Electrodynamics

— *Lecture F* —

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Course overview

Electrodynamics

A	Introduction, Maxwell equations, brush up on vector calculus, Dirac delta, potentials, Taylor expansion, Fourier transform,
B	Maxwell equations, brush up on electro- and magnetostatics, multipole expansion
C	Maxwell equations, microscopic and macroscopic, time- and frequency domain, differential and integral form, interface conditions, continuity equation, energy and momentum of electromagnetic fields
D	Wave equation, plane waves, plane harmonic electromagnetic waves, refractive index, polarization, energy transport, spherical waves
E	Reflection and transmission at interfaces, lossy materials, wave packets, dispersion, phase and group velocity
F	Maxwell equations, vectorial and scalar Helmholtz equation, 2D configurations (intermezzo), mode problems, metallic and dielectric waveguides
G	Scalar and vector potentials, gauge conditions, retarded potentials, electric and magnetic dipole radiation
H	Special relativity; transmission lines

Helmholtz equation

- Maxwell equations, homog., linear media $\epsilon(\boldsymbol{r})$, $\mu(\boldsymbol{r})$, principal fields $\boldsymbol{E}$, $\boldsymbol{H}$:

$$\epsilon_0 \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{E}) = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \mu_0 \boldsymbol{\nabla} \cdot (\mu \boldsymbol{H}) = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \epsilon_0 \epsilon \dot{\boldsymbol{E}}.$$

- Frequency domain, $\sim \mathrm{e}^{-\mathrm{i}\omega t}$:

$$\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} = \mathrm{i}\omega\mu_0\mu\tilde{\boldsymbol{H}},$$

$$\boldsymbol{\nabla} \times \tilde{\boldsymbol{H}} = -\mathrm{i}\omega\epsilon_0\epsilon\tilde{\boldsymbol{E}}.$$

(implies $\epsilon_0 \boldsymbol{\nabla} \cdot (\epsilon \tilde{\boldsymbol{E}}) = 0$, $\mu_0 \boldsymbol{\nabla} \cdot (\mu \tilde{\boldsymbol{H}}) = 0$)

- Alternatives:
  - two coupled equations of first order for $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{H}}$, or
  - one second order equation, either for $\tilde{\boldsymbol{E}}$ or $\tilde{\boldsymbol{H}}$:

$$\mu \boldsymbol{\nabla} \times \frac{1}{\mu} \boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} - k_0^2 \epsilon \mu \tilde{\boldsymbol{E}} = 0,$$

$$\epsilon \boldsymbol{\nabla} \times \frac{1}{\epsilon} \boldsymbol{\nabla} \times \tilde{\boldsymbol{H}} - k_0^2 \epsilon \mu \tilde{\boldsymbol{H}} = 0, \qquad k_0 = \frac{\omega}{\mathrm{c}},$$

vectorial Helmholtz equation, components are coupled through $\partial_r \epsilon$, $\partial_r \mu$.

Helmholtz equation

$$\mu \boldsymbol{\nabla} \times \frac{1}{\mu} \boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} - k_0^2 \epsilon \mu \tilde{\boldsymbol{E}} = 0$$

$$\epsilon \boldsymbol{\nabla} \times \frac{1}{\epsilon} \boldsymbol{\nabla} \times \tilde{\boldsymbol{H}} - k_0^2 \epsilon \mu \tilde{\boldsymbol{H}} = 0$$

- Where ϵ ~~$(\boldsymbol{r})$~~, μ ~~$(\boldsymbol{r})$~~ :

$$\Delta \tilde{\boldsymbol{E}} + k_0^2 \epsilon \mu \tilde{\boldsymbol{E}} = 0,$$

$$\Delta \tilde{\boldsymbol{H}} + k_0^2 \epsilon \mu \tilde{\boldsymbol{H}} = 0, \qquad k_0 = \frac{\omega}{\mathrm{c}}, \qquad (*)$$

  scalar Helmholtz equation, valid separately for all components of $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{H}}$.

- Plane waves $\tilde{\boldsymbol{E}}(\boldsymbol{r}) = \boldsymbol{E}_0\, \mathrm{e}^{\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}}$, $\tilde{\boldsymbol{H}}(\boldsymbol{r}) = \boldsymbol{H}_0\, \mathrm{e}^{\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}}$, $\boldsymbol{k}^2 = k_0^2 \epsilon \mu$, solve $(*)$.

