1. *Units*

   Assuming conventional use of symbols, figure out the physical units (SI) of the following quantities: $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$, ϵ_0, μ_0, $\boldsymbol{\nabla}\cdot\boldsymbol{D}$, $\boldsymbol{\nabla}\cdot\boldsymbol{B}$, $\boldsymbol{\nabla}\times\boldsymbol{E}$, $\dot{\boldsymbol{B}}$, $\boldsymbol{\nabla}\times\boldsymbol{H}$, $\dot{\boldsymbol{D}}$, $\boldsymbol{E}\times\boldsymbol{H}$, $\boldsymbol{E}\cdot\boldsymbol{D}$, $\boldsymbol{H}\cdot\boldsymbol{B}$, c, $\Delta\boldsymbol{E}$, $\ddot{\boldsymbol{E}}$.

2. *Identities from vector calculus*

   Given a vector field $\boldsymbol{A}(\boldsymbol{r})$,

   (a) evaluate $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})$, and
   (b) verify $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \Delta\boldsymbol{A}$

   explicitly in Cartesian coordinates.

3. *Curl and divergence*

   Assume that the fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ satisfy the equation $\boldsymbol{\nabla}\times\boldsymbol{E} = -\dot{\boldsymbol{B}}$, and that $\boldsymbol{B}(\boldsymbol{r},t_0) = 0$ at an arbitrary time t_0. Show that then $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ at all times t.

4. *Plane waves*

   Show that fields $\psi(\boldsymbol{r},t) = f(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) + b(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)$, with $\omega^2 = \boldsymbol{k}^2\mathrm{c}^2$, solve the wave equation

   $$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0,$$

   for arbitrary, suitably smooth functions f, b.

5. *Derivatives of harmonic plane waves*

   Given time-harmonic scalar and vector fields of the form

   $$\psi(\boldsymbol{r},t) = \psi_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \qquad \boldsymbol{F}(\boldsymbol{r},t) = \boldsymbol{F}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

   with constant ψ_0 and $\boldsymbol{F}_0$, make the substitution rules "$\partial_t \to -\mathrm{i}\omega$", "$\boldsymbol{\nabla} \to \mathrm{i}\boldsymbol{k}$" precise by evaluating the following expressions: $\dot{\psi}$, $\dot{\boldsymbol{F}}$, $\ddot{\psi}$, $\ddot{\boldsymbol{F}}$, $\boldsymbol{\nabla}\psi$, $\boldsymbol{\nabla}\cdot\boldsymbol{F}$, $\boldsymbol{\nabla}\times\boldsymbol{F}$, $\Delta\psi$, $\Delta\boldsymbol{F}$.

6. *Parseval identity*

   Show the identity

   $$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \mathrm{d}k \tag{1}$$

   for an integrable function f and its Fourier-transform $\tilde{f}$. Start by replacing the factors of $(\tilde{f}(k))^*\tilde{f}(k)$ on the right-hand side by their integral representation, and employ the Fourier representation of δ.

7. *Restriction of a functional*

   Consider the functional

   $$\mathcal{L}(u) = \int_{-1}^{1} (u(x) - \sin(x))^2\,\mathrm{d}x$$

   for functions in $\{u : [-1,1] \to \mathbb{R}\}$ where it is well defined. Minimizing this functional is an example of the least squares method to approximate the function $\sin$ by a function u from a given subset. Clearly, if any function u would be allowed, the solution of the minimization problem would be $u_{\text{min}} = \sin$.

   (a) Let P be the set of polynomials of degree at most two:

   $$P = \left\{u : \mathbb{R} \to \mathbb{R},\ u(x) = a_0 + a_1 x + a_2 x^2 \,|\, a_0, a_1, a_2 \in \mathbb{R}\right\}.$$

   This is a three dimensional space. Restrict the functional to the space P and find the corresponding function L_P (this is a function of three variables); evaluate it as far as possible (you might wish to use a computer algebra system ...).

   (b) Find the minimizer of this function (a triple of coefficients a_0, a_1, a_2), and write down the minimizing polynomial. Illustrate the approximation of the $\sin$ function by means of a suitable plot.

Hand in your solutions until Wednesday, April 29, 09:15. Good luck!