1. *Counterpropagating waves, energy transport*

   Consider a superposition of two y-polarized plane harmonic electromagnetic waves with angular frequency ω and vacuum wavenumber k. Assume that these waves propagate in opposite directions along the z-axis, through a homogeneous, nonmagnetic medium with refractive index n. This relates to an electric field of the form

   $$E_y(z,t) = a_\mathrm{f}\,\mathrm{e}^{\mathrm{i}(\omega t - knz)} + a_\mathrm{b}\,\mathrm{e}^{\mathrm{i}(\omega t + knz)}\,,$$

   with complex amplitudes a_f and a_b. Evaluate the time averaged energy density and the time averaged power flux density $\boldsymbol{S}$. Compare with the absolute squares $|\boldsymbol{E}|^2$, $|\boldsymbol{H}|^2$ of the electric and magnetic field.

2. *Reflection of a plane wave at a surface of a potentially attenuating medium*

   Assume that the x-y-plane spans the interface between two regions (1), (2) filled with linear, nonmagnetic, and uncharged media. While region (1), $z < 0$, is filled with a lossless dielectric medium with (real) permittivity $\epsilon_1 = n_1^2$ and refractive index n_1, for the half-space (2), $z > 0$, we assume a potentially complex permittivity ϵ_2. There are no free charges or currents on the interface.

   This is a 2-D problem; let the coordinates be oriented such that all electromagnetic fields are constant in the y-direction. We restrict things to s-polarized waves. Their propagation is governed by the scalar 2-D TE Helmholtz equation

   $$(\partial_x^2 + \partial_z^2 + k^2\epsilon)E_y = 0 \tag{1}$$

   for the principal electric field component $E_y(x,z)$. Continuity of this field and of its normal derivative is required across all interfaces. $k = 2\pi/\lambda$ is the vacuum wavenumber, related to the vacuum wavelength λ. All fields oscillate in time $\sim \exp(\mathrm{i}\omega t)$ with angular frequency $\omega = k\mathrm{c}$.

   Assume that a plane with complex amplitude E_I propagates in region 1 towards the interface at an angle of incidence θ. The interface causes a reflected wave in region (1) with amplitude E_R. For region (2) we start with an ansatz of a separable field and an amplitude E_0, such that the electric field can be stated in the form

   $$E_y(x,z) = \begin{cases} E_\mathrm{I}\,\mathrm{e}^{-\mathrm{i}kn_1(z\cos\theta + x\sin\theta)} + E_\mathrm{R}\,\mathrm{e}^{-\mathrm{i}kn_1(-z\cos\theta + x\sin\theta)} & \text{for } z<0, \\ E_0\,X(x)\,Z(z) & \text{for } z>0. \end{cases} \tag{2}$$

   Here the functions X and Z are yet to be determined. Without loss of generality we choose amplitudes $X(0) = 1$ and $Z(0) = 1$.

   (a) Draw a sketch to clarify the geometry. Verify that the ansatz (2) satisfies Eq. (1) in region (1).

   (b) Use the continuity of E_y across the interface to determine X, and to relate E_0 to E_I and E_R.

   (c) E_y needs to satisfy Eq. (1) in region (2). Show that Z is of the form

   $$Z(z) = \mathrm{e}^{-\mathrm{i}k\kappa z}, \quad \text{with} \quad \kappa^2 = \epsilon_2 - n_1^2\sin^2\theta, \tag{3}$$

   where κ is a potentially complex constant.

   (d) Further the normal derivative $\partial_z E_y$ needs to be continuous at the interface. Use this requirement to derive the equations

   $$E_\mathrm{R} = \frac{n_1\cos\theta - \kappa}{n_1\cos\theta + \kappa}\,E_\mathrm{I}, \qquad E_0 = \frac{2\,n_1\cos\theta}{n_1\cos\theta + \kappa}\,E_\mathrm{I} \tag{4}$$

   that relate the amplitudes of the reflected and ”transmitted” waves to the amplitude of the incoming wave. Compare with the Fresnel-equations for s-polarized waves.

(→)

(2., continued)

(e) First consider the case of a lossless medium in region (2), with $\epsilon_2 \in \mathbb{R}$. Introduce the refractive index n_2 for region (2). Distinguish between two cases with $n_2^2 > n_1^2 \sin^2\theta$ and $n_2^2 < n_1^2 \sin^2\theta$.

i. For $n_2^2 > n_1^2 \sin^2\theta$, select a sign for κ such that the field in region (2) is an outgoing travelling wave propagating at an angle θ_2 that is related to the incoming wave by Snell's law. State the field in region (2), and verify that $|E_\mathrm{R}|^2 < |E_\mathrm{I}|^2$, due to the power carried by the transmitted wave.

ii. For $n_2^2 < n_1^2 \sin^2\theta$, select a sign for κ such that the field in region (2) is damped in the $+z$-direction. State the field in region (2), and verify that $|E_\mathrm{R}|^2 = |E_\mathrm{I}|^2$, i.e. that there is total reflection at the interface.

(f) Now specialize to a lossy medium in region (2), with $\epsilon_2 \notin \mathbb{R}$. Assume a permittivity $\epsilon_2 = \epsilon' - \mathrm{i}\epsilon''$ with $\epsilon' < 0$ and $\epsilon'' > 0$, as is the case for many metals.

i. Evaluate the wavenumber κ, choosing signs of the roots such that $\mathrm{Re}\kappa > 0$, and $\mathrm{Im}\kappa < 0$. Verify that the field in region (2) is then an outwards ($+z$) propagating damped wave.

ii. Show that $|E_\mathrm{R}|^2/|E_\mathrm{I}|^2 < 1$, i.e. that the reflection at the metallic surface is always accompanied by losses.

iii. Evaluate the reflectance $R = |E_\mathrm{R}|^2/|E_\mathrm{I}|^2$ (Why is this correct here?) for the reflection of a plane wave coming in from air ($n_1 = 1$) towards a metal surface, as a function of the angle of incidence θ. Use the material properties [1] of silver $\epsilon_\mathrm{Ag} = -14.5 - \mathrm{i}1.2$ and copper $\epsilon_\mathrm{Cu} = -11.7 - \mathrm{i}2.1$ at the wavelength $\lambda = 0.633\,\mu$m of a He-Ne-laser.

[1] M. N. Polyanskiy. "Refractive index database", http://refractiveindex.info
(accessed May 05, 2016).

3. Commercial software suitable for simulations in photonics / integrated optics is being offered by a number of companies. Examples are (state of early 2019, a list without any claim to completeness):

- JCMwave,
- Optiwave,
- Synopsys,
- Photon Design,
- CST,
- Lumerical.

In view of the classes of typical simulation tasks that we discussed during the lecture: Try to get an overview of what types of solvers are offered, and try to get an idea about the numerical schemes that the programs rely on.

Hand in your solutions until Wednesday, May 27, 09:15. Good luck!