Electrodynamics

— Lecture D —

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Integrated Optical MicroSystems
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Course overview

Electrodynamics

- A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,
- B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion
- C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields
- D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves
- E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity
- F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides
- G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation
- H Special relativity; transmission lines

Wave equation

- Homogeneous macroscopic Maxwell equations: $(\rho_{\mathrm{f}} = 0,\ \boldsymbol{J}_{\mathrm{f}} = 0)$

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \dot{\boldsymbol{D}}$$

- Linear isotropic homogeneous media: $(\epsilon\,\cancel{(\boldsymbol{r})},\ \mu\,\cancel{(\boldsymbol{r})}\ !)$

$$\boldsymbol{D} = \epsilon_0\epsilon\boldsymbol{E}, \quad \boldsymbol{B} = \mu_0\mu\boldsymbol{H}$$

$$\Rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\mu_0\mu\dot{\boldsymbol{H}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{H} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \epsilon_0\epsilon\dot{\boldsymbol{E}},$$

$$\Rightarrow \quad \Delta\boldsymbol{E} = \epsilon_0\mu_0\epsilon\mu\ddot{\boldsymbol{E}}, \quad \Delta\boldsymbol{H} = \epsilon_0\mu_0\epsilon\mu\ddot{\boldsymbol{H}}.$$

Wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}_\mathrm{m}^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E} = 0, \qquad \left(\Delta - \frac{1}{\mathrm{c}_\mathrm{m}^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{H} = 0,$$

- speed of light in vacuum: $\mathrm{c} = \dfrac{1}{\sqrt{\epsilon_0\mu_0}},$
- phase velocity in the medium: $\mathrm{c}_\mathrm{m} = \dfrac{1}{\sqrt{\epsilon_0\mu_0\epsilon\mu}} = \dfrac{\mathrm{c}}{\sqrt{\epsilon\mu}} = \dfrac{\mathrm{c}}{n},$
- refractive index of the medium: $n = \sqrt{\epsilon\mu}.$

Shorter: $\Box\boldsymbol{E} = 0, \quad \Box\boldsymbol{H} = 0, \quad \Box = \left(\Delta - \frac{1}{\mathrm{c}_\mathrm{m}^2}\frac{\partial^2}{\partial t^2}\right):$ “Quabla”, (d’Alembert operator).

Plane waves

$$\left(\Delta - \frac{1}{\mathrm{c}_\mathrm{m}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0. \qquad (*)$$

$$\psi(\boldsymbol{r},t) = f(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) + b(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)$$

solves $(*)$ for $\omega^2 = \boldsymbol{k}^2\mathrm{c}_\mathrm{m}^2$, ($f$, b arbitrary (smooth), $\omega > 0$, $k = |\boldsymbol{k}|$)

- Features of f appear on planes with $\boldsymbol{k}\cdot\boldsymbol{r} - \omega t = \varphi_0$ const., i.e. move with velocity $\mathrm{c}_\mathrm{m} = \omega/k$ in direction $\boldsymbol{k}/k$. $\quad \boldsymbol{r}\cdot\frac{\boldsymbol{k}}{k} = \frac{\varphi_0}{k} + \frac{\omega}{k}t$
- Features of b appear on planes with $\boldsymbol{k}\cdot\boldsymbol{r} + \omega t = \varphi_1$ const., i.e. move with velocity $\mathrm{c}_\mathrm{m} = \omega/k$ in direction $-\boldsymbol{k}/k$. $\quad \boldsymbol{r}\cdot\frac{\boldsymbol{k}}{k} = \frac{\varphi_1}{k} - \frac{\omega}{k}t$
- Planes $\perp$ $\boldsymbol{k}$: wave fronts, phase fronts.

Plane harmonic waves

$$\left(\Delta - \frac{1}{\mathrm{c}_\mathrm{m}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0.$$

- frequency domain: $\psi(\boldsymbol{r},t) = \frac{1}{2}\tilde{\psi}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.},$

$$\left(\Delta + \frac{\omega^2}{\mathrm{c}_\mathrm{m}^2}\right)\tilde{\psi}(\boldsymbol{r}) = 0,$$ (Helmholtz equation)

- spatial oscillations: $\psi(\boldsymbol{r},t) = \frac{1}{2}\psi_0\;\mathrm{e}^{\mathrm{i}k_x x}\;\mathrm{e}^{\mathrm{i}k_y y}\;\mathrm{e}^{\mathrm{i}k_z z}\;\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.},$

$\psi_0 \in \mathbb{C}$, only $\psi_0 \neq 0$ is of interest, $\boldsymbol{k} = (k_x, k_y, k_z)$

$$\left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}_\mathrm{m}^2}\right)\psi_0 = 0,$$

$$\psi(\boldsymbol{r},t) = \mathrm{Re}\,\psi_0\;\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}.$$