2-D problems

(Frequency domain, drop $\tilde{\cdot}$)

Assume $\partial_y \epsilon = 0,\ \partial_y \mu = 0,\ \partial_y \boldsymbol{E} = 0,\ \partial_y \boldsymbol{H} = 0:$

$$-\mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{pmatrix}, \qquad \mathrm{i}\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{pmatrix}.$$

Two decoupled sets of equations:

- $\{E_y, H_x, H_z\}:$ transverse electric (TE) fields, $\boldsymbol{E} \perp$ x-z-plane.
- $\{H_y, E_x, E_z\}:$ transverse magnetic (TM) fields, $\boldsymbol{H} \perp$ x-z-plane.

(Different conventions on the use of TE, TM.)

2-D TE waves

- Principal component E_y,

$$H_x = \frac{\mathrm{i}}{\omega\mu_0\mu}\,\partial_z E_y, \qquad H_z = \frac{-\mathrm{i}}{\omega\mu_0\mu}\,\partial_x E_y, \qquad -\mathrm{i}\omega\epsilon_0\epsilon E_y = \partial_z H_x - \partial_x H_z$$

↪ $$\mu\partial_x \frac{1}{\mu}\partial_x E_y + \mu\partial_z \frac{1}{\mu}\partial_z E_y + k_0^2\epsilon\mu E_y = 0. \qquad (*)$$

- Optical regime, $\mu = 1$, $\epsilon = n^2$: $\epsilon(\boldsymbol{r})$ (!)

↪ $$\partial_x^2 E_y + \partial_z^2 E_y + k_0^2 \epsilon E_y = 0, \qquad (**)$$

scalar 2-D (TE) Helmholtz equation.

- Reflection / transmission problem: s-polarized waves satisfy $(*)$, $(**)$.

2-D TM waves

- Principal component H_y,

$$E_x = \frac{-\mathrm{i}}{\omega\epsilon_0\epsilon}\,\partial_z H_y, \quad E_z = \frac{\mathrm{i}}{\omega\epsilon_0\epsilon}\,\partial_x H_y, \quad \mathrm{i}\omega\mu_0\mu H_y = \partial_z E_x - \partial_x E_z$$

↪ $$\epsilon\partial_x \frac{1}{\epsilon}\partial_x H_y + \epsilon\partial_z \frac{1}{\epsilon}\partial_z H_y + k_0^2 \epsilon\mu H_y = 0. \qquad (*)$$

- Optical regime, $\mu = 1$, $\epsilon = n^2$: $\epsilon(\boldsymbol{r})$ (!)

↪ $$\epsilon\partial_x \frac{1}{\epsilon}\partial_x H_y + \epsilon\partial_z \frac{1}{\epsilon}\partial_z H_y + k_0^2 \epsilon H_y = 0, \qquad (**)$$

scalar 2-D (TM) Helmholtz equation.

- Reflection / transmission problem: p-polarized waves satisfy $(*)$, $(**)$.

Waves in 2-D, numerical examples:

- Gaussian beams
- Slab waveguide: forward, backward, standing waves
- Waveguide facet
- Waveguide corner
- Bent waveguides
- Evanescent coupling of waveguides
- Ring resonator
- Square resonator
- Photonic crystal waveguide
- Bend in a photonic crystal waveguide
- Exciting IOMS

Waveguides: Mode problems

(Frequency domain, $\sim\ \mathrm{e}^{-\mathrm{i}\omega t}$, drop $\tilde{\cdot}$)

$$\nabla \times \boldsymbol{E} = \mathrm{i}\omega\mu_0\mu\boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = -\mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}.$$

- Waveguide: a system that is homogeneous along its axis z;
  $\partial_z\epsilon = 0,\ \partial_z\mu = 0,\ \partial_z n = 0.$

- Look for solutions that vary harmonically with z:

  $$\boldsymbol{E}(\boldsymbol{r}) = \bar{\boldsymbol{E}}(x,y)\,\mathrm{e}^{\mathrm{i}\beta z}, \quad \boldsymbol{H}(\boldsymbol{r}) = \bar{\boldsymbol{H}}(x,y)\,\mathrm{e}^{\mathrm{i}\beta z},$$

  modes of the waveguide, mode profile $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{H}}$, propagation constant β.

