(\boldsymbol{r}),\ \bar{\boldsymbol{H}}(\boldsymbol{r}),\ \bar{\rho}_{\mathrm{f}}(\boldsymbol{r}),\ \bar{\boldsymbol{J}}_{\mathrm{f}}(\boldsymbol{r}),\ \sim\exp(-\mathrm{i}\omega_0 t),$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{D}}=\bar{\rho}_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\bar{\boldsymbol{E}}=\mathrm{i}\omega_0\bar{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\bar{\boldsymbol{B}}=0,\quad \boldsymbol{\nabla}\times\bar{\boldsymbol{H}}=\bar{\boldsymbol{J}}_{\mathrm{f}}-\mathrm{i}\omega_0\bar{\boldsymbol{D}},$$

$$\bar{\boldsymbol{D}}=\epsilon_0\epsilon_{\omega_0}\bar{\boldsymbol{E}},\quad \bar{\boldsymbol{B}}=\mu_0\mu_{\omega_0}\bar{\boldsymbol{H}}.$$

Maxwell equations, dispersion

$$\nabla \cdot \tilde{\boldsymbol{D}} = \tilde{\rho}_{\mathrm{f}}, \quad \nabla \times \tilde{\boldsymbol{E}} = \mathrm{i}\omega \tilde{\boldsymbol{B}}, \quad \nabla \cdot \tilde{\boldsymbol{B}} = 0, \quad \nabla \times \tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}}_{\mathrm{f}} - \mathrm{i}\omega \tilde{\boldsymbol{D}},$$

$$\tilde{\boldsymbol{D}} = \epsilon_0 \epsilon \tilde{\boldsymbol{E}}, \qquad \tilde{\boldsymbol{B}} = \mu_0 \mu \tilde{\boldsymbol{H}}.$$

- (Material) dispersion: $\tilde{\epsilon}(\omega)$, $\tilde{\mu}(\omega)$,
  $\tilde{\epsilon}$, $\tilde{\mu}$ are usually determined at specific frequency ω.

- $$\tilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \epsilon_0 \tilde{\epsilon}(\omega) \tilde{\boldsymbol{E}}(\boldsymbol{r},\omega), \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega) = \mu_0 \tilde{\mu}(\omega) \tilde{\boldsymbol{H}}(\boldsymbol{r},\omega)$$

↪

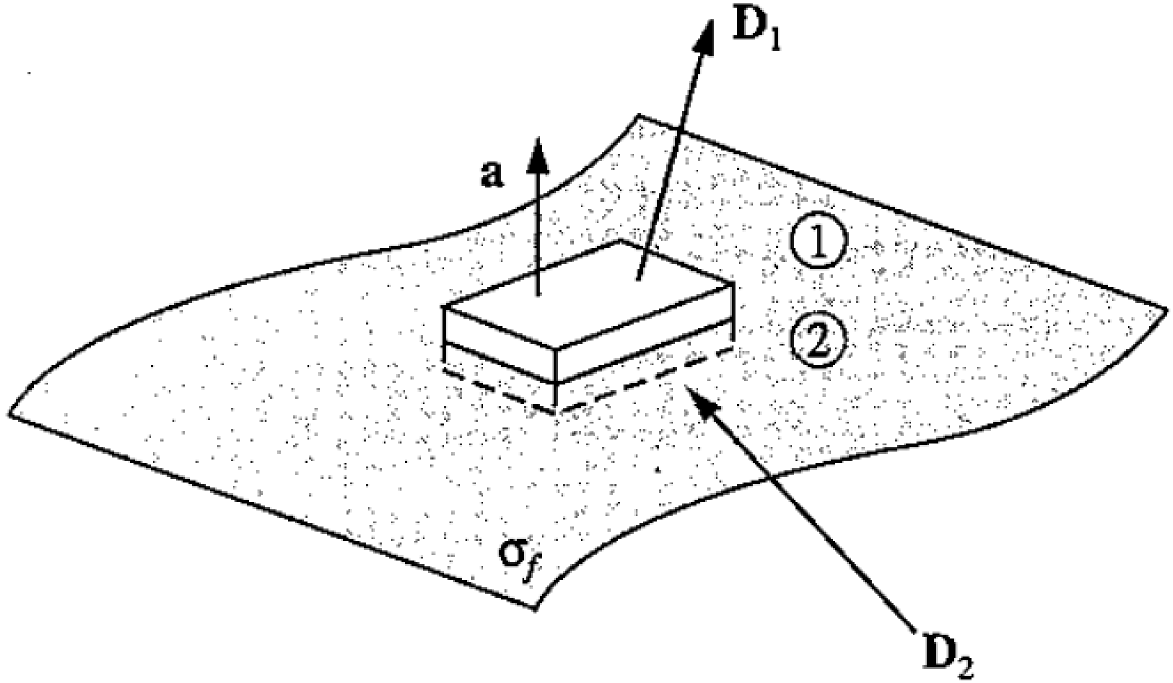
$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \int \epsilon(t-t') \boldsymbol{E}(\boldsymbol{r},t')\,\mathrm{d}t',$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \int \mu(t-t') \boldsymbol{H}(\boldsymbol{r},t')\,\mathrm{d}t'.$$

