

Optical Waveguide Theory



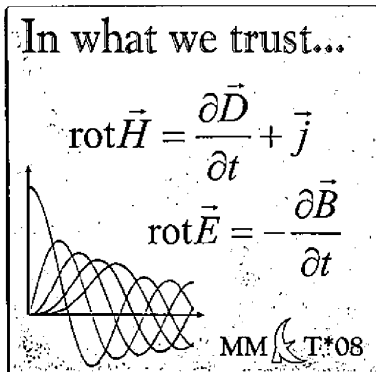
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Paderborn University — Summer Semester 2020

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MMET*08, Mathematical Methods in Electromagnetic Theory
Odesa, Ukraine, June 29 – July 2, 2008

Maxwell equations

SI, in matter, time domain, differential form:

$\nabla \cdot \mathbf{D} = \rho_f,$	$\mathbf{E}(\mathbf{r}, t)$: electric field,
$\nabla \times \mathbf{E} = -\dot{\mathbf{B}},$	$\mathbf{D}(\mathbf{r}, t)$: (di-)electric displacement,
$\nabla \cdot \mathbf{B} = 0,$	$\mathbf{B}(\mathbf{r}, t)$: magnetic induction (field, flux density),
$\nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}},$	$\mathbf{H}(\mathbf{r}, t)$: magnetic field (...),
	$\rho_f(\mathbf{r}, t)$: density of free charges,
	$\mathbf{J}_f(\mathbf{r}, t)$: density of free currents,
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},$	$\mathbf{P}(\mathbf{r}, t)$: polarization,
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}).$	$\mathbf{M}(\mathbf{r}, t)$: magnetization,
	ϵ_0 : free space permittivity,
(+ constitutive relations)	μ_0 : free space permeability.

Valid for more than a century, firm basis for further considerations.

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

Formalities

Organization of the course:



- Lectures ($\approx 14\times$)
- Homework ($7\times$)
- Tutorials, Exercises ($13\times$)
- Exam

Related textbooks (examples):

C. Vassallo, *Optical Waveguide Concepts*, Elsevier, Amsterdam (1991),
K. Okamoto, *Fundamentals of Optical Waveguides*, Academic Press, San Diego, USA (2000),
R. März, *Integrated Optics: Design and Modeling*, Artech House, Norwood, USA (1995),
A.W. Snyder, J.D. Love, *Optical Waveguide Theory*, Chapman and Hall, London, UK (1983);
& general introductory texts on classical electrodynamics.

Optical waveguides: phenomena, examples

- Beam propagation in free space
- Guided light propagation
- Waveguide end facet
- Crossing of two waveguides
- Modes of 1-D multilayer slab waveguides
- Modes of 2-D channel waveguides
- Circular step-index optical fibers
- Evanescent coupling between waveguides
- Bent waveguides
- Circular microring-resonator
- Microdisk resonator
- CROW
- Waveguide corner
- Photonic crystal waveguide
- Exciting TET !



Optical waveguide “theory”

Task: solve

$$\begin{aligned}\nabla \times \mathbf{E} &= -\dot{\mathbf{B}}, & \nabla \cdot \mathbf{D} &= \rho_f, & \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P}, \\ \nabla \times \mathbf{H} &= \mathbf{J}_f + \dot{\mathbf{D}}, & \nabla \cdot \mathbf{B} &= 0, & \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}), \quad (\& \dots).\end{aligned}$$

In this course:

- specialization to problems relevant for integrated optics,
- theoretical basis for the — mostly — numerical solution,
- approximate concepts,
- examples.

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Vector calculus, keywords

Ingredients:

(here: Cartesian coordinates)

- Space and time coordinates: $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow (x, y, z), t.$

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- Time derivatives: $\frac{\partial \phi}{\partial t}, \partial_t \phi, \dot{\phi}, \nabla_t \phi$.

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- Curl: $\text{curl}\mathbf{A} = \text{rot}\mathbf{A} = \nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$.

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- Laplacian: $\Delta = \nabla \cdot \nabla = \nabla^2$,
$$\Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi, \quad \Delta\mathbf{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$$

Dirac delta

A linear functional
that extracts the value of a function at one point:



1-D:
$$\int_a^b f(x) \delta(x - x_0) dx = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$
$$\delta(x - x_0) = 0, \text{ if } x \neq x_0.$$

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$$\text{3-D: } \int_{\mathcal{V}} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) d\mathcal{V} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(\mathbf{r} - \mathbf{r}_0) = 0, \text{ if } \mathbf{r} \neq \mathbf{r}_0.$$

Implications: manifold.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

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- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$.

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- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$.
- $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$.

Fourier transform

3-D: A field $\phi(\mathbf{r})$:

$$\phi(\mathbf{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3k, \quad \tilde{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3r.$$

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4-D: A field $\phi(\mathbf{r}, t)$:

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iint \tilde{\phi}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3k d\omega,$$
$$\tilde{\phi}(\mathbf{k}, \omega) = \frac{1}{\sqrt{2\pi}^4} \iint \phi(\mathbf{r}, t) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3r dt.$$

Directionally constant systems

A **linear** PDE in two unknowns

$$(A \partial_{xx} + B \partial_{yy} + C \partial_{xy} + D \partial_x + E \partial_y + F) \psi(x, y) = 0,$$

coefficients $A(x, y), \dots, F(x, y)$.

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If the system is **constant in x** , $\partial_x A = \dots = \partial_x F = 0$,

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↪ $\int (B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) e^{ikx} dk = 0,$

↪ $(B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) = 0, \text{ (for all } k),$

... a set of DEs in one unknown.

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(& boundary conditions, ...)

General solution of the wave equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\mathbf{r}, t) = 0, \quad \psi(\mathbf{r}, 0) = \psi_0(\mathbf{r}), \quad \partial_t \psi(\mathbf{r}, 0) = \phi_0(\mathbf{r}),$$

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• $\psi(\mathbf{r}, 0) = \psi_0(\mathbf{r}), \quad \partial_t \psi(\mathbf{r}, 0) = \phi_0(\mathbf{r}) \rightsquigarrow \dots \rightsquigarrow a_f(\mathbf{k}), a_b(\mathbf{k}).$

A touch of variational calculus

- **Functional:**
$$\begin{aligned}\mathcal{L} : U &\longrightarrow \mathbb{R}, \mathbb{C}, \\ u &\longrightarrow \mathcal{L}(u),\end{aligned}$$

a map from a space U of functions to real / complex numbers.

A touch of variational calculus

- **Functional:**
$$\begin{aligned}\mathcal{L} : U &\longrightarrow \mathbb{R}, \mathbb{C}, \\ u &\longrightarrow \mathcal{L}(u),\end{aligned}$$

a map from a space U of functions to real / complex numbers.

- **Stationary** functional:
$$\left. \frac{d}{ds} \mathcal{L}(u + s v) \right|_{s=0} = 0 \quad \text{for all } v,$$

the variation of \mathcal{L} at u vanishes for arbitrary directions v .

A touch of variational calculus

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- **Restriction** of a functional:

... to a parametrized family of functions;

➡ extremization with respect to these parameters,

➡ approximations of stationary points of the functional.

A touch of variational calculus

Example:

$$U = \{u : [0, \pi] \rightarrow \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \rightarrow \mathbb{R},$$

(...)

$$\mathcal{L}(u) = \frac{\int_0^\pi (\partial_x u)^2 dx}{\int_0^\pi u^2 dx}.$$

A touch of variational calculus

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\mathcal{L} stationary at u ,

$$\frac{d}{ds} \mathcal{L}(u + s v) \Big|_{s=0} = 0 \quad \forall v.$$



u satisfies DE & b.c.,

$$\begin{aligned} \partial_x^2 u &= -\lambda u, \quad \lambda = \mathcal{L}(u), \\ u(0) &= u(\pi) = 0. \end{aligned}$$

A touch of variational calculus

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Restrict \mathcal{L} , $L(a) = \mathcal{L}(u|_a)$.

L stationary at a , $\nabla_a L = 0$.



Approximate solution
of DE / eigenproblem.

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

...?

“This concerns time harmonic fields ... with angular frequency ..., for vacuum wavenumber ..., speed of light ..., and wavelength ...”

“The problem is governed by the Maxwell curl equations in the frequency domain for the electric field ... and magnetic field ..., for (lossless) uncharged dielectric, nonmagnetic linear (isotropic) media with (piecewise constant) relative permittivity ... :
... (.) ”

[M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016)]

Maxwell equations, Fourier transform

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}$$

$$\& \quad \mathbf{F}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\mathbf{F}}(\mathbf{r}, \omega) e^{i\omega t} d\omega, \quad \tilde{\mathbf{F}}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int \mathbf{F}(\mathbf{r}, t) e^{-i\omega t} dt$$

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$$\mathbf{E}(\mathbf{r}, t), \mathbf{D}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t), \rho_f(\mathbf{r}, t), \mathbf{J}_f(\mathbf{r}, t)$$

$$\longleftrightarrow \tilde{\mathbf{E}}(\mathbf{r}, \omega), \tilde{\mathbf{D}}(\mathbf{r}, \omega), \tilde{\mathbf{B}}(\mathbf{r}, \omega), \tilde{\mathbf{H}}(\mathbf{r}, \omega), \tilde{\rho}_f(\mathbf{r}, \omega), \tilde{\mathbf{J}}_f(\mathbf{r}, \omega),$$

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_f, \quad \nabla \times \tilde{\mathbf{E}} = -i\omega \tilde{\mathbf{B}}, \quad \nabla \cdot \tilde{\mathbf{B}} = 0, \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}_f + i\omega \tilde{\mathbf{D}}$$

(Caution: arbitrary choice of $\sim \exp(\pm i\omega t)$!).

Maxwell equations, frequency domain

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$$\mathbf{F}(\mathbf{r}, t) \in \mathbb{R} \rightsquigarrow \tilde{\mathbf{F}}(\mathbf{r}, -\omega) = (\tilde{\mathbf{F}}(\mathbf{r}, \omega))^*$$

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$$\text{"at frequency } \omega_0\text{"}: \tilde{\mathbf{F}}(\mathbf{r}, \omega) = \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}(\mathbf{r}) \delta(\omega - \omega_0) + \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}^*(\mathbf{r}) \delta(\omega + \omega_0)$$

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$$\mathbf{F}(\mathbf{r}, t) = \operatorname{Re} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} \right\},$$

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$$\hookrightarrow \bar{\mathbf{E}}(\mathbf{r}), \bar{\mathbf{D}}(\mathbf{r}), \bar{\mathbf{B}}(\mathbf{r}), \bar{\mathbf{H}}(\mathbf{r}), \bar{\rho}_f(\mathbf{r}), \bar{\mathbf{J}}_f(\mathbf{r}), \sim \exp(i\omega_0 t),$$

$$\nabla \cdot \bar{\mathbf{D}} = \bar{\rho}_f, \quad \nabla \times \bar{\mathbf{E}} = -i\omega_0 \bar{\mathbf{B}}, \quad \nabla \cdot \bar{\mathbf{B}} = 0, \quad \nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}}_f + i\omega_0 \bar{\mathbf{D}}.$$

Caution: Decorations $\sim, \bar{\cdot}, \omega_0$ are usually omitted; context determines interpretation of symbols.

Polarization

$\tilde{\mathbf{P}}$: density of electric dipole moment (bound charges).

$$\tilde{\mathbf{D}} = \epsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}, \quad [\tilde{\mathbf{D}}] = [\tilde{\mathbf{P}}] = \frac{\text{As m}}{\text{m}^3}, \quad [\tilde{\mathbf{E}}] = \frac{\text{V}}{\text{m}},$$


▶ vacuum permittivity $\epsilon_0 = 8.854187817 \dots \cdot 10^{-12} \left[\frac{\text{F}}{\text{m}} = \frac{\text{As}}{\text{Vm}} \right]$.

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• Local dipoles induced by $\tilde{\mathbf{E}} \rightsquigarrow \tilde{\mathbf{P}}(\tilde{\mathbf{E}})$.

• Linear dielectrics:

$$\tilde{\mathbf{P}} = \epsilon_0 \hat{\chi}_e \tilde{\mathbf{E}}, \quad \hat{\chi}_e: \text{dielectric susceptibility}, \quad [\hat{\chi}_e] = \hat{1}.$$

↪ $\tilde{\mathbf{D}} = \epsilon_0 (\hat{1} + \hat{\chi}_e) \tilde{\mathbf{E}} = \epsilon_0 \hat{\epsilon} \tilde{\mathbf{E}}, \quad \hat{\epsilon}: \text{relative permittivity}, \quad [\hat{\epsilon}] = \hat{1}.$

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- $\hat{\chi}_e(\mathbf{r}, \omega)$, $\hat{\epsilon}(\mathbf{r}, \omega)$ are determined in the frequency domain.
- Complications: $\text{Im } \epsilon$, $\hat{\epsilon}(T)$, $\hat{\epsilon}(\mathbf{F})$, $\chi_{jkl}^{(2)} E_k E_l$, $\chi_{jklm}^{(3)} E_k E_l E_m, \dots$
- Simpler cases: $\hat{\epsilon}(\mathbf{r})$, $\hat{\epsilon} = \epsilon \hat{1}$.

Magnetization

$\tilde{\mathbf{M}}$: density of magnetic dipole moments (bound currents).

$$\tilde{\mathbf{H}} = \frac{1}{\mu_0} \tilde{\mathbf{B}} - \tilde{\mathbf{M}}, \quad [\tilde{\mathbf{H}}] = [\mathbf{M}] = \frac{\text{A m}^2}{\text{m}^3}, \quad [\tilde{\mathbf{B}}] = \text{T} = \frac{\text{Vs}}{\text{m}^2},$$



vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{\text{N}}{\text{A}^2} = \frac{\text{Vs}}{\text{Am}} \right]$.


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$$\begin{aligned} \tilde{\mathbf{M}} &= \hat{\chi}_m \tilde{\mathbf{H}}, & \hat{\chi}_m: \text{magnetic susceptibility, } [\hat{\chi}_m] &= \hat{1}. \\ \tilde{\mathbf{B}} &= \mu_0 (\hat{1} + \hat{\chi}_m) \tilde{\mathbf{H}} = \mu_0 \hat{\mu} \tilde{\mathbf{H}}, & \hat{\mu}: \text{relative permeability, } [\hat{\mu}] &= \hat{1}. \end{aligned}$$

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- $\hat{\chi}_m(\mathbf{r}, \omega), \hat{\mu}(\mathbf{r}, \omega)$ are determined in the frequency domain.
- Complications: manifold.
- Traditional integrated optics (frequencies, media): $\hat{\mu}(\mathbf{r}) = \hat{1}$.

Maxwell equations, dispersion

(Material) **dispersion**: $\hat{\epsilon}(\mathbf{r}, \omega)$, $\hat{\mu}(\mathbf{r}, \omega)$ are frequency dependent.