- Equations for $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{H}}$, β: (drop $\bar{\cdot}$)

$$-\mathrm{i}\omega\epsilon_0\epsilon\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \partial_y H_z - \mathrm{i}\beta H_y \\ \mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix}, \qquad \mathrm{i}\omega\mu_0\mu\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} \partial_y E_z - \mathrm{i}\beta E_y \\ \mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix}.$$

Waveguides: Mode equations

(Frequency domain, $\sim\ \mathrm{e}^{-\mathrm{i}\omega t}$, drop $\tilde{\cdot}$, $\bar{\cdot}$)

$$-\mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \partial_y H_z - \mathrm{i}\beta H_y \\ \mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix}, \qquad \mathrm{i}\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} \partial_y E_z - \mathrm{i}\beta E_y \\ \mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix}.$$

- Choice: solve for E_x, E_y, H_x, H_y in terms of E_z, H_z:

$$E_x = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\,(\beta\partial_x E_z + \omega\mu_0\mu\partial_y H_z),$$

$$E_y = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\,(\beta\partial_y E_z - \omega\mu_0\mu\partial_x H_z),$$

$$H_x = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\,(\beta\partial_x H_z - \omega\epsilon_0\epsilon\partial_y E_z),$$

$$H_y = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\,(\beta\partial_y H_z + \omega\epsilon_0\epsilon\partial_x E_z).$$

& vector equation for E_z, H_z.

Waveguides: Mode equations

(Frequency domain, $\sim\ \mathrm{e}^{-\mathrm{i}\omega t}$, drop $\tilde{\cdot}$, $\bar{\cdot}$)

$$E_x = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\left(\beta\partial_x E_z + \omega\mu_0\mu\partial_y H_z\right), \qquad H_x = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\left(\beta\partial_x H_z - \omega\epsilon_0\epsilon\partial_y E_z\right),$$

$$E_y = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\left(\beta\partial_y E_z - \omega\mu_0\mu\partial_x H_z\right), \qquad H_y = \frac{\mathrm{i}}{k_0^2\epsilon\mu - \beta^2}\left(\beta\partial_y H_z + \omega\epsilon_0\epsilon\partial_x E_z\right).$$

- Where ϵ ~~$(\boldsymbol{r})$~~, μ ~~$(\boldsymbol{r})$~~:

$$\Delta\tilde{\boldsymbol{E}} + k_0^2\epsilon\mu\tilde{\boldsymbol{E}} = 0, \qquad \Delta\tilde{\boldsymbol{H}} + k_0^2\epsilon\mu\tilde{\boldsymbol{H}} = 0$$

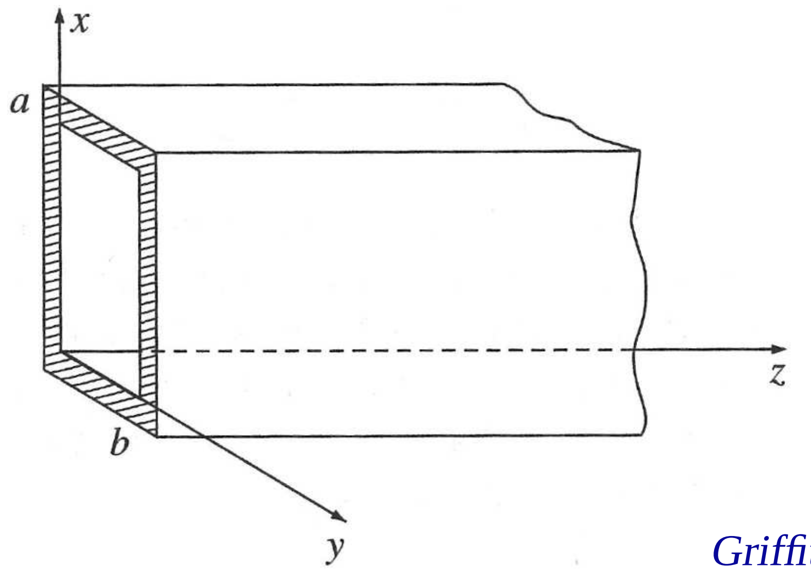
↪ $$\partial_x^2\boldsymbol{E} + \partial_y^2\boldsymbol{E} + (k_0^2\epsilon\mu - \beta^2)\boldsymbol{E} = 0, \qquad \partial_x^2\boldsymbol{H} + \partial_y^2\boldsymbol{H} + (k_0^2\epsilon\mu - \beta^2)\boldsymbol{H} = 0,$$

scalar mode equation, valid for all components of $\boldsymbol{E}$, $\boldsymbol{H}$,
to be supplemented by suitable boundary and interface conditions.