Interface conditions

$$\nabla \cdot \boldsymbol{D} = \rho_{\rm f} \quad \rightsquigarrow \quad \int_{\partial \mathcal{V}} \boldsymbol{D} \cdot {\rm d}\boldsymbol{a} = Q_{{\rm f},\mathcal{V}}$$

- interface between two media 1, 2, surface normal $\boldsymbol{n}$,
- surface charge density $\sigma_{\rm f}$,
- "Gaussian pillbox", flatten first, then shrink to a point:



(Griffiths, page 331, Fig. 7.46)

Interface conditions

$$\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}} \quad \rightsquigarrow \quad \int_{\partial \mathcal{V}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} = Q_{\mathrm{f},\mathcal{V}}$$

- interface between two media 1, 2, surface normal $\boldsymbol{n}$,
- surface charge density σ_{f},
- "Gaussian pillbox", flatten first, then shrink to a point

$$\hookrightarrow \quad \boldsymbol{n} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma_{\mathrm{f}}.$$

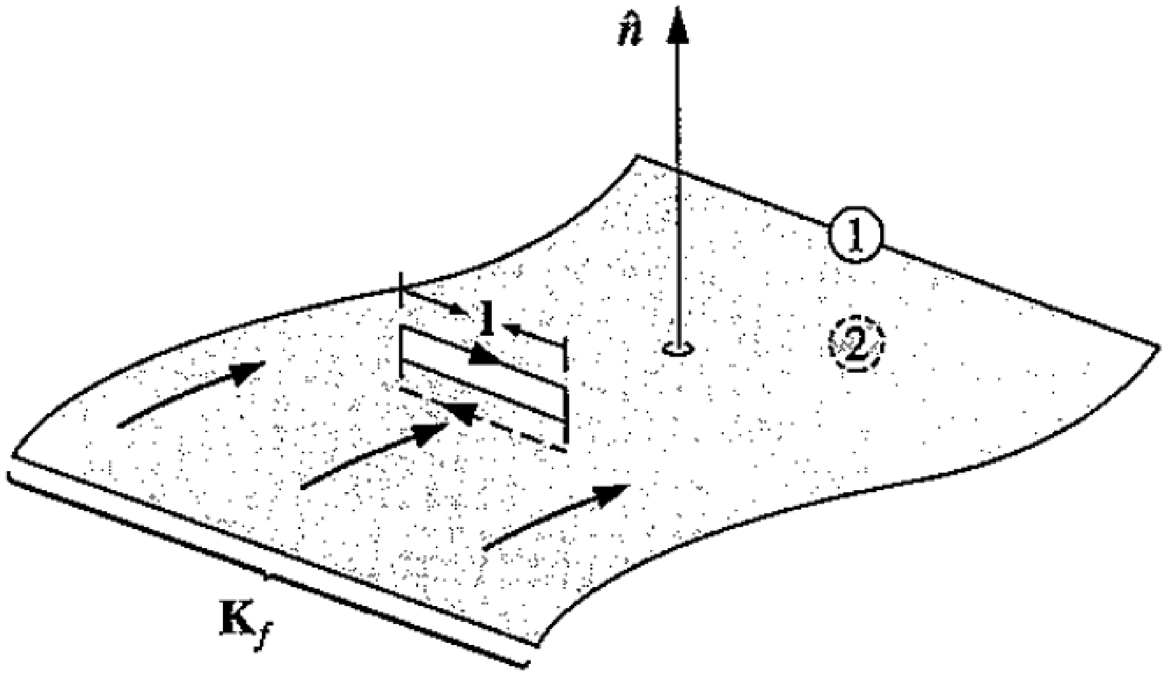
$$\nabla \cdot \boldsymbol{B} = 0 \quad \rightsquigarrow \quad \int_{\partial \mathcal{V}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = 0$$

$$\hookrightarrow \quad \boldsymbol{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0.$$