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↪
$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \int \hat{\epsilon}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t') dt',$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \int \hat{\mu}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{H}(\mathbf{r}, t') dt'.$$

Helmholtz equations


Linear dielectric media without free charges or currents,
time dependence $\sim \exp(i\omega t)$, fields $\mathbf{E}(\mathbf{r})$, $\mathbf{D}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$,
material properties $\hat{\epsilon}(\mathbf{r})$, $\hat{\mu}(\mathbf{r})$:

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -i\omega \mathbf{B}, & \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= i\omega \mathbf{D}, \\ \mathbf{D} &= \epsilon_0 \hat{\epsilon} \mathbf{E}, & \mathbf{B} &= \mu_0 \hat{\mu} \mathbf{H}.\end{aligned}$$

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$$\nabla \times \mathbf{E} = -i\omega \mu_0 \hat{\mu} \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega \epsilon_0 \hat{\epsilon} \mathbf{E}, \quad \nabla \cdot \hat{\epsilon} \mathbf{E} = 0, \quad \nabla \cdot \hat{\mu} \mathbf{H} = 0.$$

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$$\nabla \times (\hat{\mu}^{-1} \nabla \times \mathbf{E}) = \omega^2 \epsilon_0 \mu_0 \hat{\epsilon} \mathbf{E} \quad \text{or} \quad \nabla \times (\hat{\epsilon}^{-1} \nabla \times \mathbf{H}) = \omega^2 \epsilon_0 \mu_0 \hat{\mu} \mathbf{H}.$$

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$$\text{Where } \hat{\epsilon} = \epsilon \hat{1}, \quad \nabla \epsilon = 0, \quad \hat{\mu} = \mu \hat{1}, \quad \nabla \mu = 0: \quad (!)$$

$$\hookrightarrow \Delta \mathbf{E} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{E} = 0 \quad \text{or} \quad \Delta \mathbf{H} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{H} = 0, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

Plane harmonic waves

Where $\hat{\epsilon} = \epsilon \hat{1}$, $\nabla \epsilon = 0$, $\hat{\mu} = \mu \hat{1}$, $\nabla \mu = 0$: (!)

Components of \mathbf{E} , \mathbf{H} satisfy

$$\Delta \psi + \frac{\omega^2}{c^2} \epsilon \mu \psi = 0.$$

↪ $\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k}_m \cdot \mathbf{r} - \omega t)},$

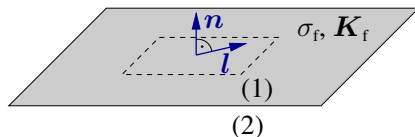
$$-k_m^2 + \frac{\omega^2}{c^2} \epsilon \mu = 0.$$

(Mixture of TD and FD expressions; \sim , \mp , Re, $1/2$, c.c. omitted; sloppy, but common.)

- | | | |
|------------------------|---------------------------|---|
| • Medium: | refractive index: | $n = \sqrt{\epsilon \mu}$ |
| • Periodicity in time: | angular frequency: | $\omega,$ |
| | frequency: | $f = \omega / (2\pi),$ |
| | period: | $T = 1/f = 2\pi / \omega,$ |
| • Spatial periodicity: | wave vector: | $\mathbf{k}_m, k_m = \mathbf{k}_m ,$ |
| | wavenumber: | $k_m = \omega / c_m = (\omega / c) n = k n,$ |
| | vacuum wavenumber: | $k = \omega / c,$ |
| | vacuum wavelength: | $\lambda = 2\pi / k = 2\pi c / \omega,$ |
| | wavelength in the medium: | $\lambda_m = 2\pi / k_m = 2\pi / (kn) = \lambda / n.$ |
| • Phase velocity: | speed of light in vacuum: | $c = 1 / \sqrt{\epsilon_0 \mu_0} = \lambda f,$ |
| | in the medium: | $c_m = c / n = \lambda_m f.$ |

(Use of symbols depends highly on context.)

Interface conditions

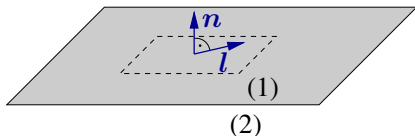


Surface between media (1) and (2), surface normal \mathbf{n} , tangents \mathbf{l} , surface charge density σ_f , surface current density \mathbf{K}_f :

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{l} \cdot (\mathbf{K}_f \times \mathbf{n}).$$

Interface conditions

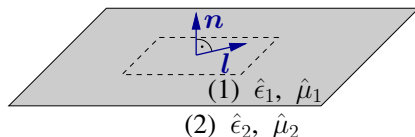


Surface between media (1) and (2), surface normal \mathbf{n} , tangents \mathbf{l} , surface without free charges or currents:

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = 0 .$$

Interface conditions



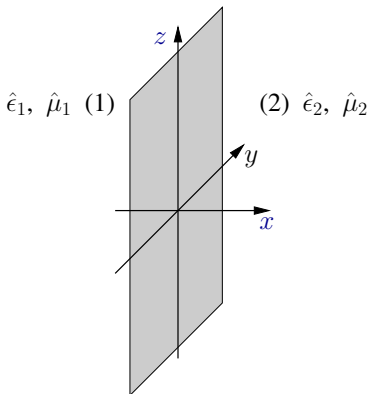
Surface between media (1) and (2), surface normal \mathbf{n} , tangents \mathbf{l} , linear media with permittivities $\hat{\epsilon}_1, \hat{\epsilon}_2$, and permeabilities $\hat{\mu}_1, \hat{\mu}_2$:

$$\mathbf{n} \cdot (\hat{\epsilon}_1 \mathbf{E}_1 - \hat{\epsilon}_2 \mathbf{E}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\hat{\mu}_1 \mathbf{H}_1 - \hat{\mu}_2 \mathbf{H}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = 0.$$

Reflection and transmission of plane waves at dielectric interfaces

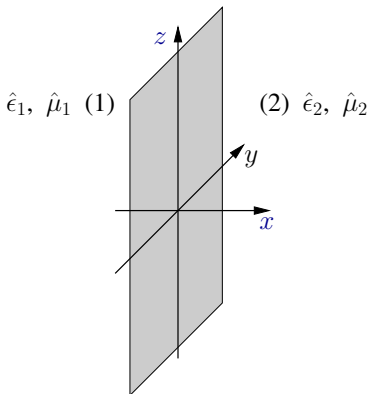
(FD)



- $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$,
 $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$
- $\hat{\epsilon}(\mathbf{r})$ and $\hat{\mu}(\mathbf{r})$
are constant along y, z

Reflection and transmission of plane waves at dielectric interfaces

(FD)



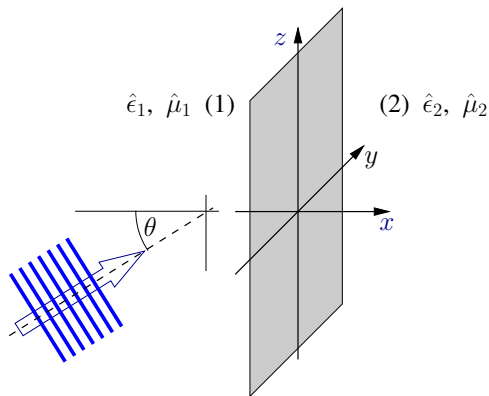
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↪ $\mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$

1-D problem for \mathbf{E}' , \mathbf{H}' .

Reflection and transmission of plane waves at dielectric interfaces

(FD)



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1-D problem for \mathbf{E}' , \mathbf{H}' .

(incoming plane wave at angle θ)

(orient coordinates ($k_y = 0$), plane of incidence, distinguish polarizations)

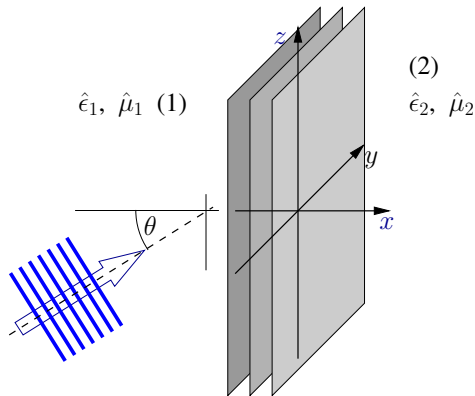
(write ansatz functions for incoming, reflected, and transmitted waves)

(interface conditions determine the amplitudes)

↪ Fresnel equations.

Dielectric multilayer structures

(FD)



- $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$,
 $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$
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 are constant along y, z

$\hookrightarrow \mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$

1-D problem for \mathbf{E}' , \mathbf{H}' .

(...)
 (...)
 (...)
 (...)

\hookrightarrow Reflectance and transmittance properties.

Energy of electromagnetic fields

(TD)

- Force on a particle with charge q , velocity \mathbf{v} , in a field \mathbf{E}, \mathbf{B} :

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

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$$dW = \mathbf{F} \cdot d\mathbf{r} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt = q\mathbf{E} \cdot \mathbf{v} dt,$$

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For a charge density $\rho_f(\mathbf{r}, t)$:

force density $\mathbf{f} = \rho_f(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$



power density $\mathbf{f} \cdot \mathbf{v} = \rho_f \mathbf{E} \cdot \mathbf{v} = \mathbf{J}_f \cdot \mathbf{E},$


total work per time unit done in \mathcal{V} :

$$\frac{dW_{\mathcal{V}}}{dt} = \int_{\mathcal{V}} \mathbf{J}_f \cdot \mathbf{E} d\mathcal{V}.$$

Power & energy density, Poynting theorem

(TD)


$$\nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$


$$\frac{d}{dt} W_{\mathcal{V}}^{\text{mech}} = \int_{\mathcal{V}} \mathbf{J}_f \cdot \mathbf{E} \, d\mathcal{V} = - \int_{\mathcal{V}} (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) \, d\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\mathbf{E} \times \mathbf{H}) \, d\mathcal{V},$$

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- $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\epsilon}(\omega)$, $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}$, $\hat{\mu}^\dagger = \hat{\mu}$, $\hat{\mu}(\omega)$, $\mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$ (!)
 $\dot{w} = (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}})$


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 $\dot{w} = (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}})$



\mathcal{V} arbitrary

$$\dot{w} + \nabla \cdot \mathbf{S} = -\mathbf{J}_f \cdot \mathbf{E}, \quad \frac{d}{dt} (W_{\mathcal{V}}^{\text{mech}} + W_{\mathcal{V}}^{\text{field}}) = - \oint_{\partial \mathcal{V}} \mathbf{S} \cdot d\mathbf{a}.$$

Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (. . .) linear media:

$$w = \frac{1}{2}(\epsilon_0 \mathbf{E} \cdot \hat{\epsilon} \mathbf{E} + \mu_0 \mathbf{H} \cdot \hat{\mu} \mathbf{H}), \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad \dot{w} + \nabla \cdot \mathbf{S} = 0,$$

$$\mathbf{E}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{H}}(\mathbf{r}) e^{i\omega t}$$

↪ \mathbf{S} , w oscillate in time.

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↪ \mathbf{S} , w oscillate in time.

Consider time-averaged quantities: $\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t') dt'$ (FD)

↪ $\bar{w} = \frac{1}{4} \text{Re} \left(\epsilon_0 \tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \tilde{\mathbf{E}} + \mu_0 \tilde{\mathbf{H}}^* \cdot \hat{\mu} \tilde{\mathbf{H}} \right), \quad \bar{\mathbf{S}} = \frac{1}{2} \text{Re} \left(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}} \right).$ (exercise)

$$\bar{\dot{w}} = \dot{\bar{w}} = 0, \quad \overline{\nabla \cdot \mathbf{S}} = \nabla \cdot \bar{\mathbf{S}} \quad \rightsquigarrow \quad \nabla \cdot \bar{\mathbf{S}} = 0, \quad \oint_{\mathcal{V}} \bar{\mathbf{S}} \cdot d\mathbf{a} = 0;$$

“power balance”, conservation of energy.

Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law $\mathbf{J}_f = \sigma \mathbf{E}$, σ : conductivity of the material.

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 $\dot{\rho}_f = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_f, \quad \rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0)\right),$

assume $\rho_f(\mathbf{r}, t_0) = 0$  $\rho_f(\mathbf{r}, t) = 0 \quad \forall t.$


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$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}.$$

Telegrapher equation

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}$$

↪
$$\Delta \mathbf{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{E}} - \mu_0 \mu \sigma \dot{\mathbf{E}} = 0, \quad \Delta \mathbf{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{H}} - \mu_0 \mu \sigma \dot{\mathbf{H}} = 0,$$

Telegrapher equation.

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Frequency domain: $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}$,

$$\rightarrow \Delta \tilde{\mathbf{E}} + \left(\frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma \right) \tilde{\mathbf{E}} = 0.$$

Nonconducting media $\sigma = 0$, $\Delta \tilde{\mathbf{E}} + \left(\frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0.$

Telegrapher equation

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$$\text{Nonconducting media } \sigma = 0, \quad \Delta \tilde{\mathbf{E}} + \left(\frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0.$$

Define $\bar{\epsilon}$ such that $\frac{\omega^2}{c^2} \bar{\epsilon} \mu = \frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma$, i.e. $\bar{\epsilon} = \epsilon - i \frac{\sigma}{\epsilon_0 \omega}$

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + k^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \text{Helmholtz equation, } \bar{\epsilon} \in \mathbb{C}, \quad k = \frac{\omega}{c}.$$

Wave attenuation

$$\Delta \tilde{E} + k^2 \bar{\epsilon} \mu \tilde{E} = 0, \quad \bar{\epsilon} \in \mathbb{C}$$

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↪ solutions $\sim e^{-i(k\bar{n}z - \omega t)}$

with refractive index $\bar{n} = n' - i n'' = \sqrt{\bar{\epsilon} \mu} \in \mathbb{C}, \quad (!)$

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with refractive index $\bar{n} = n' - i n'' = \sqrt{\bar{\epsilon} \mu} \in \mathbb{C}$, (!)

$$e^{-i(k\bar{n}z - \omega t)} = e^{-i(kn'z - \omega t)} e^{-kn''z},$$

damped plane wave solutions

↔ sign of n'' , choice of $\exp(\pm i\omega)$.

Issues:

- penetration depth,
- S and w decay with z ,
- still transverse waves,
- \mathbf{E}, \mathbf{H} no longer in phase,
- notions of wavenumber, wavelength, phase velocity $\in \mathbb{C}$.