⟷ Eigenvalue problem with eigenvalue β, eigenfunction $\boldsymbol{E}$, $\boldsymbol{H}$.

Rectangular waveguides

A rectangular metal tube, typically microwave regime



Griffiths, Fig. 9.24, page 408

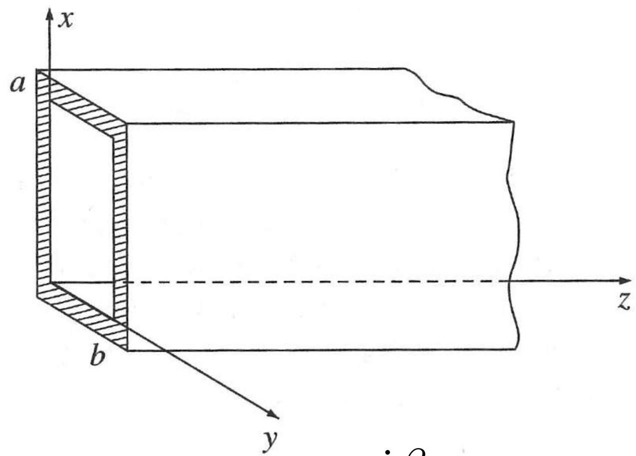
- Metal sidewalls, ideal conductors ⇝ $\boldsymbol{E} = 0$ in the walls;

  $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}} = 0, \quad \boldsymbol{B} = 0$ at some time t_0 ⇝ $\boldsymbol{B} = 0$ in the walls $\forall t$.

↪ Interface conditions: $\boldsymbol{E}^{\|} = 0, \; B^{\perp} = 0 \; \rightarrow \; H^{\perp} = 0$:

- $E_x = 0$ at $y = 0, b$,
- $E_y = 0$ at $x = 0, a$,
- $E_z = 0$ at $x = 0, a$, and at $y = 0, b$,
- $H_x = 0$ at $x = 0, a$,
- $H_y = 0$ at $y = 0, b$.

Rectangular waveguide, TE modes



Transverse electric (TE) fields:

$E_z = 0$, principal component H_z,

$$E_x = \frac{\mathrm{i}\omega\mu_0\mu}{k_0^2\epsilon\mu - \beta^2}\partial_y H_z, \quad E_y = \frac{-\mathrm{i}\omega\mu_0\mu}{k_0^2\epsilon\mu - \beta^2}\partial_x H_z, \quad H_x = \frac{\mathrm{i}\beta}{k_0^2\epsilon\mu - \beta^2}\partial_x H_z, \quad H_y = \frac{\mathrm{i}\beta}{k_0^2\epsilon\mu - \beta^2}\partial_y H_z.$$

solve (only $H_z \neq 0$ is of interest)

$$\partial_x^2 H_z + \partial_y^2 H_z + (k_0^2\epsilon\mu - \beta^2)H_z = 0 \quad \text{for } (x, y) \in [0, a] \times [0, b]$$

with $\partial_x H_z = 0$ at $x = 0, a$, and $\partial_y H_z = 0$ at $y = 0, b$.

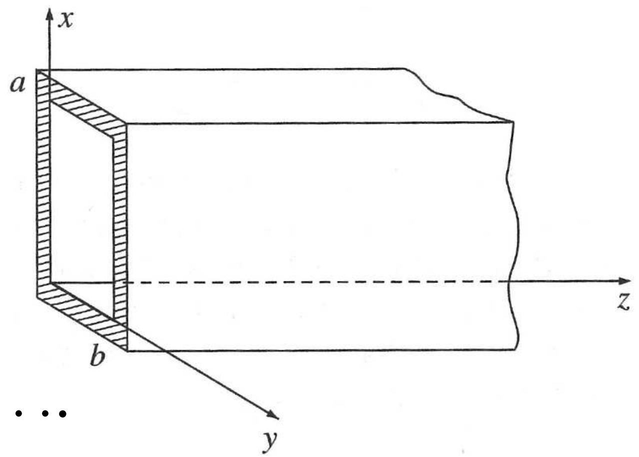
. . . separation of variables . . .

$$H_z(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right), \quad m, n = 0, 1, 2, \ldots$$

($m = n = 0$ excluded . . .)

with $\beta_{mn} = \sqrt{k_0^2\epsilon\mu - \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)}$, TE$_{mn}$ - modes.

