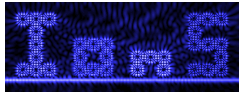


# *Electrodynamics*

## *— Lecture E —*

UNIVERSITY  
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# Course overview

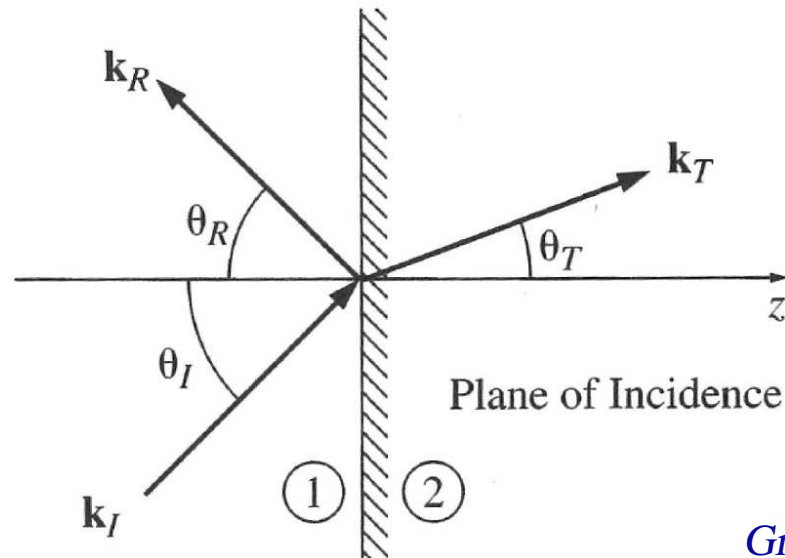
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## Electrodynamics

- A Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,
- B Maxwell equations, brush up on electro- and magnetostatics, multipole expansion
- C Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields
- D Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves
- E Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity
- F Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides
- G Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation
- H Special relativity; transmission lines

## Reflection and transmission

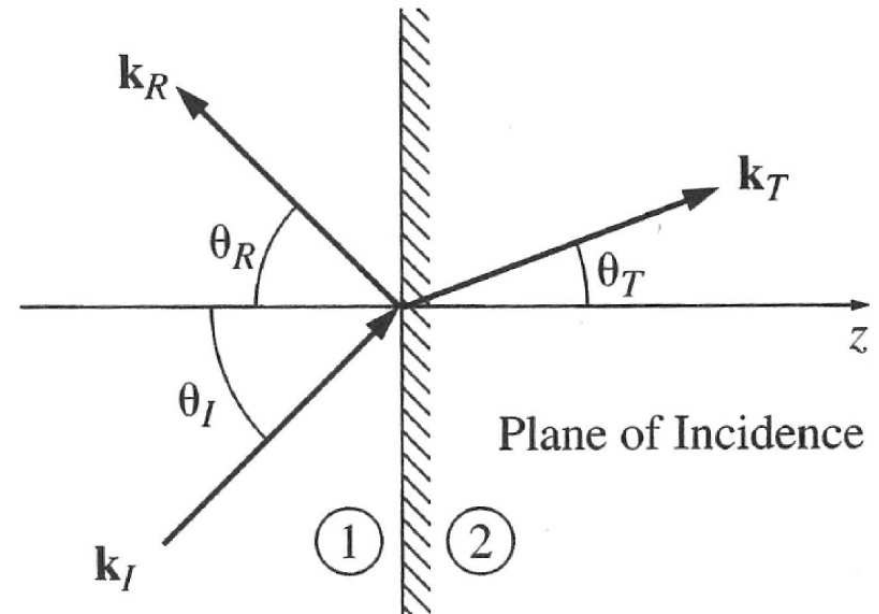
... of plane waves  
at (dielectric) interfaces.



Griffiths, Fig. 9.14, page 387

- $x$ - $y$ -plane: interface between regions 1, 2; linear homog. isotr. media; properties:  $\epsilon_j$ ,  $\mu_j$ ,  $n_j = \sqrt{\epsilon_j \mu_j}$ ,  $c_j = c/n_j$ ,  $j = 1, 2$ .
- **Expectation:**  
A given incoming wave in region 1 generates
  - a reflected wave in region 1,
  - a transmitted wave in region 2.
- **Strategy:**
  - write plane wave solutions of homogeneous ME for regions 1, 2,
  - piece these together at the  $x$ - $y$ -plane by interface conditions.