Plane harmonic waves

$$\psi(\boldsymbol{r},t) = \mathrm{Re}\,\psi_0\;\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

- Periodicity in time:
  - angular frequency: $\omega,$
  - period: $T = \dfrac{2\pi}{\omega},$
- Spatial periodicity:
  - wave vector: $\boldsymbol{k}, \quad k = |\boldsymbol{k}|,$
  - wavenumber: $k = \dfrac{\omega}{\mathrm{c_m}} = \dfrac{\omega}{\mathrm{c}}\,n = k_0\,n,$
  - vacuum wavenumber: $k_0 = \dfrac{\omega}{\mathrm{c}},$
  - refractive index: $n = \sqrt{\epsilon\mu},$
  - vacuum wavelength: $\lambda = \dfrac{2\pi}{k_0} = \dfrac{2\pi\mathrm{c}}{\omega},$
  - wavelength in the medium: $\lambda_\mathrm{m} = \dfrac{2\pi}{k} = \dfrac{2\pi}{k_0 n} = \dfrac{\lambda}{n}.$
- Electromagnetic spectrum

Plane harmonic electromagnetic waves

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\mu_0\mu\dot{\boldsymbol{H}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{H}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\epsilon_0\epsilon\dot{\boldsymbol{E}},$$

& $$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},\quad \boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

(skip $1/2$, Re, c.c.)

or

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{E}}=0,\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}=\mathrm{i}\omega\mu_0\mu\tilde{\boldsymbol{H}},\quad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{H}}=0,\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}}=-\mathrm{i}\omega\epsilon_0\epsilon\tilde{\boldsymbol{E}},$$

& $$\tilde{\boldsymbol{E}}(\boldsymbol{r})=\boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}},\quad \tilde{\boldsymbol{H}}(\boldsymbol{r})=\boldsymbol{H}_0\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

(exercise)

$$\boldsymbol{k}\cdot\boldsymbol{E}_0=0,\quad \boldsymbol{k}\times\boldsymbol{E}_0=\omega\mu_0\mu\boldsymbol{H}_0,\quad \boldsymbol{k}\cdot\boldsymbol{H}_0=0,\quad \boldsymbol{k}\times\boldsymbol{H}_0=-\omega\epsilon_0\epsilon\boldsymbol{E}_0,$$

$\boldsymbol{E}_0\perp\boldsymbol{k}$, $\boldsymbol{H}_0\perp\boldsymbol{k}$, $\boldsymbol{E}_0\perp\boldsymbol{H}_0$, transverse waves,

$\omega\epsilon_0\epsilon\boldsymbol{E}_0=\boldsymbol{H}_0\times\boldsymbol{k}$ $\rightsquigarrow$ orthogonal right-hand system $\{\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{k}\}$.

Plane harmonic electromagnetic waves

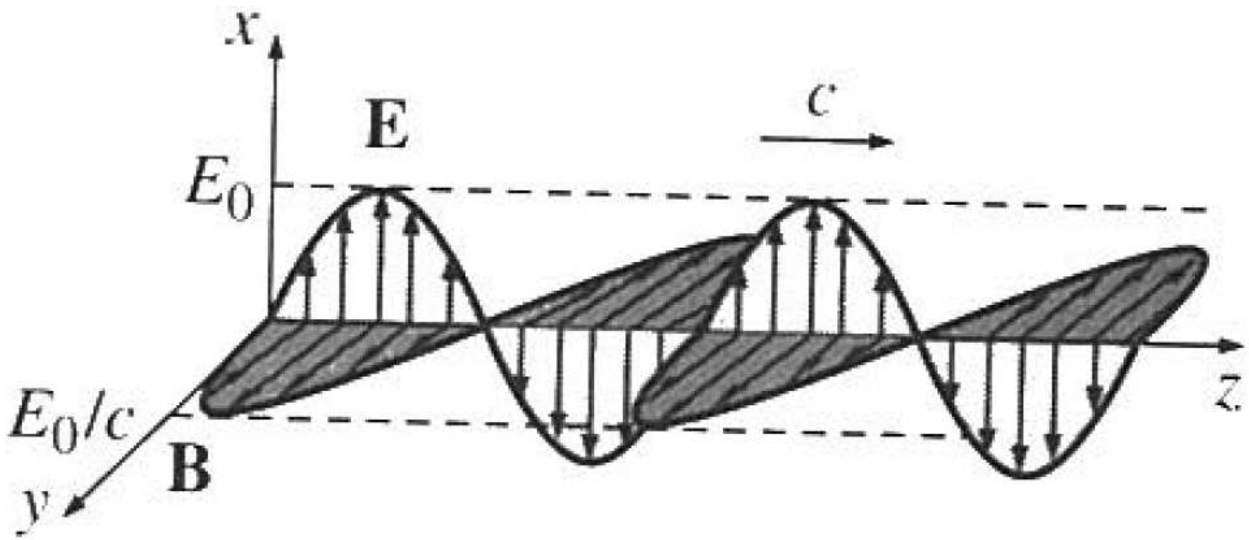
$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\ \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0\ \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}.$$