Simulations in integrated optics

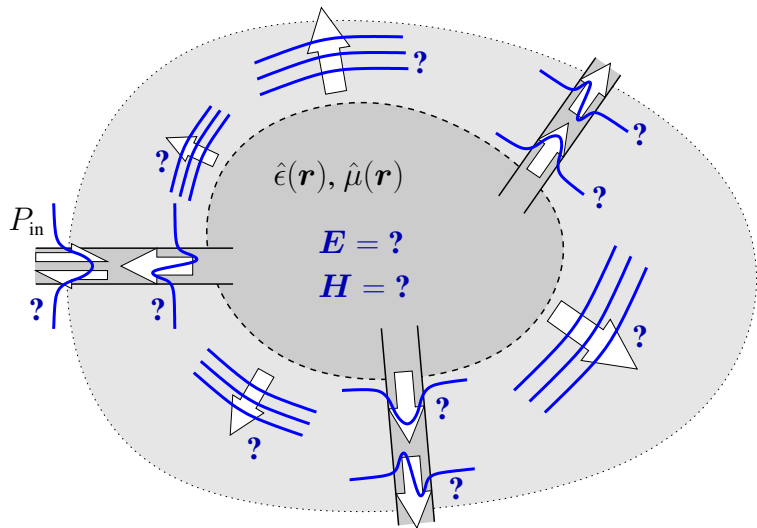
A typical setting:

- “uncharged dielectric medium”: $\nabla \cdot \mathbf{J} = 0$.
- “linear medium”: $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}$, $\mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$.
- “isotropic medium”: $\hat{\epsilon} = \epsilon \hat{1}$, $\hat{\mu} = \mu \hat{1}$.
- “nonmagnetic medium”: $\hat{\mu} = \hat{1}$.
- “lossless medium”: $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\mu}^\dagger = \hat{\mu}$, $(\epsilon, \mu \in \mathbb{R})$.
- “piecewise constant” \rightarrow “dependent on position”.
- “electric and magnetic field”: eliminate \mathbf{D} and \mathbf{B} , retain \mathbf{E} and \mathbf{H} .
- “governed by the curl equations”: divergence eqns. are satisfied.
- “frequency domain, time harmonic fields, frequency, wavelength”:
... as discussed.

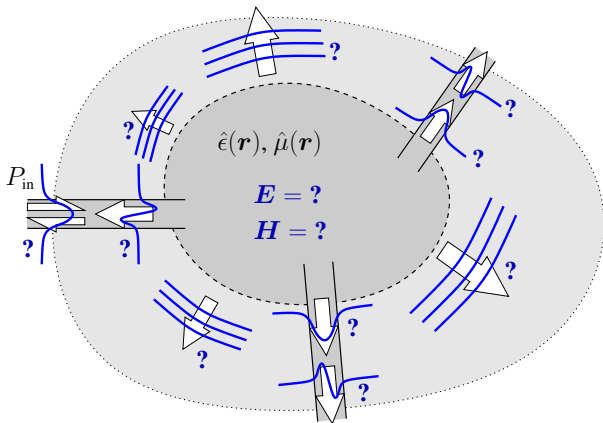
Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

Guided wave scattering problems, schematically



Guided wave scattering problems, schematically



Given $\hat{\epsilon}(\mathbf{r}), \hat{\mu}(\mathbf{r})$ & external excitation (incoming guided mode),
determine \mathbf{E}, \mathbf{H} within the computational domain
& determine the optical power carried by outgoing waves.

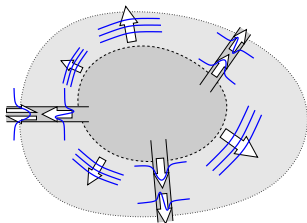
Scattering problems, time domain

(TD)

$$\mathbf{E}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t),$$

$$\nabla \times \mathbf{E} = -\mu_0 \hat{\mu} \dot{\mathbf{H}},$$

$$\nabla \times \mathbf{H} = \epsilon_0 \hat{\epsilon} \dot{\mathbf{E}}.$$



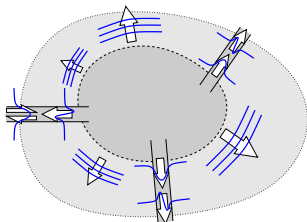
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$$\nabla \times \mathbf{H} = \epsilon_0 \hat{\epsilon} \dot{\mathbf{E}}.$$



- $\begin{pmatrix} 3\text{-D} \\ 2\text{-D} \\ 1\text{-D} \end{pmatrix}$ computational domain \times time interval.
- Initial & boundary conditions \longleftrightarrow incident waves.
- “Local” time-explicit iterative schemes possible (e.g. FDTD).
- Time evolution available; direct modeling of pulse propagation.
- Dispersion (...?).
- Guided wave excitation (...?).
- Fourier transform \longrightarrow spectral information.

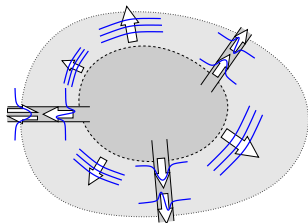
Scattering problems, frequency domain

(FD)

$$\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}), \sim \exp(i\omega t),$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H},$$

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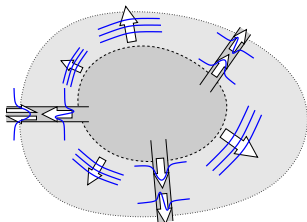
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$$\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}.$$

- $\begin{pmatrix} 3\text{-D} \\ 2\text{-D} \\ 1\text{-D} \end{pmatrix}$ computational domain.
- “ $\mathbf{M}(\vec{\text{field}}) = \vec{\text{excitation}}$ ”;
matrix needs to be determined, stored; system needs to be solved.
- Spectral information directly available.
- Dispersion — straightforward.
- Guided wave excitation — straightforward.
- Fourier transform \longrightarrow time evolution / pulse propagation.



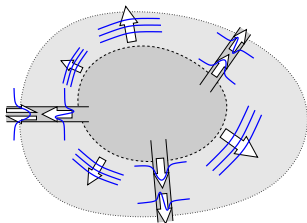
Open problems

(TD & FD)

“Open” spatial computational domain

~> boundary conditions need to

- permit outgoing radiated fields
& outgoing (reflected) guided modes to exit the domain,
 - launch the incoming external excitation.
- ~<~> simulate a nonexistent boundary, an unlimited domain.



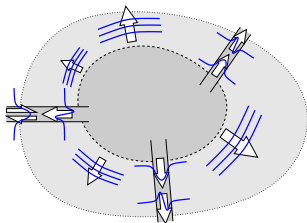
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Keywords:

- transparent-influx boundary conditions,
- absorbing boundary conditions,
- perfectly matched layers (PMLs).



2-D problems

$$\hat{\epsilon} = \epsilon \hat{1}, \quad \hat{\mu} = \mu \hat{1}, \quad \sim \exp(i\omega t) \quad (\text{FD})$$

$$\begin{pmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

2-D problems

$$\hat{\epsilon} = \epsilon \hat{1}, \quad \hat{\mu} = \mu \hat{1}, \quad \sim \exp(i\omega t) \quad (\text{FD})$$

Assume $\partial_y \epsilon = 0$, $\partial_y \mu = 0$; consider solutions $\partial_y \mathbf{E} = 0$, $\partial_y \mathbf{H} = 0$:

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Two decoupled sets of equations:

- $\{E_y, H_x, H_z\}$: **transverse electric (TE)** fields, $\mathbf{E} \perp x\text{-}z\text{-plane}$.
- $\{H_y, E_x, E_z\}$: **transverse magnetic (TM)** fields, $\mathbf{H} \perp x\text{-}z\text{-plane}$.

(Different conventions on the use of TE, TM.)

(Applies also to the TD.)

2-D TE waves

$$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0 \quad (\text{FD})$$

- Principal component E_y ,

$$H_x = \frac{-i}{\omega\mu_0\mu} \partial_z E_y, \quad H_z = \frac{i}{\omega\mu_0\mu} \partial_x E_y, \quad i\omega\epsilon_0\epsilon E_y = \partial_z H_x - \partial_x H_z$$

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(Reflection / transmission problems: s-polarized waves satisfy (*), (**).)

2-D TM waves

$$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0 \quad (\text{FD})$$

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$$E_x = \frac{i}{\omega\epsilon_0\epsilon} \partial_z H_y, \quad E_z = \frac{-i}{\omega\epsilon_0\epsilon} \partial_x H_y, \quad -i\omega\mu_0\mu H_y = \partial_z E_x - \partial_x E_z$$

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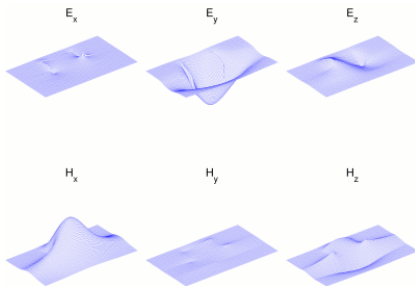
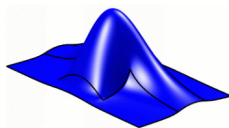
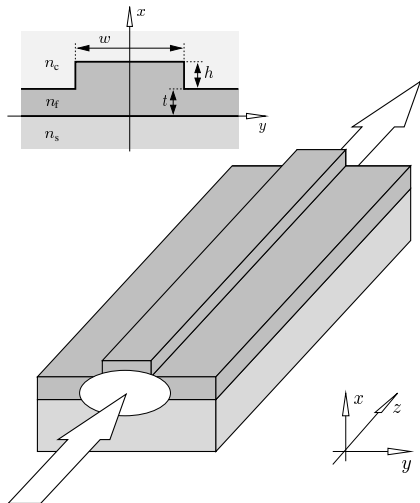
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(Reflection / transmission problems: p-polarized waves satisfy (*), (**).)

Rib waveguide

... variant of an integrated optical waveguide with 2-D confinement



Waveguides: Mode problems

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mu\mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0\epsilon\mathbf{E}. \quad \sim \exp(i\omega t) \quad (\text{FD})$$

- **Waveguide**: a system that is homogeneous along its **axis** z ,
 $\partial_z\epsilon = 0, \partial_z\mu = 0, \partial_z n = 0$.

Waveguides: Mode problems

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- Look for solutions (**modes**) that vary harmonically with z :

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mode profile $\bar{\mathbf{E}}, \bar{\mathbf{H}}$, propagation constant β .

(drop $-$)

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 vectorial mode equations, variants.

(...)

Waveguides: Mode equations

- Where $\epsilon(\mathbf{r})$, $\mu(\mathbf{r})$: $\sim \exp(i\omega t)$ (FD)

$$\Delta \tilde{\mathbf{E}} + k^2 \epsilon \mu \tilde{\mathbf{E}} = 0, \quad \Delta \tilde{\mathbf{H}} + k^2 \epsilon \mu \tilde{\mathbf{H}} = 0$$


$$\begin{aligned} \partial_x^2 \mathbf{E} + \partial_y^2 \mathbf{E} + (k^2 \epsilon \mu - \beta^2) \mathbf{E} &= 0, \\ \partial_x^2 \mathbf{H} + \partial_y^2 \mathbf{H} + (k^2 \epsilon \mu - \beta^2) \mathbf{H} &= 0, \end{aligned}$$

scalar **mode equation**, valid for all components of \mathbf{E} , \mathbf{H} ,
to be supplemented by suitable **boundary** and **interface conditions**.

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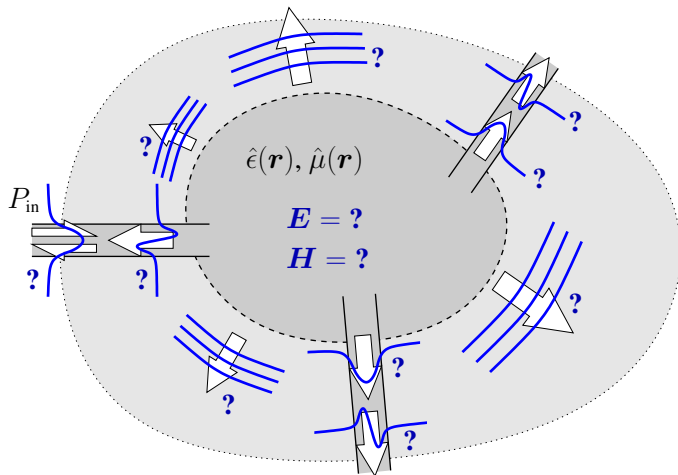
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“ $\mathbf{M}(\beta) (\overrightarrow{\text{profile}}) = 0$ ”.

- Guided modes**: discrete $\beta \in \mathbb{R}$, $\iint S_z \, dx dz < \infty$.
- Radiation modes**: continuum of $\beta^2 \in \mathbb{R}$, oscillating external fields.
- Leaky modes**: discrete $\beta \in \mathbb{C}$, outgoing wave boundary conditions.
- (...)

Guided wave scattering problems

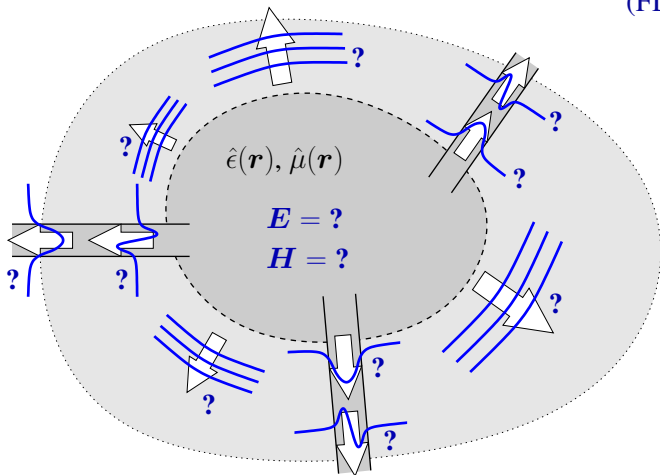
(FD)



Given external excitation $\sim \exp(i\omega t)$, $\omega \in \mathbb{R}$.

Resonance problems

(FD ...)



Omit excitation, look for nonzero solutions that decay in time.

Resonance problems

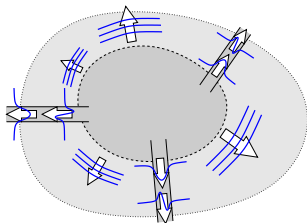
(FD ...)

$$\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}), \sim \exp(i\omega t), \omega = ?$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H},$$

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Resonance problems

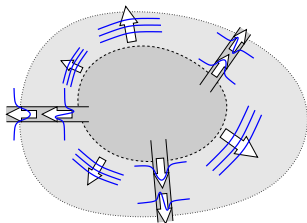
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& outgoing wave boundary conditions.



- Look for nonzero solutions with $\omega \in \mathbb{C}$ that oscillate and decay (slowly ...) in time.
- “ $\mathbf{M}(\omega) \vec{\text{field}} = 0$ ”, **eigenvalue problem**.
- Solutions: discrete eigenfrequencies ω , resonant mode profiles.

Scalar approximation

Linear, isotropic, nonmagnetic media, $\epsilon = n^2$;
a structure with “small” variations in ϵ :

A **scalar approximation** may be adequate,

$$\nabla \cdot (\epsilon \mathbf{E}) \approx \epsilon \nabla \cdot \mathbf{E}$$

$$\hookrightarrow \Delta \psi - \frac{1}{c^2} \epsilon \ddot{\psi} = 0, \quad \text{(TD)}$$

$$\Delta \psi + k^2 \epsilon \psi = 0, \quad \text{(FD)}$$

satisfied by all components ψ of \mathbf{E}, \mathbf{H} .