## Reflection and transmission, ansatz



Ansatz:

- Incident:  $\mathbf{E}_I(\mathbf{r}, t) = \mathbf{E}_{I0} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega_I t)}$ ,
  - Reflected:  $\mathbf{E}_R(\mathbf{r}, t) = \mathbf{E}_{R0} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega_R t)}$ ,
  - Transmitted:  $\mathbf{E}_T(\mathbf{r}, t) = \mathbf{E}_{T0} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega_T t)}$ ,
- $\mathbf{k}_I \perp \mathbf{E}_{I0}$ ,  $\omega_I = k_I c_1$ ,  $\mathbf{k}_R \perp \mathbf{E}_{R0}$ ,  $\omega_R = k_R c_1$ ,  $\mathbf{k}_T \perp \mathbf{E}_{T0}$ ,  $\omega_T = k_T c_2$ .

↪ Maxwell equations in regions 1, 2 are satisfied (...).

## Plane of incidence

Interface conditions, formal:

$$(\cdot) e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega_I t)} + (\cdot) e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega_R t)} = (\cdot) e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega_T t)} \Big|_{\mathbf{r}=(x,y,0)}$$

$\forall x, y, t$  (!).

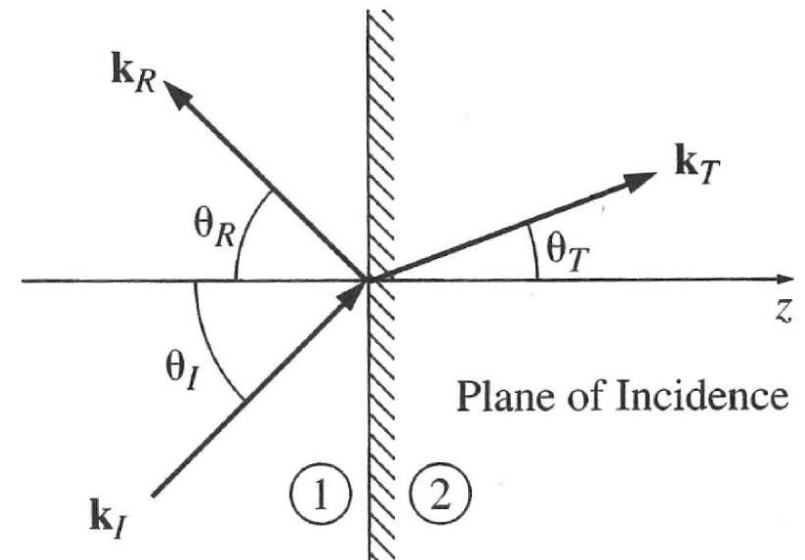
↪

$$\begin{aligned} \omega_I &= \omega_R = \omega_T =: \omega, \\ k_{Ix} &= k_{Rx} = k_{Tx} =: k_x, \\ k_{Iy} &= k_{Ry} = k_{Ty} =: k_y. \end{aligned}$$

Orient the coordinates such that  $k_{Iy} = k_y = 0$ :

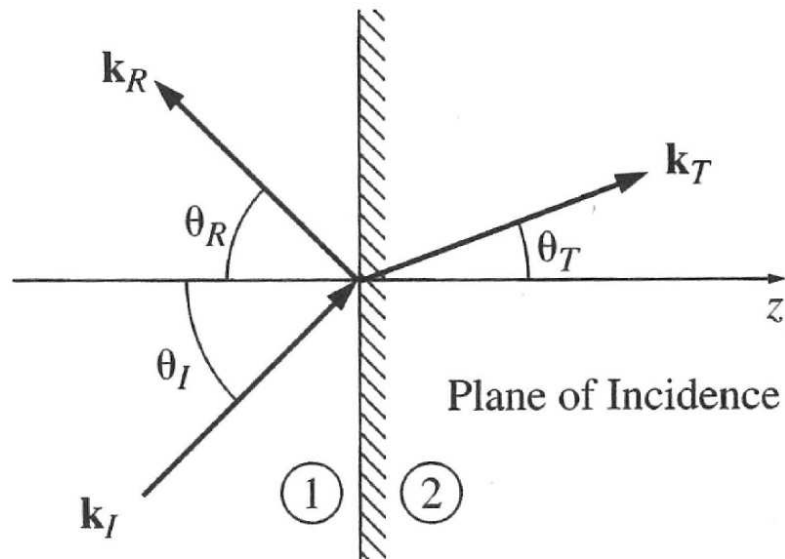
↪

$$\begin{aligned} \mathbf{k}_I &= (k_x, 0, k_{Iz}), \\ \mathbf{k}_R &= (k_x, 0, k_{Rz}), \\ \mathbf{k}_T &= (k_x, 0, k_{Tz}). \end{aligned}$$



↔ **Plane of incidence** spanned by  $\mathbf{k}_I$  and the interface normal  $\mathbf{e}_z$ .