Interface conditions

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \dot{\boldsymbol{D}} \quad \rightsquigarrow \quad \oint_{\partial\mathcal{S}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = I_{\mathrm{f},\mathcal{S}} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a}$$

- interface between two media 1, 2, surface normal $\boldsymbol{n}$,
- surface current density $\boldsymbol{K}_{\mathrm{f}}$,
- “Amperian loop”, surface tangent $\boldsymbol{l}$, flatten first, then shrink to a point:



(Griffiths, page 332, Fig. 7.47)

Interface conditions

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}} \quad \rightsquigarrow \quad \oint_{\partial\mathcal{S}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = I_{\mathrm{f},\mathcal{S}} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a}$$

- interface between two media 1, 2, surface normal $\boldsymbol{n}$,
- surface current density $\boldsymbol{K}_\mathrm{f}$,
- “Amperian loop”, surface tangent $\boldsymbol{l}$, flatten first, then shrink to a point,
- flux of $\boldsymbol{D}$ vanishes with “$\mathcal{S} \to 0$”

$$\hookrightarrow \quad \boldsymbol{l} \cdot (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{K}_\mathrm{f} \cdot (\boldsymbol{n} \times \boldsymbol{l}) = \boldsymbol{l} \cdot (\boldsymbol{K}_\mathrm{f} \times \boldsymbol{n}), \qquad \forall \boldsymbol{l},\ |\boldsymbol{l}| = 1,\ \boldsymbol{l} \parallel \text{surface.}$$

$$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}} \quad \rightsquigarrow \quad \oint_{\partial\mathcal{S}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}$$

$$\hookrightarrow \quad \boldsymbol{l} \cdot (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \qquad \forall \boldsymbol{l},\ |\boldsymbol{l}| = 1,\ \boldsymbol{l} \parallel \text{surface.}$$

Interface conditions, summary

- Surface between media 1 and 2, surface normal $\boldsymbol{n}$, surface tangents $\boldsymbol{l}$, surface charge density σ_{f}, surface current density $\boldsymbol{K}_{\mathrm{f}}$:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2)=\sigma_{\mathrm{f}}, \qquad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$

$$\boldsymbol{n}\cdot(\boldsymbol{B}_1-\boldsymbol{B}_2)=0, \qquad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=\boldsymbol{l}\cdot(\boldsymbol{K}_{\mathrm{f}}\times\boldsymbol{n}).$$

- Surface without free charges:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2)=0, \qquad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$

$$\boldsymbol{n}\cdot(\boldsymbol{B}_1-\boldsymbol{B}_2)=0, \qquad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=0.$$

- ... variants.

Continuity equation, charges

Macroscopic:

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \dot{\boldsymbol{D}}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\mathrm{f}}$$

$$\hookrightarrow \quad \dot{\rho}_{\mathrm{f}} + \boldsymbol{\nabla} \cdot \boldsymbol{J}_{\mathrm{f}} = 0, \qquad \dot{Q}_{\mathrm{f},\mathcal{V}} + \int_{\partial \mathcal{V}} \boldsymbol{J}_{\mathrm{f}} \cdot \mathrm{d}\boldsymbol{a} = 0.$$

Microscopic:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \dot{\boldsymbol{E}}, \qquad \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$$

$$\hookrightarrow \quad \dot{\rho} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0, \qquad \dot{Q}_{\mathcal{V}} + \int_{\partial \mathcal{V}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} = 0.$$

Energy of electromagnetic fields

- Force on a particle with charge q, velocity $\boldsymbol{v}$, in a field $\boldsymbol{E}$, $\boldsymbol{B}$:
  $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

- work for shifting the particle by $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$:
  $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\,\mathrm{d}t = q\boldsymbol{E} \cdot \boldsymbol{v}\,\mathrm{d}t$,

- respective power: $\dfrac{\mathrm{d}W}{\mathrm{d}t} = q\boldsymbol{E} \cdot \boldsymbol{v}$.