(Applicable to basically all types of problems.)

Beam propagation method

- Starting point: $\Delta\psi + k^2\epsilon\psi = 0$, $\sim \exp(i\omega t)$ (FD)
“small” changes in $\epsilon = n^2$ along a propagation coordinate z .

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- Ansatz: $\psi(x, y, z) = \psi_0(x, y, z) e^{-ikn_r z}$,
reference effective index n_r ,
assume that ψ_0 varies “slowly” along z \longleftrightarrow neglect $\partial_z^2\psi_0$.

$\hookrightarrow -i2kn_r\partial_z\psi_0 + (\partial_x^2 + \partial_y^2)\psi_0 + k^2(\epsilon - n_r^2)\psi_0 = 0,$

PDE of first order in z , solved as an initial value problem.

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PDE of first order in z , solved as an initial value problem.

- Restriction to unidirectional propagation, reflections are neglected.
- Paraxial propagation, errors for waves with effective indices $\neq n_r$.

(Many variants (vectorial, wide-angle, bi-directional, ...) have been proposed.)

(Other ways of motivating the approximation exist.)

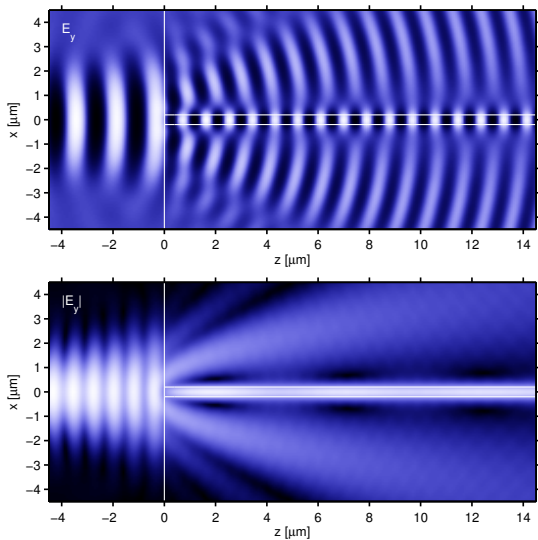
(Term “BPM” in use also for other types of methods.)

- Keywords: Paraxial approximation,
Slowly-varying-envelope approximation (SVEA),
Beam propagation method (BPM).

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

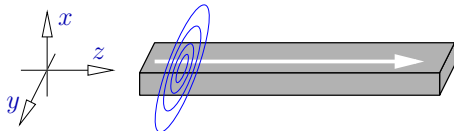
Context: Relevance of guided modes



(2-D, TE)

Butt-coupling to a waveguide facet.

Waveguides: Mode problems



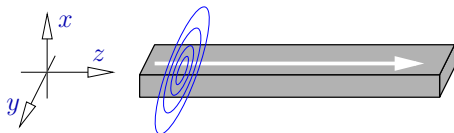
$$\mu = 1, \epsilon = n^2, \sim \exp(i\omega t) \quad (\text{FD})$$

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$$\nabla \times \mathbf{H} = i\omega\epsilon_0\epsilon\mathbf{E}.$$

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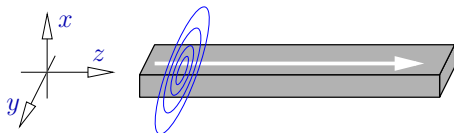
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mode profile $\bar{\mathbf{E}}, \bar{\mathbf{H}}$,
propagation constant β ,
effective index $n_{\text{eff}} = \beta/k$.

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$$\partial_z \longrightarrow -i\beta,$$

(& boundary conditions)

- **Eigenvalue** problem with eigenvalue β , eigenfunction $\bar{\mathbf{E}}, \bar{\mathbf{H}}$,
 “ $\mathbf{M}(\beta) (\overrightarrow{\text{profile}}) = 0$ ”.

Mode equations

(drop $-$)


$$\begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

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- Express E_x, E_y, E_z, H_z through principal components H_x, H_y :


$$\hookrightarrow \begin{aligned} \partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x &= 0, \\ \epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y &= 0, \end{aligned}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{1}{\omega\epsilon_0} \begin{pmatrix} \beta H_y - \beta^{-1}(\partial_{yx} H_x + \partial_y^2 H_y) \\ -\beta H_x + \beta^{-1}(\partial_{xy} H_y + \partial_x^2 H_x) \\ -i(\partial_x H_y - \partial_y H_x) \end{pmatrix}, \quad \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ -i\beta^{-1}(\partial_x H_x + \partial_y H_y) \end{pmatrix}.$$


(H_x, H_y are continuous for all x, y .)

Mode equations

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- Express H_x, H_y, H_z, E_z through principal components E_x, E_y :


$$(\dots).$$


(E_x, E_y are discontinuous at specific interfaces.)

Mode equations

(drop ∇)


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- Express E_x, E_y, H_x, H_y through principal components E_z, H_z :


$$(\dots).$$

(E_z, H_z are usually small components.)

Plane mode profiles

- Modes are eigenfunctions
 - ➡ profiles are determined up to a complex constant only.

Plane mode profiles

- Modes are eigenfunctions
 \longleftrightarrow profiles are determined up to a complex constant only.

- Propagating modes, $\beta \in \mathbb{R}$, lossless structures, $\epsilon \in \mathbb{R}$:

$E_z := iE'_z$, $H_z := iH'_z$ \rightsquigarrow real PDE for $E_x, E_y, E'_z, H_x, H_y, H'_z$:

$$\begin{pmatrix} \partial_y E'_z + \beta E_y \\ -\beta E_x - \partial_x E'_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ -H'_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H'_z + \beta H_y \\ -\beta H_x - \partial_x H'_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ -E'_z \end{pmatrix};$$

it is possible to choose a phase such that

E_x, E_y, H_x, H_y are real,

E_z, H_z are imaginary

\longleftrightarrow plane mode profiles.


(It makes sense to prepare real plots of mode profile components.)

(That requires a suitable adjustment of the global phase.)

Guided modes

- **Guided modes:** profiles located “around” the waveguide core

 discrete $\beta \in \mathbb{R}$, $\iint S_z \, dx \, dy < \infty$.

- In general: **Hybrid modes**, all six field components present.
Planar-like waveguides  adapt 2-D naming scheme;
“TE-like” / “TM-like” modes.

(\leftrightarrow 5-component **semivectorial** approximations, plane \perp x-axis:

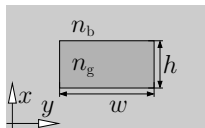
quasi-TE: tiny E_x , dominant E_y , small E_z ; major H_x , small H_y , minor H_z ,

quasi-TM: tiny H_x , dominant H_y , small H_z ; major E_x , small E_y , minor E_z .)

- Mode indices mostly relate to numbers of nodal lines in the dominant electric or magnetic field component.

(Naming schemes are highly context dependent.)

A rectangular strip waveguide, fundamental mode profiles

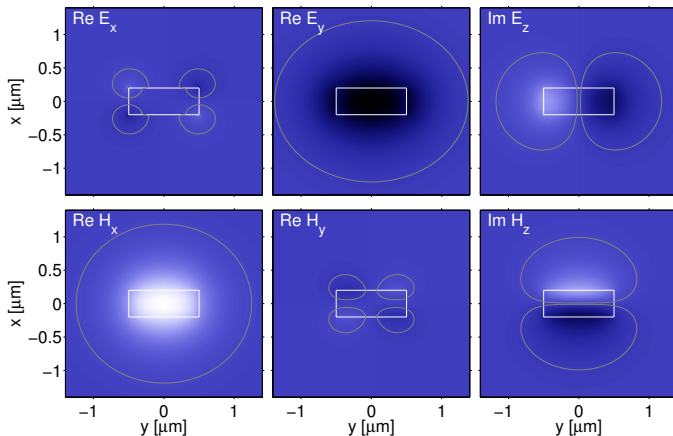


$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $w = 1.0 \mu\text{m}$,
 $h = 0.4 \mu\text{m}$;

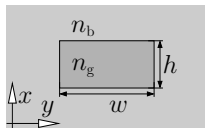
$x \in [-2, 2] \mu\text{m}$,
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$n_{\text{eff}} = 1.63554$

(q-)TE₀₀ [JCMwave].



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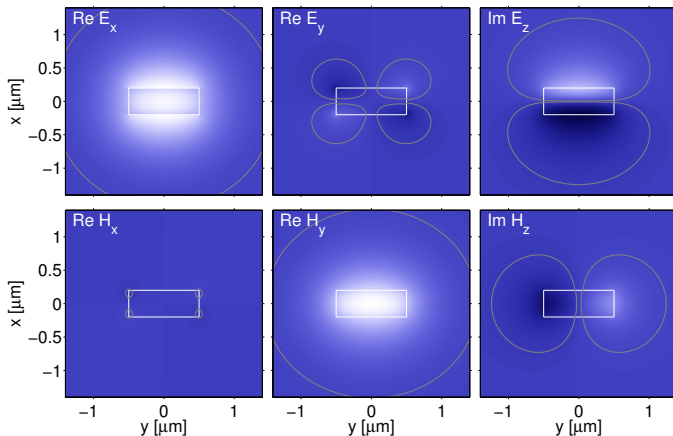


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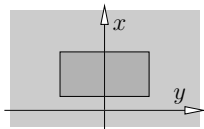
$x \in [-2, 2] \mu\text{m}$,
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(q-) TM_{00} [JCMwave].

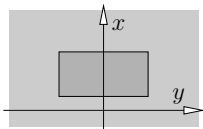


Symmetric waveguides



Waveguide with mirror symmetry $y \rightarrow -y$:
↔ modes have a definite parity.

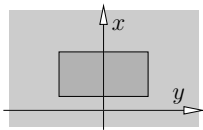
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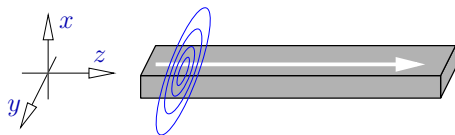


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↪ Equal parity of H_x, E_y, H_z , reversed parity of E_x, H_y, E_z .

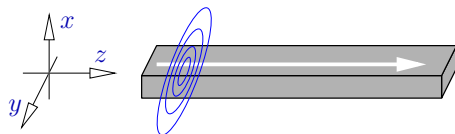
Directional modes



$$(\text{FD}) \sim \exp(i\omega t)$$

Longitudinally homogeneous waveguide: mirror symmetry $z \rightarrow -z$.

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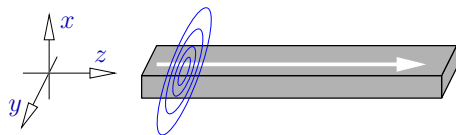


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forward: $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^f \\ \bar{\mathbf{H}}^f \end{pmatrix} (x, y) e^{-i\beta z}, \quad \begin{matrix} \bar{\mathbf{E}}^f = (E_x, E_y, E_z), \\ \bar{\mathbf{H}}^f = (H_x, H_y, H_z), \end{matrix}$



backward: $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^b \\ \bar{\mathbf{H}}^b \end{pmatrix} (x, y) e^{+i\beta z}, \quad \begin{matrix} \bar{\mathbf{E}}^b = (E_x, E_y, -E_z), \\ \bar{\mathbf{H}}^b = (-H_x, -H_y, H_z). \end{matrix}$

Modal power

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 $\mathbf{S} = \frac{|a|^2}{2} \begin{pmatrix} 0 \\ 0 \\ \bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x \end{pmatrix},$
or $S_x = 0, S_y = 0, S_z = \frac{1}{2} \text{Re} (E_x^* H_y - E_y^* H_x).$ ($S_z(x, y)$)

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- Power carried by the mode:

$$P = \iint S_z \, dx \, dy = \frac{1}{4} \iint (E_x^* H_y - E_y^* H_x + E_x H_y^* - E_y H_x^*) \, dx \, dy.$$

(backward mode, $E_x \rightarrow E_x, E_y \rightarrow E_y, H_x \rightarrow -H_x, H_y \rightarrow -H_y$: $P \rightarrow -P$)

Mode orthogonality

- A set of guided modes of the same waveguide (ϵ):

$$\beta \in \mathbb{R}$$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z},$$
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↪
$$0 = i(\beta_l - \beta_m) \left\{ \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m + \bar{\mathbf{E}}_m \times \bar{\mathbf{H}}_l^*)_z dx dy \right\} e^{i(\beta_l - \beta_m)z},$$

$$(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \iint (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx dy$$

Mode orthogonality

- A set of guided modes of the same waveguide (ϵ):

$$\beta \in \mathbb{R}$$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}, \quad \begin{aligned} \nabla \times \mathbf{E}_m &= -i\omega\mu_0\mathbf{H}_m, \\ \nabla \times \mathbf{H}_m &= i\omega\epsilon_0\epsilon\mathbf{E}_m, \\ \beta_l &\neq \beta_m, \text{ if } l \neq m. \end{aligned}$$

- $$P_m = \frac{1}{4} \iint (E_{mx}^* H_{my} - E_{my}^* H_{mx} + E_{mx} H_{my}^* - E_{my} H_{mx}^*) dx dy.$$

- $$\mathbf{E}_m, \mathbf{H}_m \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.$$

- $$\nabla \cdot (\mathbf{E}_l^* \times \mathbf{H}_m + \mathbf{E}_m \times \mathbf{H}_l^*) = 0 \quad \text{for all } l, m$$

↪
$$0 = i(\beta_l - \beta_m) \left\{ \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m + \bar{\mathbf{E}}_m \times \bar{\mathbf{H}}_l^*)_z dx dy \right\} e^{i(\beta_l - \beta_m)z},$$

$$(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \iint (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx dy$$

$$(\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m) = \begin{cases} 0, & \text{if } l \neq m, \\ P_m, & \text{otherwise.} \end{cases}$$

(The modes are “power orthogonal”.)
 (Statements hold for propagating guided modes.)
 ((., .; ., .) is frequently used for mode normalization.)

Power transport by a mode superposition

- A set of guided modes of the same waveguide (ϵ):

$$\beta \in \mathbb{R}$$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}, \quad P_m = (\mathbf{E}_m, \mathbf{H}_m; \mathbf{E}_m, \mathbf{H}_m).$$

- Superposition with amplitudes $a_m \in \mathbb{C}$:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_m a_m \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \sum_m a_m \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}.$$

Power flow along the waveguide :

$$\begin{aligned} \iint S_z dx dy &= (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) \\ &= \sum_l \sum_m a_l^* a_m (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m) \\ &= \sum_m |a_m|^2 P_m. \end{aligned}$$

(Forward / backward modes: $P \geq 0$.)