## Reflection and refraction

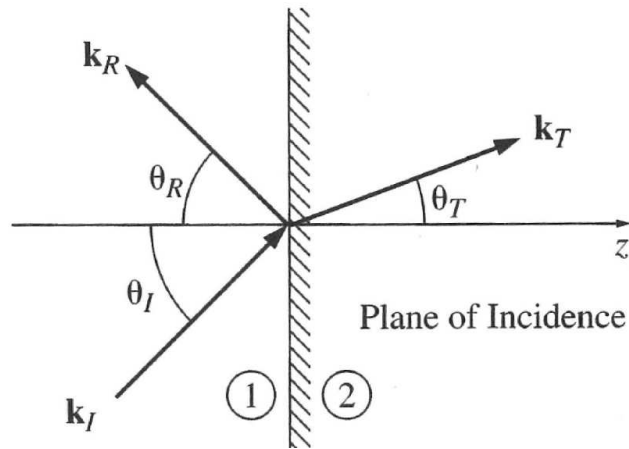


$$k_I = \frac{\omega}{c_1} = k_0 n_1 = k_R, \quad k_T = \frac{\omega}{c_2} = k_0 n_2$$

$$\begin{aligned} \mathbf{k}_I &= k_0 n_1 (\sin \theta_1, 0, \cos \theta_1), \\ \mathbf{k}_R &= k_0 n_1 (\sin \theta_1, 0, -\cos \theta_1), \\ \mathbf{k}_T &= k_0 n_2 (\sin \theta_2, 0, \cos \theta_2). \end{aligned}$$

- $\theta_I = \theta_R = \theta_1$ , angle of incidence = angle of reflection.
- $n_1 \sin \theta_1 = n_2 \sin \theta_2$ , Snell's law, law of refraction.

## Reflection and transmission



$$\begin{aligned} \mathbf{E}_1(\mathbf{r}, t) &= \mathbf{E}_{I0} e^{i(k_0 n_1 (\sin \theta_1 x + \cos \theta_1 z) - \omega t)} \\ &\quad + \mathbf{E}_{R0} e^{i(k_0 n_1 (\sin \theta_1 x - \cos \theta_1 z) - \omega t)}, \\ \mathbf{E}_2(\mathbf{r}, t) &= \mathbf{E}_{T0} e^{i(k_0 n_2 (\sin \theta_2 x + \cos \theta_2 z) - \omega t)}. \end{aligned}$$

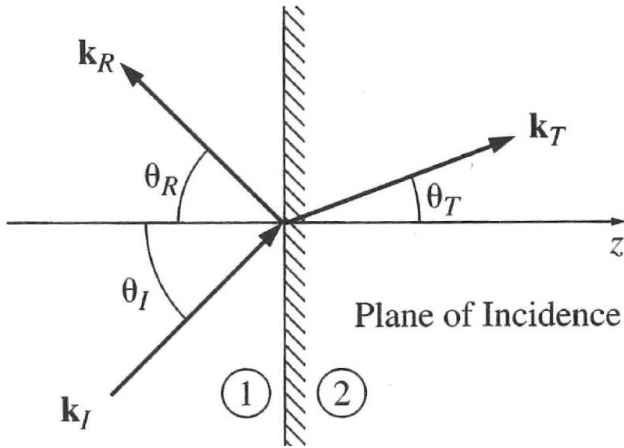
- $\nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}} \quad \rightsquigarrow \quad \mathbf{H}_{j0} = \frac{1}{\omega \mu_0 \mu_{1,2}} \mathbf{k}_j \times \mathbf{E}_{j0}, \quad j = I, R, T.$
- Assume  $\mathbf{E}_{I0} = E_{I0} \mathbf{e}_y$ ,  $\mathbf{E}_{R0} = E_{R0} \mathbf{e}_y$ ,  $\mathbf{E}_{T0} = E_{T0} \mathbf{e}_y$  (Lecture F, 2-D TE)  
s-polarization, polarization of  $\mathbf{E}_I \perp$  plane of incidence.