Example:

$$\boldsymbol{k} = k\,\boldsymbol{e}_z, \quad \boldsymbol{E}_0 = E_0\,\boldsymbol{e}_x, \quad \boldsymbol{H}_0 = \frac{1}{\omega\mu_0\mu}\,\boldsymbol{k}\times\boldsymbol{E}_0 = \sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\,E_0\,\boldsymbol{e}_y \qquad (E_0 \in \mathbb{R})$$

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0\,\boldsymbol{e}_x\ \cos(kz-\omega t), \quad \boldsymbol{H}(\boldsymbol{r},t) = \sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\,E_0\,\boldsymbol{e}_y\ \cos(kz-\omega t).$$

$(\hat{\epsilon} = \epsilon\,\cancel{(\boldsymbol{r})}\,1,\ \hat{\mu} = \mu\,\cancel{(\boldsymbol{r})}\,1\ !)$



Griffiths, Fig. 9.10, page 379

$$\boldsymbol{B} = \mu_0\mu\boldsymbol{H}, \quad \boldsymbol{B}(\boldsymbol{r},t) = \frac{E_0}{c_{\mathrm{m}}}\,\boldsymbol{e}_y\ \cos(kz-\omega t).$$

Polarization

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\ \mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)}, \qquad \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0\ \mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)}.$$

- Polarization $\longleftrightarrow$ orientation of the electric part of the wave.
- Complex notation: $\boldsymbol{E}_0, \boldsymbol{H}_0 \in \mathbb{C}^3$ in general.
- $\boldsymbol{k}$, $\boldsymbol{E}_0$ given, $\boldsymbol{k} \perp \boldsymbol{E}_0$ $\rightsquigarrow$ $\boldsymbol{H}_0 = \omega\mu_0\mu\boldsymbol{k} \times \boldsymbol{E}_0$.

Assume $\boldsymbol{k} = k\,\boldsymbol{e}_z$, $E_{0,x} = |E_{0,x}|$, $E_{0,y} = |E_{0,y}|\,\mathrm{e}^{\mathrm{i}\delta}$

(common phase in $E_{0,x}$ and $E_{0,y}$ omitted)

$$\hookrightarrow \quad \boldsymbol{E}(\boldsymbol{r},t) = |E_{0,x}|\,\boldsymbol{e}_x\ \cos(kz - \omega t) + |E_{0,y}|\,\boldsymbol{e}_y\ \cos(kz - \omega t + \delta)$$

Polarization

$$\boldsymbol{k} = k\,\boldsymbol{e}_z,\;\; E_{0,x} = |E_{0,x}|,\;\; E_{0,y} = |E_{0,y}|\,\mathrm{e}^{\mathrm{i}\,\delta}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = |E_{0,x}|\,\boldsymbol{e}_x\,\cos(kz-\omega t) + |E_{0,y}|\,\boldsymbol{e}_y\,\cos(kz-\omega t+\delta) \qquad (*)$$

- $\delta = 0,\;\; \delta = \pm\pi$:
  $\boldsymbol{E}(\boldsymbol{r},t) = (|E_{0,x}|\,\boldsymbol{e}_x + |E_{0,y}|\,\boldsymbol{e}_y)\,\cos(kz-\omega t)$,
  linear polarized wave, $\boldsymbol{E}$ oscillates in the plane $\parallel\;\{|E_{0,x}|\,\boldsymbol{e}_x + |E_{0,y}|\,\boldsymbol{e}_y,\;\boldsymbol{k}\}$

- $\delta = \pm\pi/2,\;\; |E_{0,x}| = |E_{0,y}| = E_0$:
  $\boldsymbol{E}(\boldsymbol{r},t) = E_0\,(\cos(kz-\omega t)\,\boldsymbol{e}_x \pm \sin(kz-\omega t)\,\boldsymbol{e}_y)$
  circularly polarized wave, z fixed: $\boldsymbol{E}(t)$ describes a circle in time.

- otherwise: elliptical polarization,
- $(*)$ is the sum of two linearly polarized waves.