Rectangular waveguide, TE modes



TE$_{mn}$ - modes, propagation constants:

$$\beta_{mn} = \sqrt{k_0^2 \epsilon\mu - \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)}, \quad m, n = 0, 1, 2, \ldots.$$

- Requirement for propagating waves:

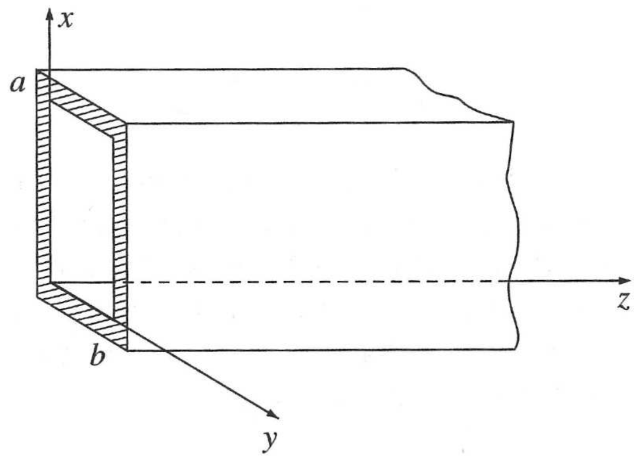
$$k_0^2 \epsilon\mu > \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \qquad \text{or} \qquad \omega^2 > \frac{c^2\pi^2}{\epsilon\mu}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) =: \omega_{mn}^2$$

ω_{mn} : cutoff frequency for mode TE$_{mn}$, no propagating mode for $\omega < \omega_{mn}$.

- Fundamental mode, principal mode TE$_{10}$, $(a > b)$

cutoff frequency $\omega_{10} = \dfrac{c\pi}{\sqrt{\epsilon\mu}\, a}$.

Rectangular waveguide, TE modes, dispersion



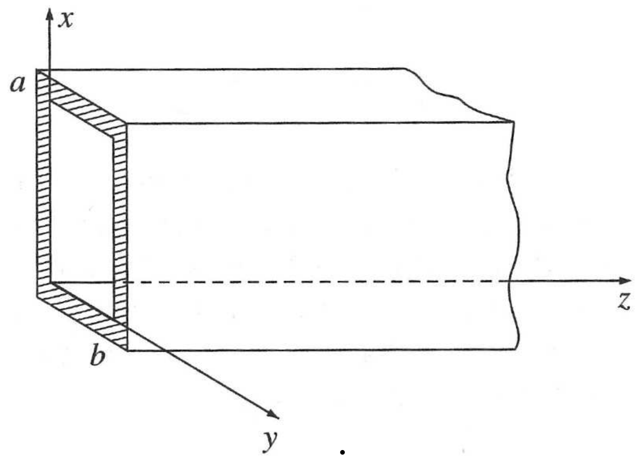
TE_{mn} - modes, propagation constants:

$$\beta_{mn} = \frac{\sqrt{\epsilon\mu}}{\mathrm{c}} \sqrt{\omega^2 - \omega_{mn}^2}, \quad m, n = 0, 1, 2, \ldots .$$

- $\beta(\omega)$ ⟷ $\omega(\beta)$, propagation $\sim \mathrm{e}^{\mathrm{i}\,(\beta z - \omega(\beta)\,t)}$, a dispersive system.
- Phase velocity $\mathrm{c}_\square = \dfrac{\omega}{\beta} = \dfrac{\mathrm{c}}{\sqrt{\epsilon\mu}} \left(\sqrt{1 - \dfrac{\omega_{mn}^2}{\omega^2}} \right)^{-1}$ (!)
- Group velocity $v_{\mathrm{g}} = \dfrac{\mathrm{d}\omega}{\mathrm{d}\beta} = \left(\dfrac{\mathrm{d}\beta}{\mathrm{d}\omega} \right)^{-1} = \dfrac{\mathrm{c}}{\sqrt{\epsilon\mu}} \sqrt{1 - \dfrac{\omega_{mn}^2}{\omega^2}}$ (!)

  velocity for signal transport, power transport.