For a charge density $\rho_\mathrm{f}(\boldsymbol{r}, t)$:

force density $\boldsymbol{f} = \rho_\mathrm{f}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

↪ power density $\boldsymbol{f} \cdot \boldsymbol{v} = \rho_\mathrm{f}\boldsymbol{E} \cdot \boldsymbol{v} = \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}$,

total work per time unit done in $\mathcal{V}$: $$\frac{\mathrm{d}W_\mathcal{V}}{\mathrm{d}t} = \int_\mathcal{V} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}\,\mathrm{d}\mathcal{V}.$$

Poynting theorem

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \dot{\boldsymbol{D}}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$$

$$\hookrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} W_{\mathcal{V}}^{\mathrm{mech}} = \int_{\mathcal{V}} \boldsymbol{J}_{\mathrm{f}} \cdot \boldsymbol{E}\, \mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})\, \mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H})\, \mathrm{d}\mathcal{V},$$

- Poynting vector: $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H},$ (energy flux density, power density)
- energy density: $u = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}),$ $W_{\mathcal{V}}^{\mathrm{field}} = \int_{\mathcal{V}} u\, \mathrm{d}\mathcal{V},$
- $\boldsymbol{D} = \epsilon_0 \epsilon \boldsymbol{E},\ \boldsymbol{B} = \mu_0 \mu \boldsymbol{H} \quad \rightsquigarrow \quad \dot{u} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$ (!)

$\mathcal{V}$ arbitrary

$$\hookrightarrow \quad \dot{u} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J}_{\mathrm{f}} \cdot \boldsymbol{E}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(W_{\mathcal{V}}^{\mathrm{mech}} + W_{\mathcal{V}}^{\mathrm{field}}\right) = -\oint_{\partial\mathcal{V}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}.$$

Momentum of electromagnetic fields

- Density of forces on a charge density ρ_f, velocity $\boldsymbol{v}$, in a field $\boldsymbol{E}, \boldsymbol{B}$:

  $$\boldsymbol{f} = \rho_\mathrm{f}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \rho_\mathrm{f}\boldsymbol{E} + \boldsymbol{J}_\mathrm{f} \times \boldsymbol{B},$$

- change of mechanical momentum in $\mathcal{V}$ with time:

  $$\boldsymbol{F}_\mathcal{V} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}_\mathcal{V}^\mathrm{mech} = \int_\mathcal{V} (\rho_\mathrm{f}\boldsymbol{E} + \boldsymbol{J}_\mathrm{f} \times \boldsymbol{B})\,\mathrm{d}\mathcal{V}.$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_\mathrm{f}, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

↪ $$\boldsymbol{f} = (\boldsymbol{E}\,\boldsymbol{\nabla} \cdot \boldsymbol{D} - \boldsymbol{D} \times \boldsymbol{\nabla} \times \boldsymbol{E} + \boldsymbol{H}\,\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{\nabla} \times \boldsymbol{H}) - \frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}).$$

Momentum of electromagnetic fields

$$\begin{aligned} \boldsymbol{f} &= \rho_{\rm f}\boldsymbol{E} + \boldsymbol{J}_{\rm f} \times \boldsymbol{B} \\ &= (\boldsymbol{E}\,\boldsymbol{\nabla}\cdot\boldsymbol{D} - \boldsymbol{D}\times\boldsymbol{\nabla}\times\boldsymbol{E} + \boldsymbol{H}\,\boldsymbol{\nabla}\cdot\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{\nabla}\times\boldsymbol{H}) - \frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}). \end{aligned}$$

$\ldots \quad \boldsymbol{D} = \epsilon_0\epsilon\boldsymbol{E}, \;\; \boldsymbol{B} = \mu_0\mu\boldsymbol{H}, \quad \ldots$ (!)

$$(\boldsymbol{E}\,\boldsymbol{\nabla}\cdot\boldsymbol{D} - \boldsymbol{D}\times\boldsymbol{\nabla}\times\boldsymbol{E})_i = \epsilon_0\epsilon\sum_{j=1}^{3}\frac{\partial}{\partial r_j}\left(E_iE_j - \frac{1}{2}\boldsymbol{E}^2\delta_{ij}\right),$$

$$(\boldsymbol{H}\,\boldsymbol{\nabla}\cdot\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{\nabla}\times\boldsymbol{H})_i = \frac{1}{\mu_0\mu}\sum_{j=1}^{3}\frac{\partial}{\partial r_j}\left(B_iB_j - \frac{1}{2}\boldsymbol{B}^2\delta_{ij}\right).$$