Mode interference

- Two modes $m = 1, 2$:

$$\beta \in \mathbb{R}$$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}.$$

- Superposition with amplitudes a_1, a_2 :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = a_1 \begin{pmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{H}}_1 \end{pmatrix} (x, y) e^{-i\beta_1 z} + a_2 \begin{pmatrix} \bar{\mathbf{E}}_2 \\ \bar{\mathbf{H}}_2 \end{pmatrix} (x, y) e^{-i\beta_2 z}.$$

- Fix a position x, y and component F :

Omit (x, y) .

$$F(z) = a_1 \bar{F}_1 e^{-i\beta_1 z} + a_2 \bar{F}_2 e^{-i\beta_2 z}, \quad r e^{-i\phi} := a_1^* a_2 \bar{F}_1^* \bar{F}_2,$$

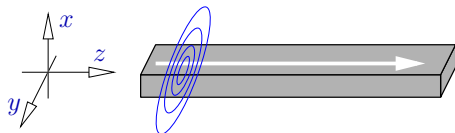
$$\hookrightarrow |F|^2(z) = |a_1|^2 |\bar{F}_1|^2 + |a_2|^2 |\bar{F}_2|^2 + 2r \cos((\beta_1 - \beta_2)z + \phi).$$

Periodic beating pattern with half-beat-length $L_c = \frac{\pi}{|\beta_1 - \beta_2|}$.

(Supermodes ) (Evanescent coupling )

(“Coupling length” L_c .)

Polarization of a guided wave field



Unidirectional guided waves in a “long” dielectric channel that supports fundamental TE- and TM-like modes only:

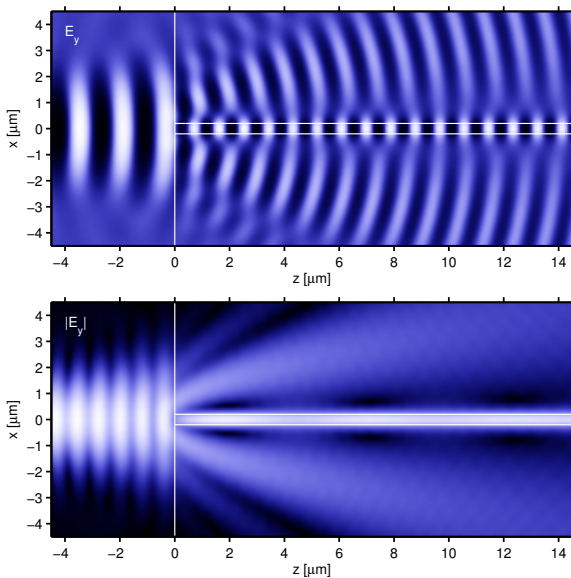
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = a_{\text{TE}} \begin{pmatrix} \bar{\mathbf{E}}_{\text{TE}} \\ \bar{\mathbf{H}}_{\text{TE}} \end{pmatrix} (x, y) e^{-i\beta_{\text{TE}}z} + a_{\text{TM}} \begin{pmatrix} \bar{\mathbf{E}}_{\text{TM}} \\ \bar{\mathbf{H}}_{\text{TM}} \end{pmatrix} (x, y) e^{-i\beta_{\text{TM}}z},$$

amplitudes $a_{\text{TE}}, a_{\text{TM}} \in \mathbb{C}$.

- $E_{\text{TE}z} \neq 0, E_{\text{TM}z} \neq 0$.
- $\bar{\mathbf{E}}_{\text{TE}}(x, y) \neq \bar{\mathbf{E}}_{\text{TM}}(x, y)$.
- At (x, y) : adjust $\mathbf{E}/|\mathbf{E}|$ via $a_{\text{TE}}, a_{\text{TM}}$.
- $a_{\text{TE}}, a_{\text{TM}}$ fixed: $(\mathbf{E}/|\mathbf{E}|)(x, y)$ varies.

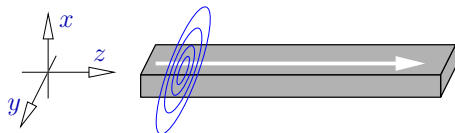
“Polarization” frequently indicates the presence of only one mode.

What about non-guided fields?



(2-D, TE)

Normal modes: real mode problems



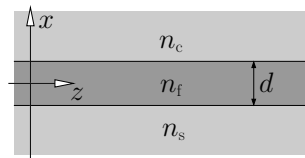
- lossless waveguide, $\epsilon \in \mathbb{R}$,
- “real” boundary conditions at x, y “far away” from the core,
- “real” vectorial mode equations:

$$\partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x = 0,$$

$$\epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y = 0,$$

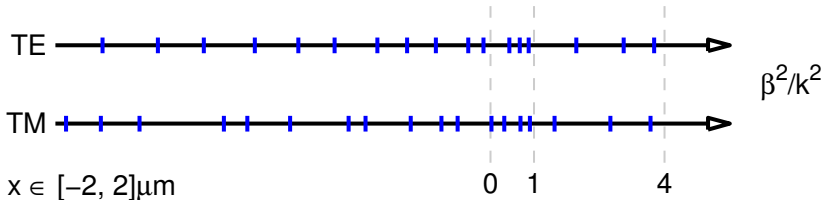
↪ real principal components $H_x(x, y), H_y(x, y)$, $\beta^2 \in \mathbb{R}$.

2-D slab waveguide, normal mode spectrum



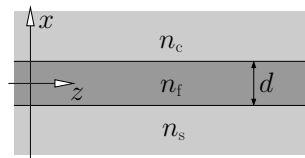
$$n_s = n_c = 1.0, \quad n_f = 2.0,$$
$$d = 1.3 \mu\text{m}, \quad \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm 2 \mu\text{m}.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided modes**.
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation modes**.
- $\beta^2/k^2 < 0$: **evanescent radiation modes**.

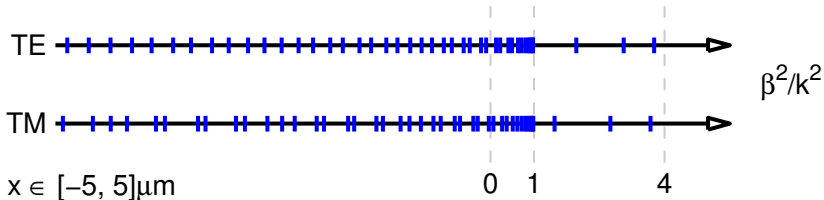
2-D slab waveguide, normal mode spectrum



$$n_s = n_c = 1.0, \quad n_f = 2.0,$$

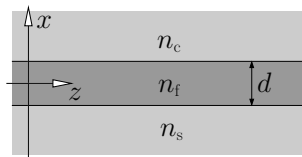
$$d = 1.3 \mu\text{m}, \quad \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm 5 \mu\text{m}.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided modes**.
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation modes**.
- $\beta^2/k^2 < 0$: **evanescent radiation modes**.

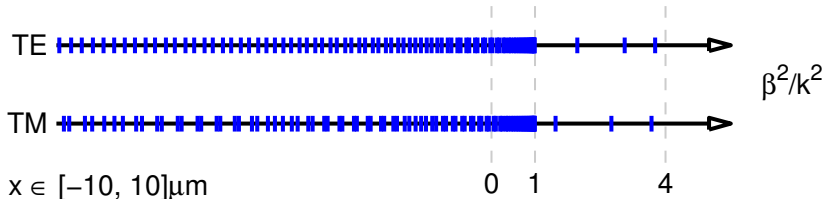
2-D slab waveguide, normal mode spectrum



$$n_s = n_c = 1.0, \quad n_f = 2.0,$$

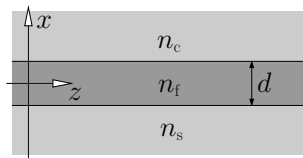
$$d = 1.3 \mu\text{m}, \quad \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm 10 \mu\text{m}.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided modes**.
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation modes**.
- $\beta^2/k^2 < 0$: **evanescent radiation modes**.

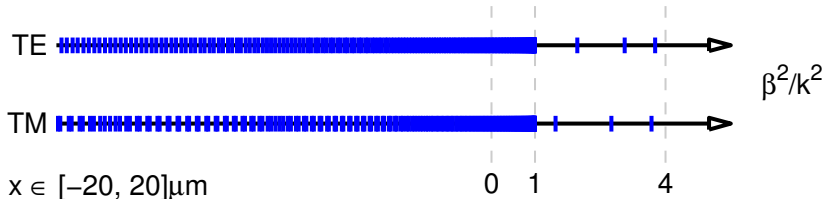
2-D slab waveguide, normal mode spectrum



$$n_s = n_c = 1.0, \quad n_f = 2.0,$$

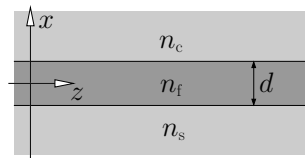
$$d = 1.3 \mu\text{m}, \quad \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm 20 \mu\text{m}.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided modes**.
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation modes**.
- $\beta^2/k^2 < 0$: **evanescent radiation modes**.

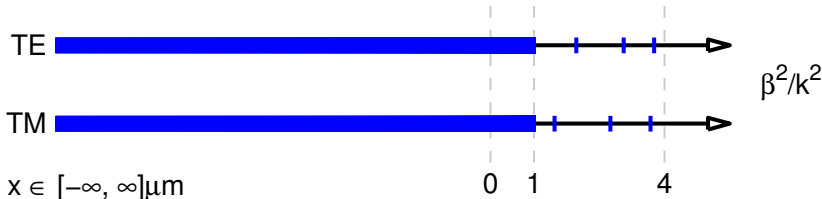
2-D slab waveguide, normal mode spectrum



$$n_s = n_c = 1.0, \quad n_f = 2.0,$$

$$d = 1.3 \mu\text{m}, \quad \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm\infty.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided** modes (discrete spectrum).
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation** modes (continuous spec.).
- $\beta^2/k^2 < 0$: **evanescent radiation** modes (continuous spec.).

Propagating & evanescent modes

$$\begin{pmatrix} E \\ H \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{E}^{\text{f,b}} \\ \bar{H}^{\text{f,b}} \end{pmatrix} (x, y) e^{\mp i \beta z}. \quad \sim \exp(i \omega t) \quad (\text{FD})$$

- $\beta^2 > 0 \iff \beta = \sqrt{\beta^2}, \beta \in \mathbb{R}, \beta > 0,$
 $\sim e^{\mp i \beta z},$ a forward / backward **propagating** mode.

(Physical relevance of individual modes.)

Propagating & evanescent modes

$$\begin{pmatrix} E \\ H \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{E}^{\text{f,b}} \\ \bar{H}^{\text{f,b}} \end{pmatrix} (x, y) e^{\mp i \beta z}. \quad \sim \exp(i \omega t) \quad (\text{FD})$$

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 $\sim e^{\mp i \beta z},$ a forward/backward **propagating** mode.
(Physical relevance of individual modes.)

- $\beta^2 < 0 \iff \beta = -i \sqrt{|\beta^2|} = -i \alpha, \alpha = \sqrt{|\beta^2|} \in \mathbb{R}, \alpha > 0,$
 $\sim e^{\mp \alpha z},$ a forward/backward traveling **evanescent** mode.

“forward”: $\sim e^{-\alpha z},$ field decays with $z,$
“backward”: $\sim e^{+\alpha z},$ field grows with $z.$

(Relevant for purposes of field expansions.)

Propagating & evanescent modes

$$\begin{pmatrix} E \\ H \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{E}^{\text{f,b}} \\ \bar{H}^{\text{f,b}} \end{pmatrix} (x, y) e^{\mp i \beta z}. \quad \sim \exp(i \omega t) \quad (\text{FD})$$

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“forward”: $\sim e^{-\alpha z},$ field decays with $z,$
“backward”: $\sim e^{+\alpha z},$ field grows with $z.$

(Relevant for purposes of field expansions.)

- $\{\text{forward \& backward, propagating \& evanescent modes}\}$
= the set of **normal modes**.

Evanescent modes

$$\epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Evanescent modes

$$\beta = -i\alpha, \alpha \in \mathbb{R}$$

$$\epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Evanescent modes

$$\beta = -i\alpha, \alpha \in \mathbb{R}$$

$$\epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- **Plane mode profiles:** real PDE for $E_x, E_y, E_z, iH_x, iH_y, iH_z$;
common phase with real E_x, E_y, E_z , imaginary H_x, H_y, H_z .

Evanescent modes

$$\beta = -i\alpha, \alpha \in \mathbb{R} \qquad \epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- **Plane mode profiles:** real PDE for $E_x, E_y, E_z, iH_x, iH_y, iH_z$;
common phase with real E_x, E_y, E_z , imaginary H_x, H_y, H_z .
- **Directional evanescent modes:**
 $\{E_x, E_y, E_z, H_x, H_y, H_z; \alpha\}^f \rightsquigarrow \{E_x, E_y, -E_z, -H_x, -H_y, H_z; -\alpha\}^b$.

Evanescent modes

$$\beta = -i\alpha, \quad \alpha \in \mathbb{R} \qquad \epsilon \in \mathbb{R}$$

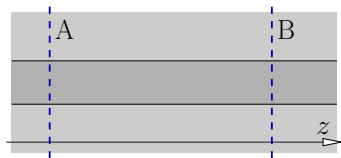
$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- **Plane mode profiles:** real PDE for $E_x, E_y, E_z, iH_x, iH_y, iH_z$; common phase with real E_x, E_y, E_z , imaginary H_x, H_y, H_z .
- **Directional evanescent modes:**
 $\{E_x, E_y, E_z, H_x, H_y, H_z; \alpha\}^f \rightsquigarrow \{E_x, E_y, -E_z, -H_x, -H_y, H_z; -\alpha\}^b$.
- **Modal power:**

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-\alpha z}, \quad \begin{aligned} \bar{\mathbf{E}} &= a(E'_x, E'_y, E'_z), \\ \bar{\mathbf{H}} &= ia(H'_x, H'_y, H'_z), \\ E'_x, \dots, H'_z &\in \mathbb{R}, \quad a \in \mathbb{C} \end{aligned}$$

$$\hookrightarrow S_z = \frac{1}{2} \operatorname{Re} (E_x^* H_y - E_y^* H_x) = 0, \quad \iint S_z \, dx \, dy = 0.$$

Completeness of normal modes



$$\epsilon \in \mathbb{R}, \sim \exp(i\omega t) \text{ (FD)}$$

A lossless, z -homogeneous waveguide configuration; **general solution** of the Maxwell equations between cross sectional planes A and B:

$$\begin{aligned} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) &= \sum_{m \in \mathcal{N}} F_m \begin{pmatrix} \bar{\mathbf{E}}_m^f \\ \bar{\mathbf{H}}_m^f \end{pmatrix} (x, y) e^{-i\beta_m z} \\ &+ \sum_{m \in \mathcal{N}} B_m \begin{pmatrix} \bar{\mathbf{E}}_m^b \\ \bar{\mathbf{H}}_m^b \end{pmatrix} (x, y) e^{+i\beta_m z}, \end{aligned} \quad \Sigma \rightarrow \not\Sigma$$

\mathcal{N} : the set of forward **normal modes** supported by the waveguide.