$$\mathbf{H}_{I0} = \sqrt{\frac{\epsilon_0}{\mu_0} \frac{n_1}{\mu_1}} E_{I0} (-\cos \theta_1, 0, \sin \theta_1), \quad \mathbf{H}_{R0} = \sqrt{\frac{\epsilon_0}{\mu_0} \frac{n_1}{\mu_1}} E_{R0} (\cos \theta_1, 0, \sin \theta_1),$$

$$\mathbf{H}_{T0} = \sqrt{\frac{\epsilon_0}{\mu_0} \frac{n_2}{\mu_2}} E_{T0} (-\cos \theta_2, 0, \sin \theta_2).$$

## Reflection and transmission, fields



$$\begin{aligned} \mathbf{E}_1(\mathbf{r}, t) &= E_{I0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{i(k_0 n_1 (\sin \theta_1 x + \cos \theta_1 z) - \omega t)} \\ &+ E_{R0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{i(k_0 n_1 (\sin \theta_1 x - \cos \theta_1 z) - \omega t)}, \end{aligned}$$

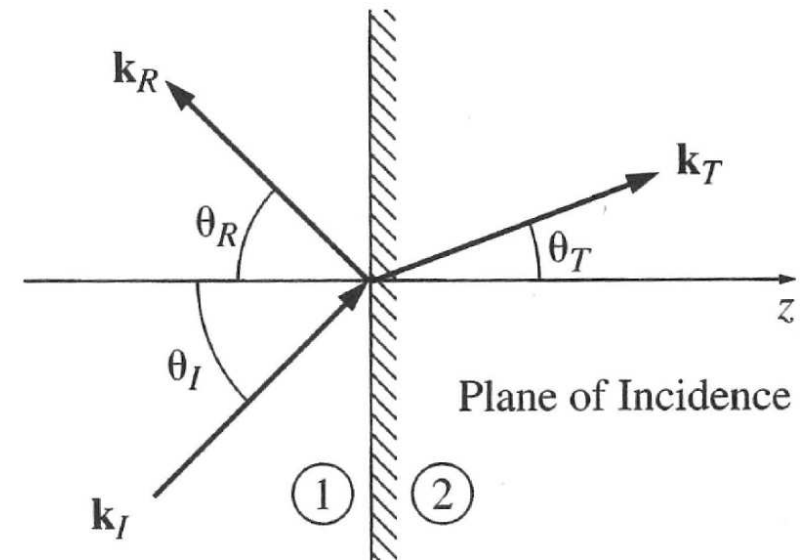
$$\mathbf{E}_2(\mathbf{r}, t) = E_{T0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{i(k_0 n_2 (\sin \theta_2 x + \cos \theta_2 z) - \omega t)},$$

$$\begin{aligned} \mathbf{H}_1(\mathbf{r}, t) &= \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_1}{\mu_1} E_{I0} \begin{pmatrix} -\cos \theta_1 \\ 0 \\ \sin \theta_1 \end{pmatrix} e^{i(k_0 n_1 (\sin \theta_1 x + \cos \theta_1 z) - \omega t)} \\ &+ \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_1}{\mu_1} E_{R0} \begin{pmatrix} \cos \theta_1 \\ 0 \\ \sin \theta_1 \end{pmatrix} e^{i(k_0 n_1 (\sin \theta_1 x - \cos \theta_1 z) - \omega t)}, \end{aligned}$$

$$\mathbf{H}_2(\mathbf{r}, t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{n_2}{\mu_2} E_{T0} \begin{pmatrix} -\cos \theta_2 \\ 0 \\ \sin \theta_2 \end{pmatrix} e^{i(k_0 n_2 (\sin \theta_2 x + \cos \theta_2 z) - \omega t)}.$$



## Reflection and transmission, interface conditions



Interface conditions at  $z = 0$  ( $\sigma_f = 0$ ,  $\mathbf{K}_f = 0$ ):