- Maxwell stress tensor: (momentum flux density)

$$T_{ij} = \epsilon_0\epsilon E_iE_j + \frac{1}{\mu_0\mu}B_iB_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0\epsilon\boldsymbol{E}^2 + \frac{1}{\mu_0\mu}\boldsymbol{B}^2\right).$$

$$\mathsf{T} = (T_{ij}), \;\; T_{ij} = T_{ji}, \;\; (\boldsymbol{a}\cdot\mathsf{T})_i = \sum_{j=1}^{3} a_jT_{ij}, \;\; (\boldsymbol{\nabla}\cdot\mathsf{T})_i = \sum_{j=1}^{3}\frac{\partial}{\partial r_j}T_{ij}.$$

Momentum of electromagnetic fields

$$\begin{aligned} \boldsymbol{f} &= \rho_\mathrm{f}\boldsymbol{E} + \boldsymbol{J}_\mathrm{f} \times \boldsymbol{B} \\ &= (\boldsymbol{E}\,\nabla\cdot\boldsymbol{D} - \boldsymbol{D}\times\nabla\times\boldsymbol{E} + \boldsymbol{H}\,\nabla\cdot\boldsymbol{B} - \boldsymbol{B}\times\nabla\times\boldsymbol{H}) - \frac{\partial}{\partial t}(\boldsymbol{D}\times\boldsymbol{B}). \end{aligned}$$

- Maxwell stress tensor: (momentum flux density)

$$T_{ij} = \epsilon_0\epsilon E_i E_j + \frac{1}{\mu_0\mu} B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0\epsilon \boldsymbol{E}^2 + \frac{1}{\mu_0\mu}\boldsymbol{B}^2\right).$$

- density of momentum: $\boldsymbol{D}\times\boldsymbol{B} = \epsilon_0\epsilon\mu_0\mu\,\boldsymbol{S},\qquad \boldsymbol{p}_\mathcal{V}^\mathrm{field} = \epsilon_0\mu_0\int_\mathcal{V}\epsilon\mu\,\boldsymbol{S}\,\mathrm{d}\mathcal{V}$

$$\epsilon_0\epsilon\mu_0\mu\dot{\boldsymbol{S}} - \nabla\cdot\mathsf{T} = -(\rho_\mathrm{f}\boldsymbol{E} + \boldsymbol{J}_\mathrm{f}\times\boldsymbol{B}),\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_\mathcal{V}^\mathrm{mech} + \boldsymbol{p}_\mathcal{V}^\mathrm{field}\right) = \oint_{\partial\mathcal{V}}\mathsf{T}\cdot\mathrm{d}\boldsymbol{a}.$$

$\mathsf{T}\cdot\mathrm{d}\boldsymbol{a}$: force per unit area acting on the surface element $\mathrm{d}\boldsymbol{a}$,
$T_{ij} = T_{ji}$: component i of the force per unit area acting on a surface element in direction j,
T_{ii} : pressures, $T_{i\neq j}$: shear forces.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}_\mathcal{V}^\mathrm{mech} = \boldsymbol{F}_\mathcal{V}$$

Energy and momentum, microscopic

$(\epsilon = 1,\, \mu = 1,\, \rho_{\mathrm{f}} \to \rho,\, \boldsymbol{J}_{\mathrm{f}} \to \boldsymbol{J})$

Energy:

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}, \qquad u = \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right),$$

$$\dot{u} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = -\boldsymbol{J} \cdot \boldsymbol{E}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(W_{\mathcal{V}}^{\mathrm{mech}} + W_{\mathcal{V}}^{\mathrm{field}}\right) = -\oint_{\partial\mathcal{V}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}.$$

Momentum:

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right), \qquad \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} = \epsilon_0 \mu_0\, \boldsymbol{S},$$

$$\epsilon_0 \mu_0 \dot{\boldsymbol{S}} - \boldsymbol{\nabla} \cdot \mathsf{T} = -(\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_{\mathcal{V}}^{\mathrm{mech}} + \boldsymbol{p}_{\mathcal{V}}^{\mathrm{field}}\right) = \oint_{\partial\mathcal{V}} \mathsf{T} \cdot \mathrm{d}\boldsymbol{a}.$$

- . . . variants.

Upcoming

Next lecture:

- Wave equation, plane waves, refractive index, polarization, energy transport, spherical waves