Energy transport

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}), \quad \boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H},$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\,\tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t} \quad (\boldsymbol{D},\ \boldsymbol{B},\ \boldsymbol{H}\ \text{analogously})$$

$\hookrightarrow$ $\boldsymbol{S}$, $\boldsymbol{u}$ oscillate in time.

Consider time-averaged quantities: $\bar{f}(t) = \dfrac{1}{T}\displaystyle\int_t^{t+T} f(t')\,\mathrm{d}t'$ *(exercise)*

$$\hookrightarrow \quad \overline{u} = \frac{1}{4}\,\mathrm{Re}\left(\tilde{\boldsymbol{E}}^*\cdot\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{H}}^*\cdot\tilde{\boldsymbol{B}}\right), \quad \overline{\boldsymbol{S}} = \frac{1}{2}\,\mathrm{Re}\left(\tilde{\boldsymbol{E}}^*\times\tilde{\boldsymbol{H}}\right).$$

Specifically: *(exercise)*

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

$$\hookrightarrow \quad \overline{u} = \frac{1}{2}\,\epsilon_0\epsilon\,|\boldsymbol{E}_0|^2 = \frac{1}{2}\,\mu_0\mu\,|\boldsymbol{H}_0|^2, \quad \overline{\boldsymbol{S}} = \frac{1}{2}\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\,|\boldsymbol{E}_0|^2\,\frac{\boldsymbol{k}}{k}, \qquad \overline{\boldsymbol{S}} = \mathrm{c_m}\,\overline{u}\,\frac{\boldsymbol{k}}{k},$$

Intensity $|\overline{\boldsymbol{S}}|$: average power per unit area transported by the wave.

Spherical waves

$$\left(\Delta - \frac{1}{\mathrm{c}_{\mathrm{m}}^2} \frac{\partial^2}{\partial t^2} \right) \psi(\boldsymbol{r}, t) = 0. \qquad (*)$$

Spherical coordinates r, ϑ, φ:

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left\{ \frac{1}{\sin\vartheta} \frac{\partial}{\partial \vartheta} \left(\sin\vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial \varphi^2} \right\}.$$

(exercise)

Spherically symmetric solutions of $(*)$:

$$\psi(\boldsymbol{r}, t) = \frac{1}{r} f_{\mathrm{out}}(kr - \omega t) + \frac{1}{r} f_{\mathrm{in}}(kr + \omega t)$$

with $\omega^2 = k^2 \mathrm{c}_{\mathrm{m}}^2$ (f_{out}, f_{in} arbitrary (smooth), $\omega > 0$).

Spherical waves

$$\psi(\boldsymbol{r},t) = \frac{1}{r}\, f_{\text{out}}(kr-\omega t) + \frac{1}{r}\, f_{\text{in}}(kr+\omega t).$$

- Wave fronts: spheres around the origin.
- Features of f_{out} move with velocity $\mathrm{c_m} = \omega/k$ outward, damped $\sim \frac{1}{r}$.
- Features of f_{in} move with velocity $\mathrm{c_m} = \omega/k$ inward, growing $\sim \frac{1}{r}$.

Harmonic spherical electromagnetic waves,

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0(\boldsymbol{r})\,\frac{1}{r}\,\mathrm{e}^{\mathrm{i}(kr-\omega t)}, \qquad \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_0(\boldsymbol{r})\,\frac{1}{r}\,\mathrm{e}^{\mathrm{i}(kr-\omega t)},$$

... as for the plane waves:

- period, wavenumber, wavelength, phase velocity,
- transverse waves, $\{\boldsymbol{E}_0(\boldsymbol{r}), \boldsymbol{H}_0(\boldsymbol{r}), \boldsymbol{r}\}$ form a right-hand system,
- polarization.

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}_\mathrm{m}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde{\psi}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

$$\hookrightarrow \quad \left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}_\mathrm{m}^2}\right)\tilde{\psi}(\boldsymbol{k},\omega) = 0,$$

$$\hookrightarrow \quad \tilde{\psi}(\boldsymbol{k},\omega) = a_\mathrm{f}(\boldsymbol{k})\,\delta(\omega-\omega_k) + a_\mathrm{b}(\boldsymbol{k})\,\delta(\omega+\omega_k), \qquad \omega_k = \mathrm{c}_\mathrm{m}k,$$

$$\hookrightarrow \quad \psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int\left(a_\mathrm{f}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + a_\mathrm{b}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}+\omega_k t)}\right)\mathrm{d}^3k,$$

- $\psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}) \rightsquigarrow \ \ldots \ \rightsquigarrow a_\mathrm{f}(\boldsymbol{k}),\ a_\mathrm{b}(\boldsymbol{k}).$

Upcoming

Next lecture:

- Reflection and transmission at interfaces, lossy materials, wave packets, phase and group velocity, dispersion