(“Solution”: obvious; “general”: without proof.)

Completeness of normal modes

Stronger statement:

“any” transverse 2-component field on a cross sectional plane can be expanded into alternatively

- the transverse electric components of forward normal modes,
- the transverse magnetic components of forward normal modes,
- the transverse electric components of backward normal modes,
- the transverse magnetic components of backward normal modes.

Orthogonality of normal modes

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}. \quad \sim \exp(i\omega t) \quad (\text{FD})$$

	$\bar{\mathbf{E}}$	$\bar{\mathbf{H}}$	β
[prop., f]	(E'_x, E'_y, iE'_z)	(H'_x, H'_y, iH'_z)	$\beta > 0$
[prop., b]	$(E'_x, E'_y, -iE'_z)$	$(-H'_x, -H'_y, iH'_z)$	$\beta < 0$
[evan., f]	(E'_x, E'_y, E'_z)	(iH'_x, iH'_y, iH'_z)	$\beta = -i\alpha, \alpha > 0$
[evan., b]	$(E'_x, E'_y, -E'_z)$	$(-iH'_x, -iH'_y, iH'_z)$	$\beta = i\alpha, \alpha > 0$

individual $E'_x, \dots, H'_z \in \mathbb{R}$.

$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) := \frac{1}{4} \iint (E_{ax}^* H_{by} - E_{ay}^* H_{bx} + H_{ay}^* E_{bx} - H_{ax}^* E_{by}) \, dx \, dy$$

$$\begin{pmatrix} \mathbf{E}_{1,2} \\ \mathbf{H}_{1,2} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_{1,2} \\ \bar{\mathbf{H}}_{1,2} \end{pmatrix} (x, y) e^{-i\beta_{1,2}z}, \quad \begin{aligned} \nabla \times \mathbf{E}_{1,2} &= -i\omega\mu_0\mathbf{H}_{1,2}, \\ \nabla \times \mathbf{H}_{1,2} &= i\omega\epsilon_0\epsilon\mathbf{E}_{1,2}, \end{aligned}$$

$$\nabla \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) = 0 \rightsquigarrow 0 = (\beta_1^* - \beta_2) (\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2).$$



Orthogonality of normal modes

Nondegenerate directional normal modes of the same waveguide (ϵ):

$$\begin{pmatrix} \mathbf{E}_m^{\text{f,b}} \\ \mathbf{H}_m^{\text{f,b}} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m^{\text{f,b}} \\ \bar{\mathbf{H}}_m^{\text{f,b}} \end{pmatrix} (x, y) e^{-i\beta_m^{\text{f,b}} z}, \quad \begin{aligned} \nabla \times \mathbf{E}_m &= -i\omega\mu_0\mathbf{H}_m, \\ \nabla \times \mathbf{H}_m &= i\omega\epsilon_0\epsilon\mathbf{E}_m, \\ \beta_l &\neq \beta_m, \text{ if } l \neq m. \end{aligned}$$

- A propagating mode m :

$$\begin{aligned} (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) &=: P_m, \quad (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) = -P_m, \quad P_m \in \mathbb{R}, \\ (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) &= (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) = 0, \\ (\bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}; \bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}) &= (\bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}; \bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}) = 0 \quad \text{for all } l \neq m, \text{ d,r = f,b.} \end{aligned}$$

- An evanescent mode m :

$$\begin{aligned} (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) &= (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) = 0, \\ (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) &=: P_m, \quad (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) = -P_m, \quad P_m \notin \mathbb{R}, \\ (\bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}; \bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}) &= (\bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}; \bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}) = 0 \quad \text{for all } l \neq m, \text{ d,r = f,b.} \end{aligned}$$

(This implies orthogonality of propagating and evanescent modes.)

($1/\sqrt{|P_m|}$ is frequently used for mode normalization.)

Power flow associated with a normal mode expansion

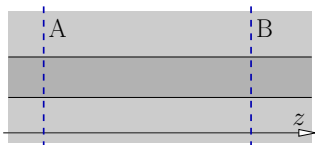
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_{m \in \mathcal{N}} \left\{ F_m \begin{pmatrix} \bar{\mathbf{E}}_m^f \\ \bar{\mathbf{H}}_m^f \end{pmatrix} (x, y) e^{-i\beta_m z} + B_m \begin{pmatrix} \bar{\mathbf{E}}_m^b \\ \bar{\mathbf{H}}_m^b \end{pmatrix} (x, y) e^{+i\beta_m z} \right\}$$

Power carried along z :

$$\begin{aligned} P &= \iint S_z \, dx \, dy = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) \\ &= \sum_{m \text{ propag.}} (|F_m|^2 - |B_m|^2) P_m + \sum_{m \text{ evanesc.}} (F_m^* B_m - B_m^* F_m) P_m. \end{aligned}$$

- P is independent of z .
- Individual contributions from forward and backward propagating modes.
- Contributions from evanescent modes require forward and backward fields to be present.
- Unidirectional field (forward: $B_m = 0$ for all m): Only propagating modes carry power.

Projection onto normal modes



E, H : a solution of the Maxwell equations for the z -homogeneous waveguide between two cross sectional planes A and B.

↪ Extract local mode amplitudes by **projection onto normal modes**:

- A propagating mode m , $\beta_m > 0$:

$$(\bar{E}_m^f, \bar{H}_m^f; E, H) = F_m P_m e^{-i\beta z},$$

$$(\bar{E}_m^b, \bar{H}_m^b; E, H) = -B_m P_m e^{i\beta z}.$$

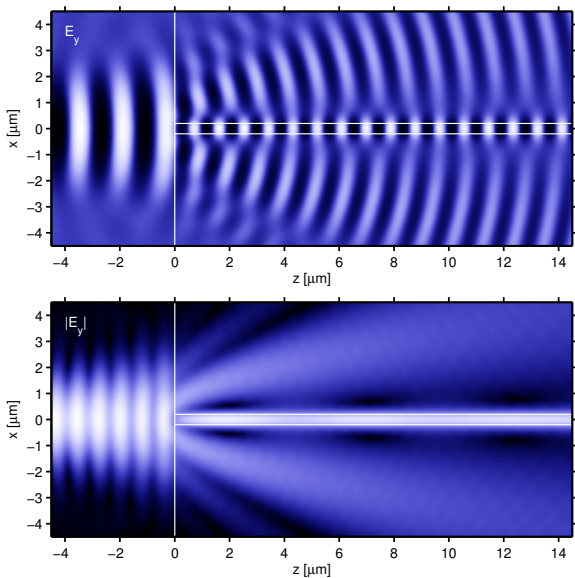
$$F_m e^{-i\beta z} = \frac{(\bar{E}_m^f, \bar{H}_m^f; E, H)}{(\bar{E}_m^f, \bar{H}_m^f; \bar{E}_m^f, \bar{H}_m^f)}$$

- An evanescent mode m , $\beta_m = -i\alpha_m$, $\alpha_m > 0$:

$$(\bar{E}_m^f, \bar{H}_m^f; E, H) = B_m P_m e^{\alpha z}, \quad (\bar{E}_m^b, \bar{H}_m^b; E, H) = -F_m P_m e^{-\alpha z}.$$

↔ **Ports** of a photonic integrated circuit.

Waveguide facet: Port definition

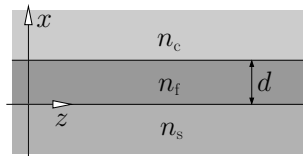


(2-D, TE)

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

2-D waveguide configurations



$$\epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i\omega t) \text{ (FD)}$$

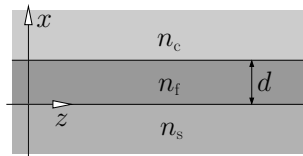
- 2-D waveguide, 1-D cross section.
- Permittivity $\epsilon = n^2$,
refractive index $n(x)$. (1-D waveguide)

- $\partial_y \epsilon = 0 \iff \partial_y \mathbf{E} = 0, \partial_y \mathbf{H} = 0$, 2-D TE/TM setting.
- $\partial_z \epsilon = 0 \iff$ Modal solutions that vary harmonically with z :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x) e^{-i\beta z},$$

mode profile $\bar{\mathbf{E}}, \bar{\mathbf{H}}$,
propagation constant β ,
effective index $n_{\text{eff}} = \beta/k$.

2-D waveguide configurations



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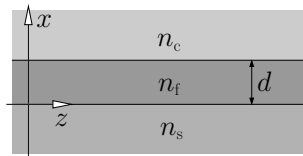
mode profile $\bar{\mathbf{E}}, \bar{\mathbf{H}}$,
propagation constant β ,
effective index $n_{\text{eff}} = \beta/k$.

(TE): principal component \bar{E}_y , $\partial_x^2 \bar{E}_y + (k^2 \epsilon - \beta^2) \bar{E}_y = 0$,

$$\bar{E}_x = 0, \quad \bar{E}_z = 0, \quad \bar{H}_x = \frac{-\beta}{\omega \mu_0} \bar{E}_y, \quad \bar{H}_y = 0, \quad \bar{H}_z = \frac{i}{\omega \mu_0} \partial_x \bar{E}_y,$$

\bar{E}_y & $\partial_x \bar{E}_y$ continuous at dielectric interfaces.

2-D waveguide configurations



$$\epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i\omega t) \text{ (FD)}$$

- 2-D waveguide, 1-D cross section.
- Permittivity $\epsilon = n^2$,
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$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x) e^{-i\beta z},$$

mode profile $\bar{\mathbf{E}}, \bar{\mathbf{H}}$,
propagation constant β ,
effective index $n_{\text{eff}} = \beta/k$.

(TM): principal component \bar{H}_y , $\epsilon \partial_x \frac{1}{\epsilon} \partial_x \bar{H}_y + (k^2 \epsilon - \beta^2) \bar{H}_y = 0$,

$$\bar{E}_x = \frac{\beta}{\omega \epsilon_0 \epsilon} \bar{H}_y, \quad \bar{E}_y = 0, \quad \bar{E}_z = \frac{-i}{\omega \epsilon_0 \epsilon} \partial_x \bar{H}_y, \quad \bar{H}_x = 0, \quad \bar{H}_z = 0,$$

\bar{H}_y & $\epsilon^{-1} \partial_x \bar{H}_y$ continuous at dielectric interfaces.

Guided 2-D TE/TM modes, orthogonality properties

(→ Exercise.)

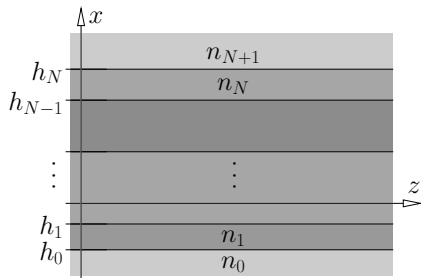
- A set (index m) of guided modes of a 2-D waveguide (ϵ),
 $\psi_m^p = (\bar{E}_m, \bar{H}_m)$, $p=\text{TE, TM}$ & β_m , $\beta_m \neq \beta_l$, if $l \neq m$.
- $(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \int (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx$.
- Power P_m per lateral (y) unit length carried by mode ψ_m^p , β_m :

$$P_m := \int S_z dx = (\psi_m^p; \psi_m^p) = \begin{cases} \frac{\beta_m}{2\omega\mu_0} \int |E_{m,y}|^2 dx, & \text{if } p = \text{TE}, \\ \frac{\beta_m}{2\omega\epsilon_0} \int \frac{1}{\epsilon} |H_{m,y}|^2 dx, & \text{if } p = \text{TM}. \end{cases}$$

$$(\psi_l^{\text{TE}}; \psi_m^{\text{TM}}) = 0, \quad (\psi_l^{\text{TE}}; \psi_m^{\text{TE}}) = \frac{\beta_m}{2\omega\mu_0} \int E_{l,y}^* E_{m,y} dx = \delta_{lm} P_m,$$

$$(\psi_l^{\text{TM}}; \psi_m^{\text{TE}}) = 0, \quad (\psi_l^{\text{TM}}; \psi_m^{\text{TM}}) = \frac{\beta_m}{2\omega\epsilon_0} \int \frac{1}{\epsilon} H_{l,y}^* H_{m,y} dx = \delta_{lm} P_m.$$

Dielectric multilayer slab waveguide



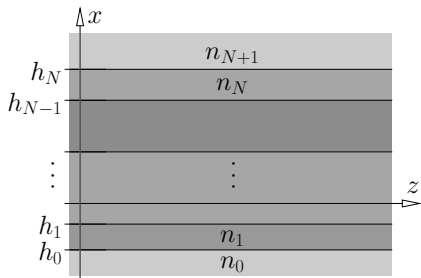
$\epsilon \in \mathbb{R}$, $\mu = 1$, $\sim \exp(i\omega t)$ (2-D, FD)

- N interior layers,
piecewise constant $\epsilon = n^2$:

$$n(x) = \begin{cases} n_{N+1} & \text{if } h_N < x, \\ n_l & \text{if } h_{l-1} < x < h_l, \\ n_0 & \text{if } x < h_0. \end{cases}$$

- Principal component $\phi(x)$ (TE: $\phi = \bar{E}_y$, TM: $\phi = \bar{H}_y$).
- $\partial_x^2 \phi + (k^2 n_l^2 - \beta^2) \phi = 0$, $x \in \text{layer } l$, $l = 0, \dots, N+1$
(Half-infinite substrate ($l = 0$) and cover ($l = N+1$) layers.)
- ϕ & $\eta \partial_x \phi$ continuous at $x = h_l$, (TE: $\eta = 1$, TM: $\eta = n^{-2}$).

Dielectric multilayer slab waveguide



- Interior layer l ,
 $h_{l-1} < x < h_l$,
 local refractive index n_l ,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_l^2) \phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_l^2 \quad \rightsquigarrow \quad \partial_x^2 \phi = -\kappa_l^2 \phi, \quad \kappa_l := \sqrt{k^2 n_l^2 - \beta^2},$

$$\phi(x) = A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x).$$

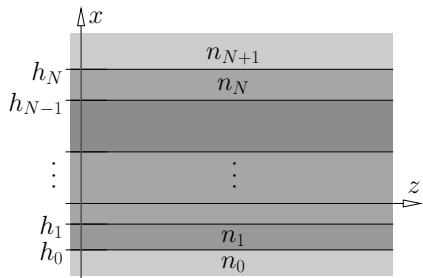
- $\beta^2 > k^2 n_l^2 \quad \rightsquigarrow \quad \partial_x^2 \phi = \kappa_l^2 \phi, \quad \kappa_l := \sqrt{\beta^2 - k^2 n_l^2},$

$$\phi(x) = A_l e^{\kappa_l x} + B_l e^{-\kappa_l x}.$$

- Unknowns $A_l, B_l \in \mathbb{C}$.

(Local coordinate offsets required to cope with the exponentials.)

