

1. Units

Assuming conventional use of symbols, figure out the physical units (SI) of the following quantities: \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} , ϵ_0 , μ_0 , $\nabla \cdot \mathbf{D}$, $\nabla \cdot \mathbf{B}$, $\nabla \times \mathbf{E}$, $\dot{\mathbf{B}}$, $\nabla \times \mathbf{H}$, $\dot{\mathbf{D}}$, $\mathbf{E} \times \mathbf{H}$, $\mathbf{E} \cdot \mathbf{D}$, $\mathbf{H} \cdot \mathbf{B}$, c , $\Delta \mathbf{E}$, $\ddot{\mathbf{E}}$.

2. Identities from vector calculus

Given a vector field $\mathbf{A}(\mathbf{r})$,

- (a) evaluate $\nabla \cdot (\nabla \times \mathbf{A})$, and
- (b) verify $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$

explicitly in Cartesian coordinates.

3. Curl and divergence

Assume that the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ satisfy the equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, and that $\mathbf{B}(\mathbf{r}, t_0) = 0$ at an arbitrary time t_0 . Show that then $\nabla \cdot \mathbf{B} = 0$ at all times t .

4. Plane waves

Show that fields $\psi(\mathbf{r}, t) = f(\mathbf{k} \cdot \mathbf{r} - \omega t) + b(\mathbf{k} \cdot \mathbf{r} + \omega t)$, with $\omega^2 = \mathbf{k}^2 c^2$, solve the wave equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\mathbf{r}, t) = 0,$$

for arbitrary, suitably smooth functions f, b .

5. Derivatives of harmonic plane waves

Given time-harmonic scalar and vector fields of the form

$$\psi(\mathbf{r}, t) = \psi_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad \mathbf{F}(\mathbf{r}, t) = \mathbf{F}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

with constant ψ_0 and \mathbf{F}_0 , make the substitution rules “ $\partial_t \rightarrow -i\omega$ ”, “ $\nabla \rightarrow i\mathbf{k}$ ” precise by evaluating the following expressions: $\dot{\psi}$, $\dot{\mathbf{F}}$, $\ddot{\psi}$, $\ddot{\mathbf{F}}$, $\nabla \psi$, $\nabla \cdot \mathbf{F}$, $\nabla \times \mathbf{F}$, $\Delta \psi$, $\Delta \mathbf{F}$.

6. Parseval identity

Show the identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk \tag{1}$$

for an integrable function f and its Fourier-transform \tilde{f} . Start by replacing the factors of $(\tilde{f}(k))^* \tilde{f}(k)$ on the right-hand side by their integral representation, and employ the Fourier representation of δ .

7. Restriction of a functional

Consider the functional

$$\mathcal{L}(u) = \int_{-1}^1 (u(x) - \sin(x))^2 dx$$

for functions in $\{u : [-1, 1] \rightarrow \mathbb{R}\}$ where it is well defined. Minimizing this functional is an example of the least squares method to approximate the function \sin by a function u from a given subset. Clearly, if any function u would be allowed, the solution of the minimization problem would be $u_{\min} = \sin$.

- (a) Let P be the set of polynomials of degree at most two:

$$P = \{u : \mathbb{R} \rightarrow \mathbb{R}, u(x) = a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}.$$

This is a three dimensional space. Restrict the functional to the space P and find the corresponding function L_P (this is a function of three variables); evaluate it as far as possible (you might wish to use a computer algebra system ...).

- (b) Find the minimizer of this function (a triple of coefficients a_0, a_1, a_2), and write down the minimizing polynomial. Illustrate the approximation of the \sin function by means of a suitable plot.

Hand in your solutions until Wednesday, April 24, 09:15. Good luck!