

1. Plane harmonic waves

Consider the following scalar harmonic plane waves. Here $\psi_0 \in \mathbb{R}$ is a constant amplitude, $\mathbf{k} \in \mathbb{R}^3$ and $\omega > 0$ are the wave-vector and frequency associated with the waves:

- (a) $\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$,
- (b) $\psi(\mathbf{r}, t) = \psi_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$,
- (c) $\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k} \cdot \mathbf{r} + \omega t)}$,
- (d) $\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \psi_0 e^{-i(\mathbf{k} \cdot \mathbf{r} + \omega t)}$,
- (e) $\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \frac{1}{2} \psi_0 e^{-i(\mathbf{k} \cdot \mathbf{r} + \omega t)}$.

Using the conventions for complex notation of time-harmonic fields, evaluate the real physical field, simplify as far as reasonable. Clarify the notions of a *forward traveling wave*, *backward traveling wave*, *standing wave*, and *partly traveling, partly standing wave*.

2. Time-averaged Poynting vector and energy density

- (a) Suppose that $a(\mathbf{r}, t)$ and $b(\mathbf{r}, t)$ are scalar fields (or components of vector fields) of the form $a(\mathbf{r}, t) = \tilde{a}(\mathbf{r}) e^{i\omega t}$, $b(\mathbf{r}, t) = \tilde{b}(\mathbf{r}) e^{i\omega t}$. Using the conventions for the complex notation of time-harmonic fields, here for angular frequency $\omega = 2\pi/T$, with time period T , write out the (real, physical) product $a b$, and show that the time-average

$$\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t') dt'$$

of that product evaluates to $\overline{a b} = \frac{1}{2} \text{Re}(\tilde{a}^* \tilde{b})$, where $*$ denotes complex conjugation.

- (b) Show that, for time-harmonic electromagnetic fields of the form $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}$ (\mathbf{D} , \mathbf{B} , \mathbf{H} analogously), the time-averaged Poynting vector $\bar{\mathbf{S}}$ and the time averaged energy density \bar{w} take the forms

$$\bar{\mathbf{S}} = \frac{1}{2} \text{Re}(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}), \quad \text{and} \quad \bar{w} = \frac{1}{4} \text{Re}(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} + \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}}).$$

3. Plane harmonic electromagnetic waves

Consider an electromagnetic wave with an electric field of the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)},$$

with frequency ω , wave vector \mathbf{k} , and amplitude vector \mathbf{E}_0 (remember the conventions for complex notation of time harmonic fields). The wave is assumed to propagate through a lossless isotropic homogeneous linear medium without free charges or currents, characterized by the relative permittivity ϵ , relative permeability μ , and refractive index $n = \sqrt{\epsilon\mu}$.

- (a) Write out the remaining parts \mathbf{D} , \mathbf{B} , \mathbf{H} of the electromagnetic field in terms of the quantities introduced above. State conditions such that all Maxwell equations are satisfied.
- (b) Show that, for this wave, the time averaged energy density \bar{w} and the time-averaged Poynting vector $\bar{\mathbf{S}}$ take the following alternative forms:

$$\bar{w} = \frac{1}{2} \epsilon_0 \epsilon |\mathbf{E}_0|^2 = \frac{1}{2} \mu_0 \mu |\mathbf{H}_0|^2,$$

$$\bar{\mathbf{S}} = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon}{\mu_0 \mu}} |\mathbf{E}_0|^2 \frac{\mathbf{k}}{k} = \frac{1}{2} \sqrt{\frac{\mu_0 \mu}{\epsilon_0 \epsilon}} |\mathbf{H}_0|^2 \frac{\mathbf{k}}{k} = \bar{w} c_m \frac{\mathbf{k}}{k}.$$

Here $c_m = (\sqrt{\epsilon_0 \epsilon \mu_0 \mu})^{-1}$ is the phase velocity in the medium, and \mathbf{H}_0 is the amplitude vector associated with the magnetic part of the wave field. (\rightarrow)

(3., continued)

Now consider a lossy medium, with the attenuation given by the imaginary part of the complex permittivity $\epsilon = \epsilon' - i\epsilon''$, with $\epsilon'' > 0$.