Rectangular waveguide, TM modes



Transverse magnetic (TM) fields:

$H_z = 0$, principal component E_z,

$$E_x = \frac{\mathrm{i}\beta}{k_0^2\epsilon\mu - \beta^2}\partial_x E_z, \quad E_y = \frac{\mathrm{i}\beta}{k_0^2\epsilon\mu - \beta^2}\partial_y E_z, \quad H_x = \frac{-\mathrm{i}\omega\epsilon_0\epsilon}{k_0^2\epsilon\mu - \beta^2}\partial_y E_z, \quad H_y = \frac{\mathrm{i}\omega\epsilon_0\epsilon}{k_0^2\epsilon\mu - \beta^2}\partial_x E_z.$$

solve (only $E_z \neq 0$ is of interest)

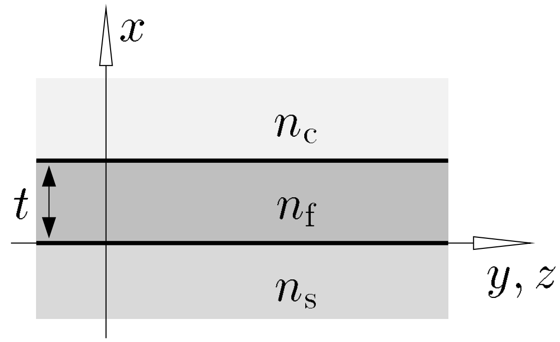
$$\partial_x^2 E_z + \partial_y^2 E_z + (k_0^2\epsilon\mu - \beta^2)E_z = 0 \quad \text{for } (x, y) \in [0, a] \times [0, b]$$

with $E_z = 0$ at $x = 0, a$, and $E_z = 0$ at $y = 0, b$.

(exercise)

Dielectric waveguides

- Confinement effected by locally higher refarctive index, $n_{\rm core} > n_{\rm cladding}$, total internal reflection.
- Realizations in the optical regime $\mu = 1,\ n^2 = \epsilon$:
  - optical fibers,
  - waveguides of integrated optics.



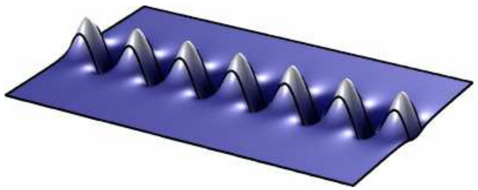
- Simplest example: planar slab waveguide:
  - 2-D TE configuration,

$$\partial_x^2 E_y + \partial_y^2 E_y + k_0^2 n^2 E_y = 0.$$

  - Ansatz: $E_y(x, z) = \varphi(x)\, {\rm e}^{{\rm i}\beta z}$.

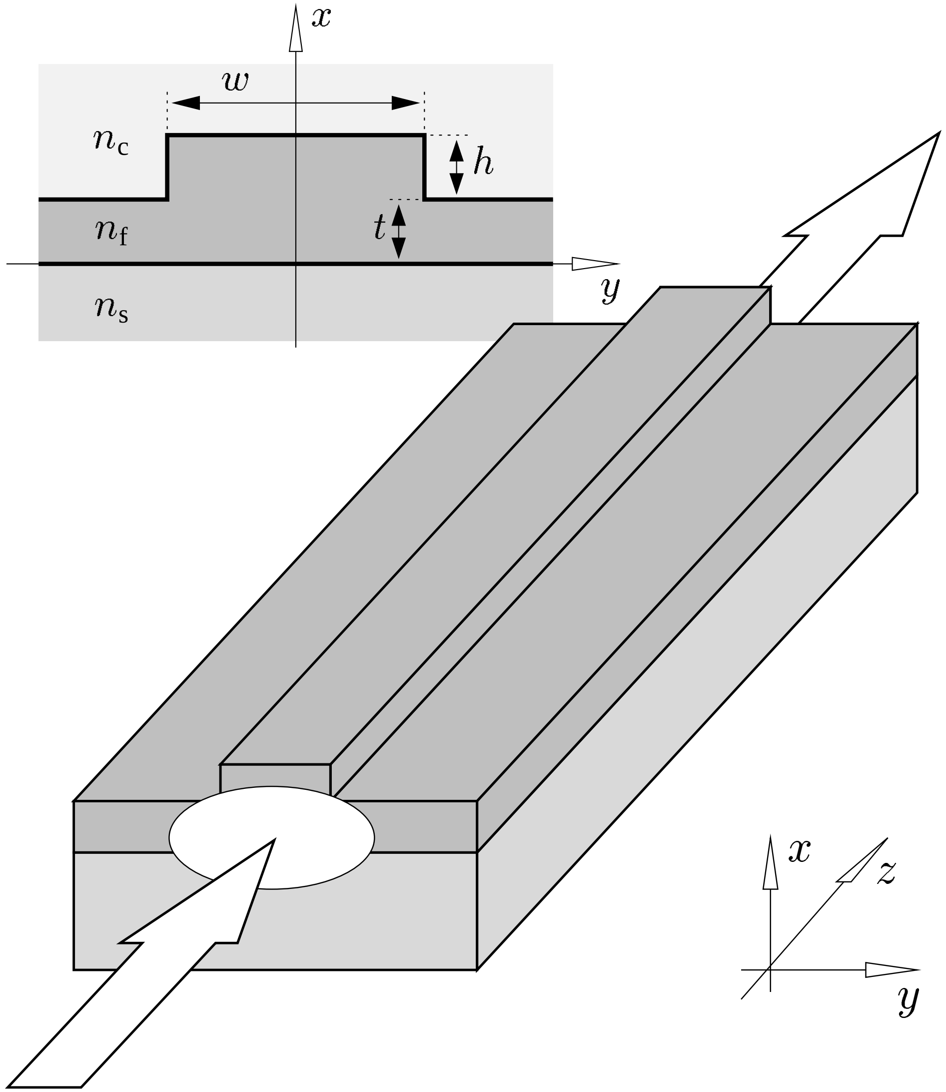
↳ $$\partial_x^2 \varphi + (k_0^2 n^2 - \beta^2)\varphi = 0,$$ scalar TE mode equation,