- 1.)  $H^{\parallel}$  continuous,  $H_{1x} = H_{2x}$ ,  $H_{1y} = H_{2y}$ ,
- 2.)  $B^{\perp}$  continuous,  $\mu_0\mu_1 H_{1z} = \mu_0\mu_2 H_{2z}$ ,
- 3.)  $E^{\parallel}$  continuous,  $E_{1x} = E_{2x}$ ,  $E_{1y} = E_{2y}$ ,
- 4.)  $D^{\perp}$  continuous,  $\epsilon_0\epsilon_1 E_{1z} = \epsilon_0\epsilon_2 E_{2z}$ .

$$\curvearrowright E_{I0} + E_{R0} = E_{T0}, \quad \frac{n_1}{\mu_1} (E_{I0} - E_{R0}) \cos \theta_1 = \frac{n_2}{\mu_2} E_{T0} \cos \theta_2.$$

## Reflection and transmission, amplitudes

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s-polarization,  $\mathbf{E} \perp$  plane of incidence:

$$E_{T0}^{(s)} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos \theta_2} E_{I0}^{(s)}, \quad E_{R0}^{(s)} = \frac{n_1 \cos \theta_1 - \frac{\mu_1}{\mu_2} n_2 \cos \theta_2}{n_1 \cos \theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos \theta_2} E_{I0}^{(s)}.$$

p-polarization,  $\mathbf{E} \parallel$  plane of incidence:

(exercise)

$$E_{T0}^{(p)} = \frac{2n_1 \cos \theta_1}{\frac{\mu_1}{\mu_2} n_2 \cos \theta_1 + n_1 \cos \theta_2} E_{I0}^{(p)}, \quad E_{R0}^{(p)} = \frac{\frac{\mu_1}{\mu_2} n_2 \cos \theta_1 - n_1 \cos \theta_2}{\frac{\mu_1}{\mu_2} n_2 \cos \theta_1 + n_1 \cos \theta_2} E_{I0}^{(p)}.$$

- $n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \rightsquigarrow \quad \cos \theta_2 = \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_1}.$

## **Fresnel equations**

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For  $\mu_1 = \mu_2$ , specifically for  $\omega \in$  **optical regime**,  $\mu_1 = \mu_2 = 1$ :

- **s-polarization,  $\mathbf{E} \perp$  plane of incidence:**

$$\frac{E_{T0}^{(s)}}{E_{I0}^{(s)}} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2}, \quad \frac{E_{R0}^{(s)}}{E_{I0}^{(s)}} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2}.$$

- **p-polarization,  $\mathbf{E} \parallel$  plane of incidence:**

$$\frac{E_{T0}^{(p)}}{E_{I0}^{(p)}} = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2}, \quad \frac{E_{R0}^{(p)}}{E_{I0}^{(p)}} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2}.$$

Alternative form: eliminate  $n_1, n_2$  using Snell's law.

At **normal incidence**  $\theta_1 = \theta_2 = 0$ , (s)  $\simeq$  (p):

$$\frac{E_{T0}}{E_{I0}} = \frac{2n_1}{n_1 + n_2}, \quad \frac{E_{R0}}{E_{I0}} = \pm \frac{n_1 - n_2}{n_1 + n_2}.$$

## Transmittance and reflectance

- Power flow density  $\bar{\mathbf{S}}$  for the plane waves:  $\bar{\mathbf{S}} = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon}{\mu_0 \mu}} |E_0|^2 \frac{\mathbf{k}}{k}$ .
- Intensity  $I$ , power flow per unit area through the interface, normal  $\mathbf{e}_z$ :

$$I = |\bar{\mathbf{S}} \cdot \mathbf{e}_z| = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon}{\mu_0 \mu}} |E_0|^2 \left| \frac{\mathbf{k}}{k} \cdot \mathbf{e}_z \right|.$$

↪  $I_I = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon_1}{\mu_0 \mu_1}} |E_{I0}|^2 \cos \theta_1, \quad I_R = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon_1}{\mu_0 \mu_1}} |E_{R0}|^2 \cos \theta_1, \quad I_T = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon_2}{\mu_0 \mu_2}} |E_{T0}|^2 \cos \theta_2.$