Dielectric multilayer slab waveguide, guided modes



- Substrate region,
 $x < h_0$,
 local refractive index n_0 ,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_0^2) \phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_0^2 \rightsquigarrow \partial_x^2 \phi = -\kappa_0^2 \phi, \quad \kappa_0 := \sqrt{k^2 n_0^2 - \beta^2},$

$$\phi(x) = A_0 \sin(\kappa_0 x) + B_0 \cos(\kappa_0 x).$$

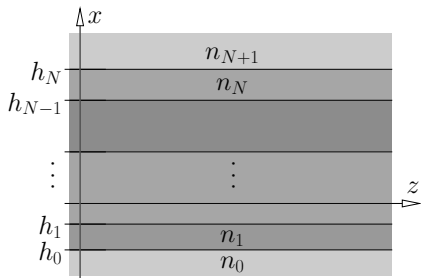
- $\beta^2 > k^2 n_0^2 \rightsquigarrow \partial_x^2 \phi = \kappa_0^2 \phi, \quad \kappa_0 := \sqrt{\beta^2 - k^2 n_0^2},$

$$\phi(x) = A_0 e^{\kappa_0 x} + B_0 e^{-\kappa_0 x}.$$

- Unknown $A_0 \in \mathbb{C}$.

Guided modes: $n_{\text{eff}} = \beta/k > n_0$.

Dielectric multilayer slab waveguide, guided modes



- Cover region,
 $h_N < x$,
 local refractive index n_{N+1} ,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_{N+1}^2) \phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_{N+1}^2 \rightsquigarrow \partial_x^2 \phi = -\kappa_{N+1}^2 \phi, \quad \kappa_{N+1} := \sqrt{k^2 n_{N+1}^2 - \beta^2},$

$$\phi(x) = A_{N+1} \sin(\kappa_{N+1} x) + B_{N+1} \cos(\kappa_{N+1} x).$$

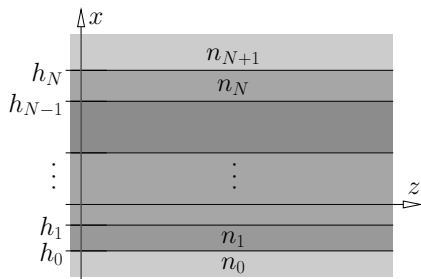
- $\beta^2 > k^2 n_{N+1}^2 \rightsquigarrow \partial_x^2 \phi = \kappa_{N+1}^2 \phi, \quad \kappa_{N+1} := \sqrt{\beta^2 - k^2 n_{N+1}^2},$

$$\phi(x) = A_{N+1} e^{\kappa_{N+1} x} + B_{N+1} e^{-\kappa_{N+1} x}.$$

- Unknown $B_{N+1} \in \mathbb{C}$.

Guided modes: $n_{\text{eff}} = \beta/k > n_{N+1}$.

Dielectric multilayer slab waveguide



Trial value $\beta^2 \in \mathbb{R}$,

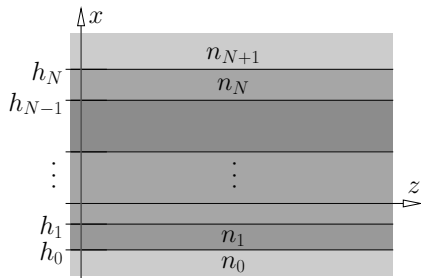
$\beta/k > n_0, n_{N+1}$,

$\rightsquigarrow \kappa_l, \quad l = 0, \dots, N+1.$

$$\phi(x) = \begin{cases} B_{N+1} e^{-\kappa_{N+1} x}, & \text{for } h_N < x, \\ \left\{ \begin{array}{ll} A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x), & \text{if } \beta^2 < k^2 n_l^2, \\ A_l e^{\kappa_l x} + B_l e^{-\kappa_l x}, & \text{if } \beta^2 > k^2 n_l^2, \end{array} \right\} & \text{for } h_{l-1} < x < h_l, \\ A_0 e^{\kappa_0 x}, & \text{for } x < h_0. \end{cases}$$

- $2N + 2$ unknowns $A_0, A_1, B_1, \dots, A_N, B_N, B_{N+1}$.
- Continuity of $\phi, \eta \partial_x \phi$ at $N + 1$ interfaces $\rightsquigarrow 2N + 2$ equations.

Dielectric multilayer slab waveguide



Trial value $\beta^2 \in \mathbb{R}$,
 $\beta/k > n_0, n_{N+1}$.

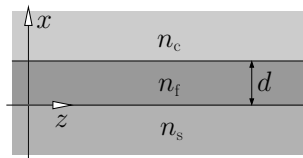
- $2N + 2$ unknowns $A_0, A_1, B_1, \dots, A_N, B_N, B_{N+1}$.
- Continuity of $\phi, \eta \partial_x \phi$ at $N + 1$ interfaces $\rightsquigarrow 2N + 2$ equations.
- Arrange as linear system of equations $\mathbf{M}(\beta^2) (A_0, \dots, B_{N+1})^T = 0$.
- Identify propagation constants where $\mathbf{M}(\beta^2)$ becomes singular.

(Equations relate to the series of interfaces \leftrightarrow A transfer-matrix technique can be applied.)

- Choose e.g. $A_0 = 1$, fill A_1, \dots, B_{N+1} , normalize. $(\cdot, \cdot; \cdot, \cdot)$

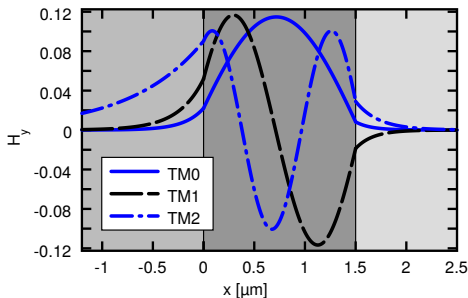
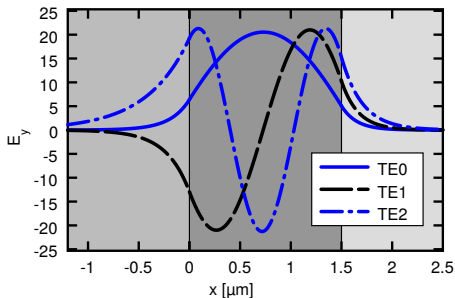
Guided modes $\{\beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m)\}$.

A nonsymmetric 3-layer slab waveguide

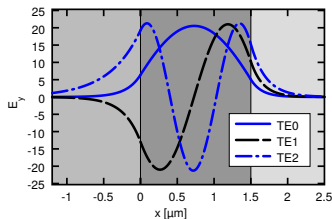


$n_s = 1.45$, $n_f = 1.99$, $n_c = 1.0$,
 $d = 1.5 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$.

TE₀: $n_{\text{eff}} = 1.944$, TM₀: $n_{\text{eff}} = 1.933$,
TE₁: $n_{\text{eff}} = 1.804$, TM₁: $n_{\text{eff}} = 1.759$,
TE₂: $n_{\text{eff}} = 1.562$, TM₂: $n_{\text{eff}} = 1.490$.



Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE/TM.)

$$\partial_x(\partial_x \phi) = -(k^2 n^2 - \beta^2)\phi.$$

$k^2 n^2 - \beta^2$ determines the rate of change of the slope of ϕ .

Imagine a numerical ODE algorithm of “shooting-type”.



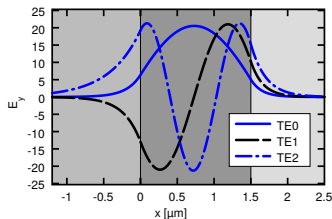
- Guided modes with a growing number of nodes (x with $\phi(x) = 0$) with decreasing effective indices

↔ mode indices = number of nodes in ϕ .



“Quantum numbers”.

Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE/TM.)

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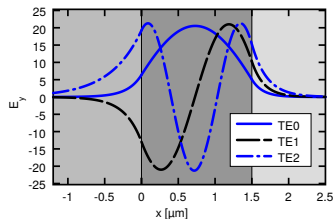
↔ mode indices = number of nodes in ϕ .



“Quantum numbers”.

- A fundamental mode with zero nodes and highest effective index.

Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE/TM.)

$$\partial_x(\partial_x \phi) = -(k^2 n^2 - \beta^2)\phi.$$

$k^2 n^2 - \beta^2$ determines the rate of change of the slope of ϕ .

Imagine a numerical ODE algorithm of “shooting-type”.



- Guided modes with a growing number of nodes (x with $\phi(x) = 0$) with decreasing effective indices

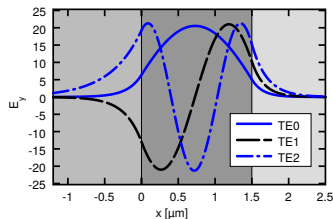
↔ mode indices = number of nodes in ϕ .



“Quantum numbers”.

- A **fundamental mode** with zero nodes and highest effective index.
- Modes of the same polarization are **non-degenerate**.

Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE/TM.)

$$\partial_x(\partial_x\phi) = -(k^2 n^2 - \beta^2)\phi.$$

$k^2 n^2 - \beta^2$ determines the rate of change of the slope of ϕ .

Imagine a numerical ODE algorithm of “shooting-type”.

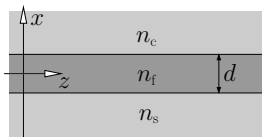


- A sign change of $\partial_x\phi$ is required to form a guided mode
 ~> There must be some region (layer) with $k^2 n^2 - \beta^2 > 0$.

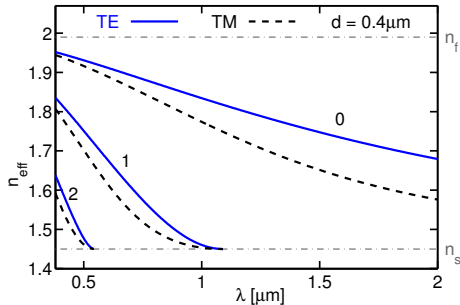
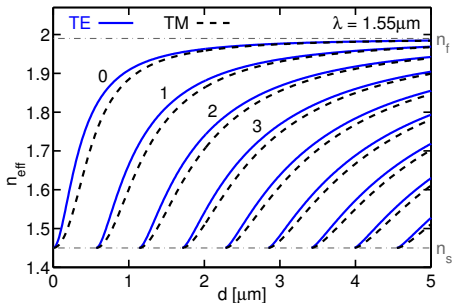
Interval for effective indices n_{eff} of guided modes:

$$\max\{n_0, n_{N+1}\} < n_{\text{eff}} < \max_l\{n_l\}.$$

3-layer slab waveguide, dispersion curves

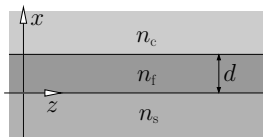


Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$.

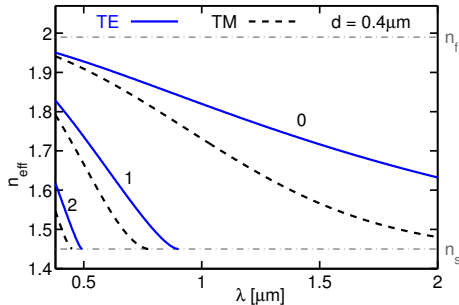
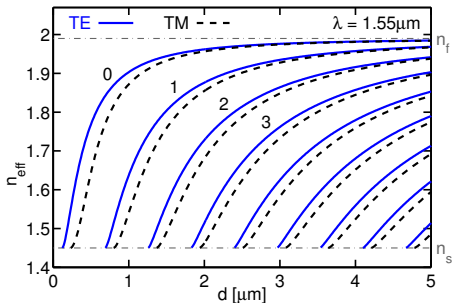


(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves

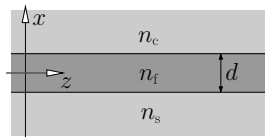


Nonsymmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.0$.

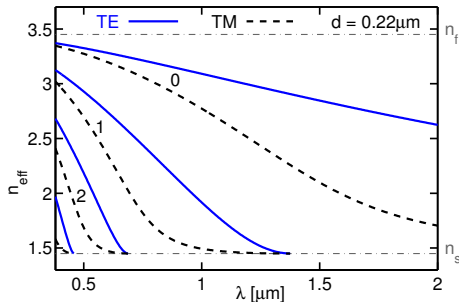
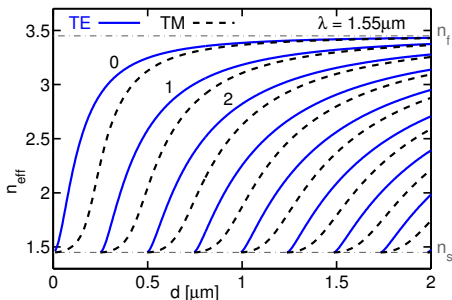


(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves

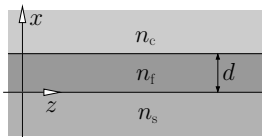


Symmetric waveguide,
high refractive index contrast,
 $n_s = 1.45$, $n_f = 3.45$, $n_c = 1.45$.

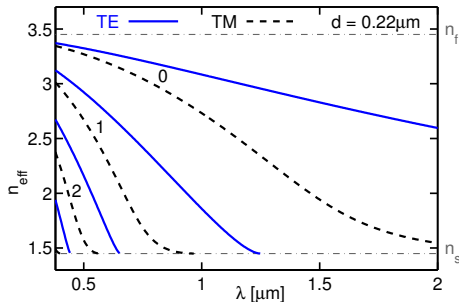
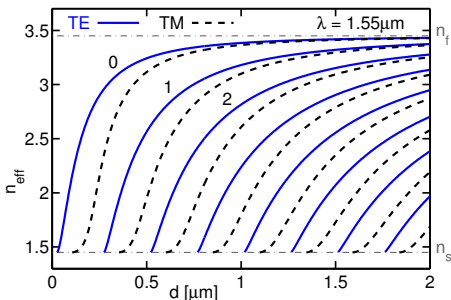


(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves



Nonsymmetric waveguide,
high refractive index contrast,
 $n_s = 1.45$, $n_f = 3.45$, $n_c = 1.0$.



(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves

Remarks / observations:

- At large core thicknesses, or short wavelengths, for all modes: n_{eff} approaches the level n_f of bulk waves in the core material.
- Modes of higher order at the same n_{eff} supported by waveguides with thickness increased by specific distances.

Guided mode, layer l with $\kappa_l^2 = (k^2 n^2 - \beta^2) > 0$, field $\phi(x) \sim \cos(\kappa_l x + \chi)$ for $x \in \text{layer } l$;
increase layer thickness by $\Delta x = \pi / \kappa_l$, such that $\kappa_l(x + \Delta x) = \kappa_l x + \pi$
→ the thicker waveguide supports a mode of order +1 with the same propagation constant.

- Cutoff thicknesses at fixed wavelength.