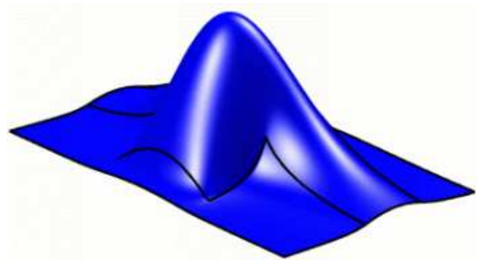
to be solved for nonzero φ with $\int \varphi\, {\rm d}x < \infty$,
for φ and $\partial_x \varphi$ continuous at $x = 0, t$.

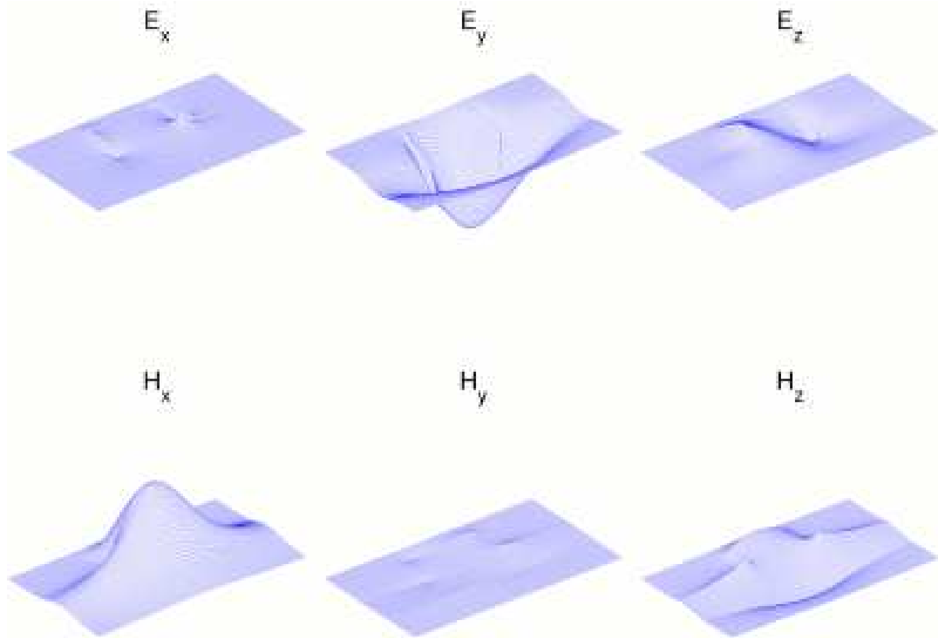


Rib waveguide

. . . variant of an integrated optical waveguide with 2-D confinement







Upcoming

Next lecture:

- scalar and vector potentials, gauge conditions,
- electric and magnetic dipole radiation