- Transmittance  $T$ , transmission coefficient:  $T = \frac{I_T}{I_I} = \sqrt{\frac{\epsilon_2 \mu_1}{\mu_2 \epsilon_1}} \left| \frac{E_{T0}}{E_{I0}} \right|^2 \frac{\cos \theta_2}{\cos \theta_1},$
- Reflectance  $R$ , reflection coefficient:  $R = \frac{I_R}{I_I} = \left| \frac{E_{R0}}{E_{I0}} \right|^2.$
- Power balance:  $I_I = I_R + I_T, \quad R + T = 1. \quad (\text{exercise})$

## Brewster's angle

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- Reflectance for (p)-polarization:

$$R^{(p)} = \frac{I_R}{I_I} = \left| \frac{E_{R0}}{E_{I0}} \right|^2 \quad \text{with} \quad \frac{E_{R0}^{(p)}}{E_{I0}^{(p)}} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2}.$$

- $R^{(p)} = 0$  for  $\theta_1$  such that  $n_2 \cos \theta_1 - n_1 \cos \theta_2 = 0$

*(exercise)*

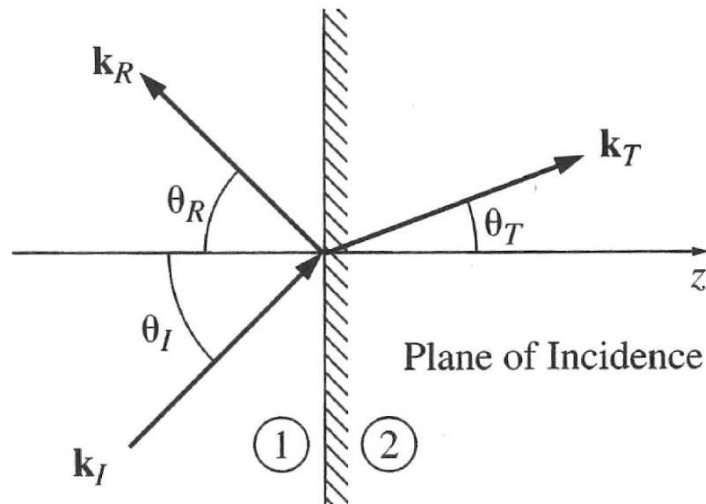
↪  $R^{(p)} = 0$  for  $\theta_1 = \theta_B$  with  $\tan \theta_B = \frac{n_2}{n_1}$ ,

$\theta_B$ : Brewster's angle.

Unpolarized incidence at  $\theta_1 = \theta_B$ :

- ↪ The reflected waves are purely s-polarized,  
p-polarized waves are fully transmitted.

# Total internal reflection



$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$
$$\sin \theta_2 = \frac{n_1}{n_2} \sin \theta_1.$$

$n_1 > n_2$ : former reasoning fails for  $\theta_1 > \theta_c$  where  $\frac{n_1}{n_2} \sin \theta_c = 1$ .

$\theta_c$ : **critical angle** for total internal reflection,  $\sin \theta_c = \frac{n_2}{n_1}$ .

Incidence at  $\theta_1 > \theta_c$ :

*(exercise)*

- no transmitted propagating wave, evanescent fields in region 2,
- the incident power is fully reflected back into region 1.

## Wave propagation in conductors

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law  $\mathbf{J}_f = \sigma \mathbf{E}$ ,  $\sigma$ : conductivity of the material.

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \dot{\mathbf{D}}.$$

$$\begin{aligned} \hookrightarrow \dot{\rho}_f &= -\frac{\sigma}{\epsilon_0 \epsilon} \rho_f, \quad \rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon} (t - t_0)\right), \\ \text{assume } \rho_f(\mathbf{r}, t_0) &= 0 \quad \rightsquigarrow \quad \rho_f(\mathbf{r}, t) = 0 \quad \forall t. \end{aligned}$$

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$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}.$$

## Telegrapher equation

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$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}$$

$$\hookrightarrow \Delta \mathbf{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{E}} - \mu_0 \mu \sigma \dot{\mathbf{E}} = 0, \quad \Delta \mathbf{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{H}} - \mu_0 \mu \sigma \dot{\mathbf{H}} = 0,$$

Telegrapher equation.