- (c) Show that the real- and imaginary parts of the complex refractive index $n = n' - in''$, defined through the relation $n^2 = \epsilon\mu$, are related to the complex permittivity ϵ and real permeability μ as

$$(n')^2 = \frac{\epsilon'\mu}{2} \left(1 + \sqrt{1 + \frac{(\epsilon'')^2}{(\epsilon')^2}} \right), \quad (n'')^2 = \frac{\epsilon'\mu}{2} \left(-1 + \sqrt{1 + \frac{(\epsilon'')^2}{(\epsilon')^2}} \right).$$

- (d) Modify your expressions from (3a) and (3b) to take into account the attenuation. Write the wavenumber $\mathbf{k} = k_0 n \boldsymbol{\kappa}$ as a product of the vacuum wavenumber $k_0 = \omega/c$, the complex refractive index n , and a direction $\boldsymbol{\kappa} \in \mathbb{R}^3$, with $|\boldsymbol{\kappa}|^2 = 1$. Show that the time-averaged Poynting vector and the time-averaged energy density can be given the form

$$\overline{\mathbf{S}}(\mathbf{r}) = \frac{k_0 n'}{2\omega\mu_0\mu} e^{-2k_0 n'' \boldsymbol{\kappa} \cdot \mathbf{r}} |\mathbf{E}_0|^2 \boldsymbol{\kappa}, \quad \overline{w}(\mathbf{r}) = \frac{\epsilon_0 (n')^2}{2\mu} e^{-2k_0 n'' \boldsymbol{\kappa} \cdot \mathbf{r}} |\mathbf{E}_0|^2,$$

such that the relation $\overline{\mathbf{S}}(\mathbf{r}) = \overline{w}(\mathbf{r}) (c/n') \boldsymbol{\kappa}$ holds. Verify that all results relate to *damped* waves, for positive ϵ'' , n'' and the time dependence $\sim \exp(i\omega t)$ introduced before.

4. 2-D Gaussian beams

The principal electric field component $E_y(x, z)$ of 2-D TE waves with vacuum wavelength $\lambda = 2\pi/k$ and vacuum wavenumber k in a homogeneous medium with refractive index n satisfies the 2-D Helmholtz equation

$$(\partial_x^2 + \partial_z^2 + k^2 n^2) E_y = 0. \quad (1)$$

Elementary solutions can be stated in the form

$$E_y(x, z) \sim e^{-i(k_x x + k_z z)}, \quad \text{with} \quad k^2 n^2 = k_x^2 + k_z^2. \quad (2)$$

We consider a Gaussian superposition of these solutions with wave vectors around the z -axis,

$$E_y(x, z) = E_0 \int e^{-k_x^2/\kappa^2} e^{-i(k_x x + k_z(k_x) z)} dk_x, \quad (3)$$

with a spectral width κ . Here E_0 is an arbitrary amplitude; the wavevector components k_x and k_z need to satisfy Eq. (2), which requires a dependence

$$k_z(k_x) = kn \sqrt{1 - \frac{k_x^2}{k^2 n^2}}. \quad (4)$$

- (a) For a well directed bundle with “narrow” spectral width κ , only waves with $|k_x^2/(k^2 n^2)| \ll 1$ contribute significantly. Use a first order approximation $\sqrt{1-x} \approx 1 - x/2$ of the squareroot in (4) to evaluate (3). Show that the 2-D Gaussian wave bundle can then be given the form

$$E_y(x, z) \approx E_0 \frac{2\sqrt{\pi}}{w(z)} e^{-x^2/w^2(z)} e^{-iknz}, \quad \text{with} \quad w(z) = \frac{2}{\kappa} \sqrt{1 - i \frac{\kappa^2}{2kn}} z. \quad (5)$$

You might wish to use the integral identity $\int_0^\infty e^{-at^2} \cos(2bt) dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-b^2/a}$, $\text{Re } a > 0$.

- (b) Split the exponent $-x^2/w^2$ of the first exponential term in Eq. (5) into real and imaginary parts. Interpret the functional dependence of the wave bundle on the coordinates x and z .
- (c) Visualize the field behaviour by means of suitable plots, e.g. plots of $|E_y|$ and $\text{Re} E_y$ versus x , y , for parameters $n = 1.0$, $\lambda = 1.0 \mu\text{m}$, and for beams with field-1/e-widths at focus $w(0) = 2/\kappa$ of $0.5 \mu\text{m}$ and $4 \mu\text{m}$.

Hand in your solutions until Wednesday, Mai 13, 09:15. Good luck!