

1. Vectorial mode equations (transverse- $H$ -variant)

Specialize the Maxwell-curl equation in the frequency domain  $\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}$ ,  $\nabla \times \mathbf{H} = i\omega\epsilon_0\epsilon\mathbf{E}$  to a waveguide configuration with its axis parallel to the Cartesian  $z$ -axis (use of symbols as in the lecture). Introduce a field in the form of a waveguide mode with mode profile  $\bar{\mathbf{E}}$ ,  $\bar{\mathbf{H}}$  and propagation constant  $\beta$ , and select the transverse magnetic components of the mode profile  $\bar{H}_x$ ,  $\bar{H}_y$  as principal fields. Show that these satisfy the vectorial mode equations

$$\partial_x^2 \bar{H}_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y \bar{H}_x + \partial_{xy} \bar{H}_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x \bar{H}_y + (k^2 \epsilon - \beta^2) \bar{H}_x = 0, \quad (1)$$

$$\epsilon \partial_x \frac{1}{\epsilon} \partial_x \bar{H}_y + \partial_y^2 \bar{H}_y + \partial_{yx} \bar{H}_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y \bar{H}_x + (k^2 \epsilon - \beta^2) \bar{H}_y = 0, \quad (2)$$

and that the electromagnetic profile components can be expressed through the principal fields as

$$\begin{pmatrix} \bar{E}_x \\ \bar{E}_y \\ \bar{E}_z \end{pmatrix} = \frac{1}{\omega\epsilon_0\epsilon} \begin{pmatrix} \beta \bar{H}_y - \beta^{-1}(\partial_{yx} \bar{H}_x + \partial_y^2 \bar{H}_y) \\ -\beta \bar{H}_x + \beta^{-1}(\partial_{xy} \bar{H}_y + \partial_x^2 \bar{H}_x) \\ -i(\partial_x \bar{H}_y - \partial_y \bar{H}_x) \end{pmatrix}, \quad \begin{pmatrix} \bar{H}_x \\ \bar{H}_y \\ \bar{H}_z \end{pmatrix} = \begin{pmatrix} \bar{H}_x \\ \bar{H}_y \\ -i\beta^{-1}(\partial_x \bar{H}_x + \partial_y \bar{H}_y) \end{pmatrix}. \quad (3)$$

## 2. 2-D mode equations

Consider 2-D TE and TM problems in the frequency domain, as discussed in the lecture, for a 2-D waveguide configuration with its axis oriented along the Cartesian  $z$ -axis (assume nonmagnetic media,  $\mu = 1$ ). The structure is constant along the  $y$ -axis; the permittivity  $\epsilon$  depends (smoothly, or with discontinuities) on the  $x$ -coordinate only. Introduce a field in the form of a waveguide mode with 1-D mode profile  $\bar{\mathbf{E}}(x)$ ,  $\bar{\mathbf{H}}(x)$  and propagation constant  $\beta$ .

- (a) For the 2-D TE setting, choose the transverse electric component  $\bar{E}_y$  of the mode profile as the principal field. Show that this component satisfies the scalar 2-D TE mode equation

$$\partial_x^2 \bar{E}_y + (k^2 \epsilon - \beta^2) \bar{E}_y = 0, \quad (4)$$

that the other electromagnetic field components can be expressed as

$$\bar{E}_x = 0, \quad \bar{E}_z = 0, \quad \bar{H}_x = \frac{-\beta}{\omega\mu_0} \bar{E}_y, \quad \bar{H}_y = 0, \quad \bar{H}_z = \frac{i}{\omega\mu_0} \partial_x \bar{E}_y, \quad (5)$$

and that continuity of  $\bar{E}_y$  and of  $\partial_x \bar{E}_y$  is required at any dielectric interfaces.

- (b) Now look at the 2-D TM setting. Choose the transverse magnetic component  $\bar{H}_y$  of the mode profile as the principal field. Show that this component satisfies the scalar 2-D TM mode equation

$$\epsilon \partial_x \frac{1}{\epsilon} \partial_x \bar{H}_y + (k^2 \epsilon - \beta^2) \bar{H}_y = 0, \quad (6)$$

that all electromagnetic field components can be expressed as

$$\bar{E}_x = \frac{\beta}{\omega\epsilon_0\epsilon} \bar{H}_y, \quad \bar{E}_y = 0, \quad \bar{E}_z = \frac{-i}{\omega\epsilon_0\epsilon} \partial_x \bar{H}_y, \quad \bar{H}_x = 0, \quad \bar{H}_z = 0, \quad (7)$$

and that continuity of  $\bar{H}_y$  and of  $\frac{1}{\epsilon} \partial_x \bar{H}_y$  is required at any dielectric interfaces.

### 3. Guided modes of dielectric multilayer slab waveguides

Consider the solver for the guided modes of dielectric multilayer slab waveguides with 1-D cross sections at [www.siio.eu/oms.html](http://www.siio.eu/oms.html).

Imagine that you would need to test the results for reliability. To that end, have the script compute the guided TE and TM modes of a waveguide with one interior core layer with refractive index 2.0 and a thickness of  $0.5\ \mu\text{m}$ , on top of a substrate with refractive index 1.5, covered by air with refractive index 1.0, at a wavelength of  $1.55\ \mu\text{m}$ . The solver output needs to satisfy the relations from problem 2, and the results need to describe guided modes (decaying fields for  $x \rightarrow \pm\infty$ ).

Outline a procedure for numerical assessment of the mode solver results:

- Check the properties of the profile components (dis- / continuity, signs) that can be verified by inspecting the respective mode profile plots.
- Describe a numerical procedure for verifying the differential equations, after exporting adequate mode profile components.

### 4. Guided modes of a rectangular dielectric channel, approximate solutions

Consider the solver for the guided modes of dielectric multilayer waveguides with rectangular 2-D cross sections at [www.siio.eu/eims.html](http://www.siio.eu/eims.html). Note that this is an explicitly *approximate* solver.

Imagine that you would need to quickly check the results for reliability. To that end, have the script compute the guided TE- and TM-like modes of a waveguide with a rectangular core of width  $1.0\ \mu\text{m}$ , thickness  $0.4\ \mu\text{m}$ , with refractive index 1.99, embedded in a material with refractive index 1.45, at a wavelength of  $1.55\ \mu\text{m}$ .

- State the properties (differential equations (the most appropriate of the many forms), dis- / continuity, symmetry, ...) that need to be satisfied by the modes that are exact solutions of the problem that the solver addresses.
- Prepare a list of issues where the solver results obviously violate these conditions.