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Frequency domain:  $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}}(\mathbf{r}) e^{-i\omega t}$ ,

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu + i\omega \mu_0 \mu \sigma \right) \tilde{\mathbf{E}} = 0.$$

$$\text{Nonconducting media } \sigma = 0, \quad \Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0.$$

$$\text{Define } \bar{\epsilon} \text{ such that } \frac{\omega^2}{c^2} \bar{\epsilon} \mu = \frac{\omega^2}{c^2} \epsilon \mu + i\omega \mu_0 \mu \sigma, \quad \text{i.e. } \bar{\epsilon} = \epsilon + i \frac{\sigma}{\epsilon_0 \omega}$$

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + k_0^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \text{Helmholtz equation, } \bar{\epsilon} \in \mathbb{C}.$$

$$k_0 = \frac{\omega}{c}$$



## Wave attenuation

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$$\Delta \tilde{\mathbf{E}} + k_0^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \bar{\epsilon} \in \mathbb{C}$$

↪ solutions  $\sim e^{i(k_0 \bar{n} z - \omega t)}$

with refractive index  $\bar{n} = n' + i n'' = \sqrt{\bar{\epsilon} \mu} \in \mathbb{C}$ , (!)

$$e^{i(k_0 \bar{n} z - \omega t)} = e^{i(k_0 n' z - \omega t)} e^{-k_0 n'' z},$$

damped plane wave solutions.

Issues:

- penetration depth,
- $S$  and  $u$  decay with  $z$ ,
- still transverse waves,
- $E$ ,  $H$  no longer in phase,
- notions of wavenumber  $\in \mathbb{C}$ , wavelength, phase velocity.

## Localized waves

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- Plane waves  $\psi(z, t) = e^{i(kz - \omega t)}$  present at all  $z$ . (1-D)
- Combine these into *localized* functions

$$\Psi(z, t) = \int f(k) e^{i(kz - \omega t)} dk,$$

wave equation  $\left( \frac{\partial^2}{\partial z^2} - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = 0$  requires  $\omega = \frac{kc}{n}$ .

- **Dispersive media:**  $\epsilon(\omega), \mu(\omega) \rightsquigarrow n(\omega)$ ,  
frequency / wavelength dependent refractive index.
- $\omega = \frac{kc}{n(\omega)}$ , solve for  $\omega(k)$ : **dispersion relation**.

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$$\Psi(z, t) = \int f(k) e^{i(kz - \omega(k)t)} dk.$$

## Wave packets

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$$\Psi(z, t) = \int f(k) e^{i(kz - \omega(k)t)} dk.$$

- Assume  $f(k)$  is sharply peaked around  $k_p$ , (  $\leftarrow \rightsquigarrow$  weak localization of  $\Psi$  vs.  $z$ )  
such that only  $\omega(k)$  for  $k$  close to  $k_p$  contribute; expand  $\omega(k)$  around  $k_p$ :

$$\omega(k) \approx \omega(k_p) + (k - k_p) \left. \frac{d\omega}{dk} \right|_{k_p}, \quad \left. \frac{d\omega}{dk} \right|_{k_p} = v_g, \text{ group velocity.}$$

$$\begin{aligned} \Psi(z, t) &= e^{i(k_p z - \omega(k_p)t)} \hat{\Psi}(z - v_g t), \\ \hat{\Psi}(z - v_g t) &= \int f(k_p + q) e^{iq(z - v_g t)} dq. \end{aligned}$$

**Wave packet**  $\Psi(z, t)$ : here a plane wave  $(k_p, \omega(k_p))$ , modulated by an **envelope**  $\hat{\Psi}$  that moves with velocity  $v_g$  in direction  $z$ .

## Wave packets

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$$\Psi(z, t) = e^{i(k_p z - \omega(k_p) t)} \hat{\Psi}(z - v_g t)$$

- Simplest case: *no dispersion*,  $\omega(k) = \frac{k c}{n}$ ,  $n$  const.

↪  $\frac{d\omega}{dk} = \frac{c}{n} = c_m = \frac{\omega(k)}{k}$ , phase velocity  $c_m =$  group velocity  $v_g$ ,

$$\Psi(z, t) = e^{i k_p (z - c_m t)} \hat{\Psi}(z - c_m t).$$

- Dispersive case:  $v_g = \frac{d\omega}{dk} \neq \frac{\omega(k)}{k} = c_m$ .
- Group velocity  $v_g$ :
  - velocity for energy transport of the wave packet,
  - velocity for signal / information transport,
  - $v_g \leq c$ .

# Upcoming

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Next lecture:

- Maxwell equations, vectorial and scalar Helmholtz equation,
- 2D configurations (examples),
- mode problems, metallic and dielectric waveguides (examples).