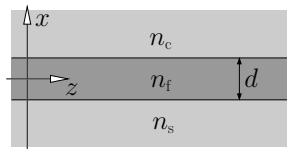
Nonsymmetric 3-layer waveguide $n_s \neq n_c$: There exist cutoff thicknesses for all modes.

Symmetric 3-layer waveguide $n_s = n_c$: Cutoff thicknesses exist for all modes of order ≥ 1 ,
no cutoff thickness for the fundamental TE/TM modes.

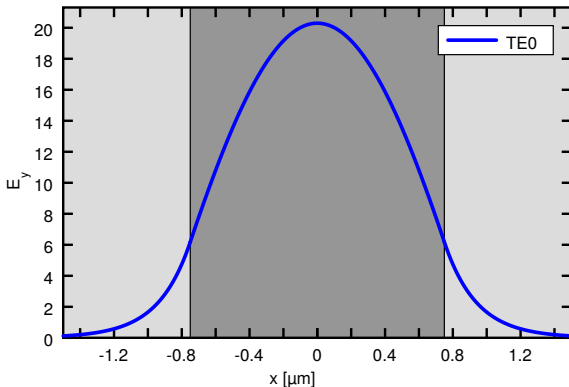
- λ is the “length-defining” quantity; wavelength scaling, factor a :
 $n_{\text{eff}}(\lambda, d) = n_{\text{eff}}(a\lambda, ad)$, $\beta(\lambda, d) = a^{-1} \beta(a\lambda, ad)$.
- Cutoff wavelengths for waveguides with fixed thickness.

For all modes; exception: no cutoff wavelength for the fundamental TE/TM modes in a symmetric 3-layer waveguide.

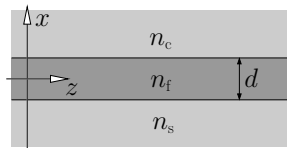
3-layer slab waveguide, mode confinement



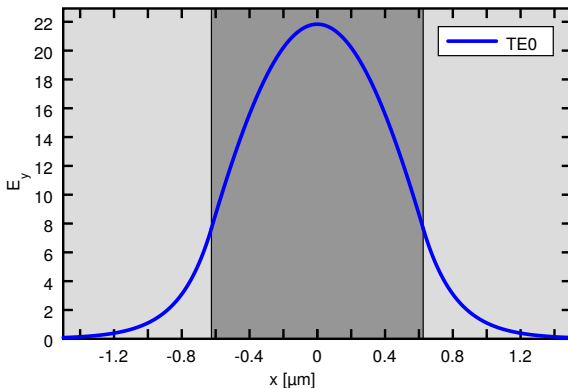
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 1.50 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.946$.



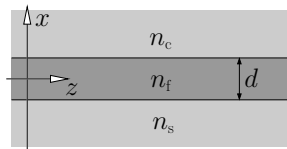
3-layer slab waveguide, mode confinement



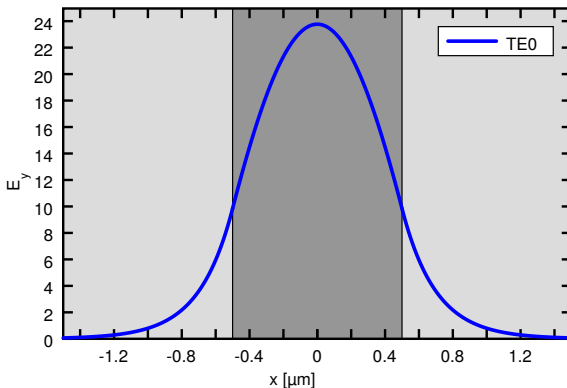
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 1.25 \mu\text{m}$, TE_0 : $n_{\text{eff}} = 1.932$.



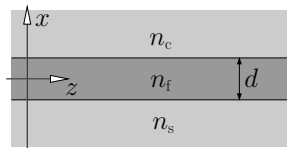
3-layer slab waveguide, mode confinement



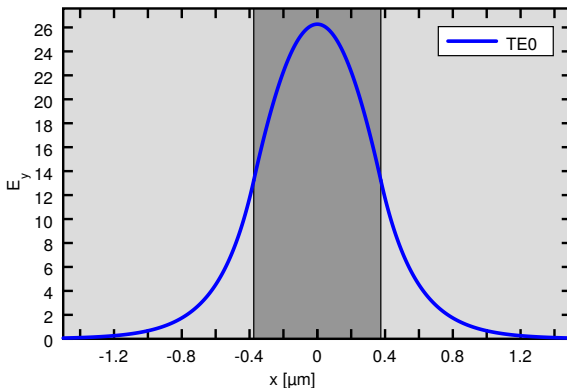
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 1.00 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.908$.



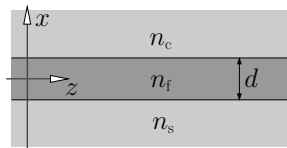
3-layer slab waveguide, mode confinement



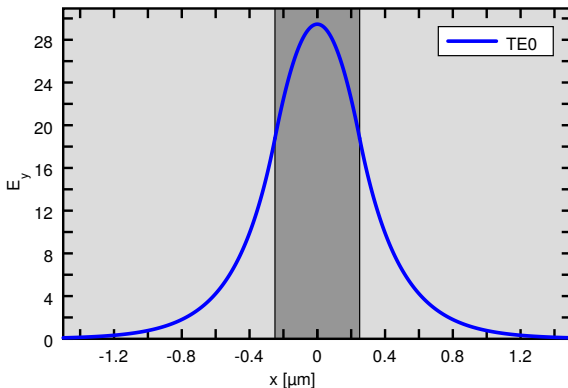
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.75 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.868$.



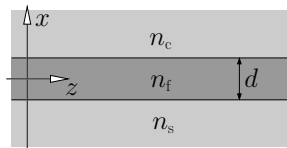
3-layer slab waveguide, mode confinement



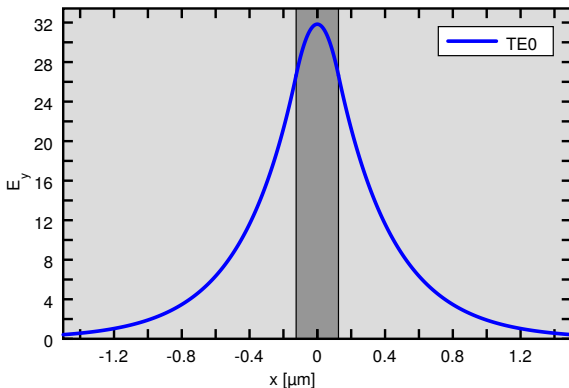
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.50 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.791$.



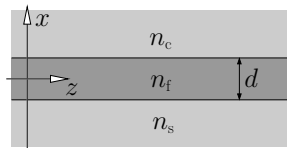
3-layer slab waveguide, mode confinement



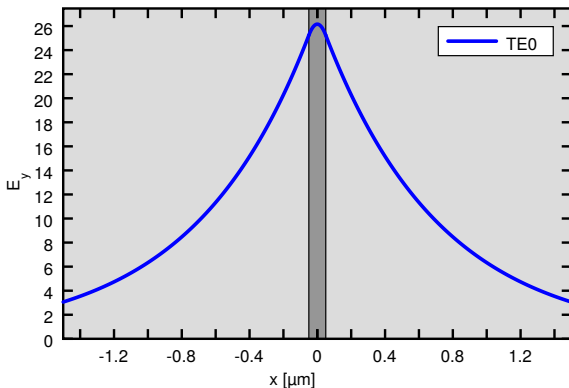
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.25 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.630$.



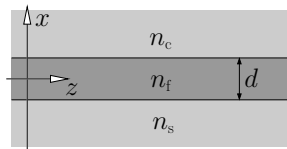
3-layer slab waveguide, mode confinement



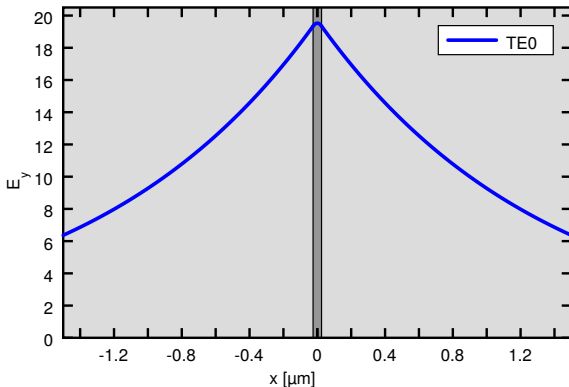
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.10 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.494$.



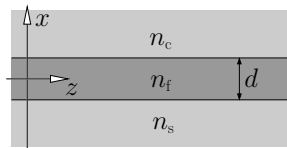
3-layer slab waveguide, mode confinement



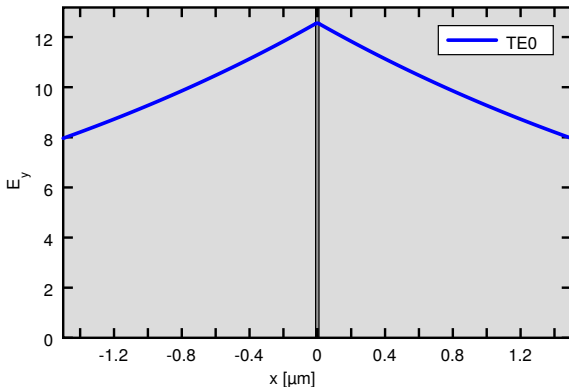
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.05 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.462$.



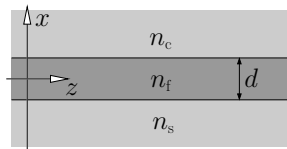
3-layer slab waveguide, mode confinement



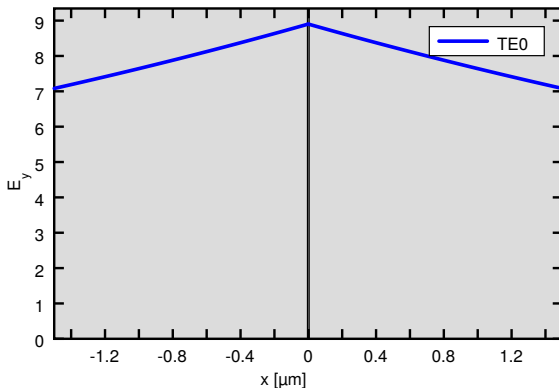
Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.02 \mu\text{m}$, TE_0 : $n_{\text{eff}} = 1.452$.



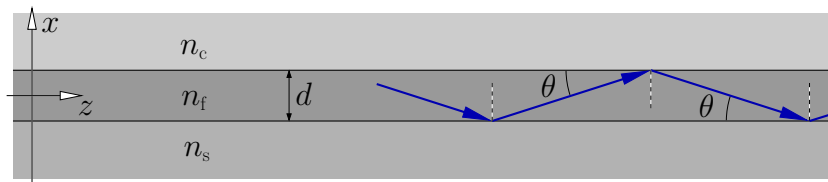
3-layer slab waveguide, mode confinement



Symmetric waveguide,
moderate refractive index contrast,
 $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \mu\text{m}$,
 $d = 0.01 \mu\text{m}$, TE₀: $n_{\text{eff}} = 1.450$.



3-layer slab waveguide, ray model



Field in the core:

$$\sim a_u e^{-i(\kappa x + \beta z)} + a_d e^{-i(-\kappa x + \beta z)}, \quad k^2 n_f^2 = \beta^2 + \kappa^2$$

↔ propagation angle θ with $\beta = k n_f \cos \theta$, $\kappa = k n_f \sin \theta$.

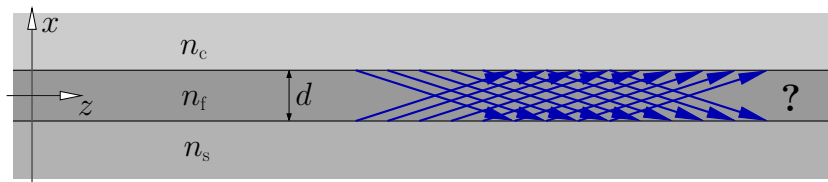


Guided mode formation:

- Repeated total internal reflection of waves in the core at upper and lower interfaces
- Calculate optical phase gain, including phase jumps at interfaces (polarization dependent).
- Phase gain of 2π for one “round trip”, “transverse resonance condition” \longleftrightarrow constructive interference of waves.

(A frequently encountered intuitive model . . . of very limited applicability.)

3-layer slab waveguide, ray model



Field in the core:

$$\sim a_u e^{-i(\kappa x + \beta z)} + a_d e^{-i(-\kappa x + \beta z)}, \quad k^2 n_f^2 = \beta^2 + \kappa^2$$

↔ propagation angle θ with $\beta = kn_f \cos \theta$, $\kappa = kn_f \sin \theta$.

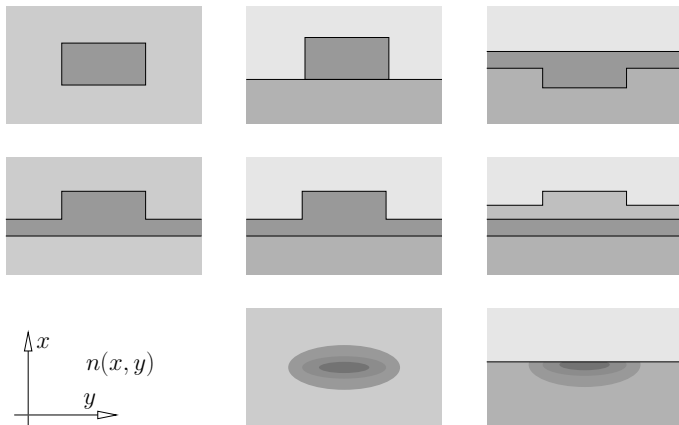


Guided mode formation:

- Repeated total internal reflection of waves in the core at upper and lower interfaces
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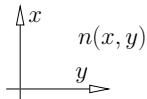
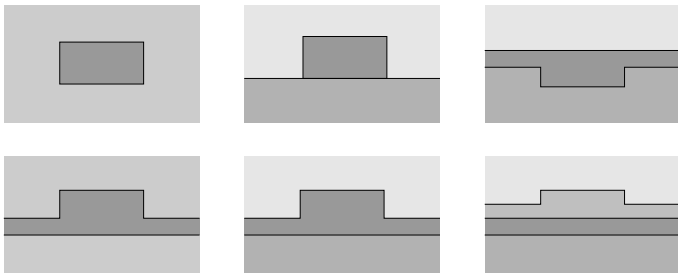
(A frequently encountered intuitive model . . . of very limited applicability.)

3-D waveguides



Cross sections (2-D) of typical integrated-optical waveguides.

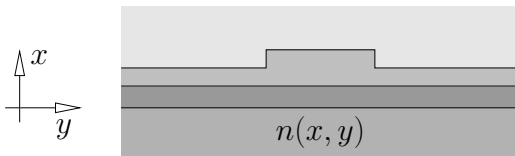
3-D rectangular waveguides



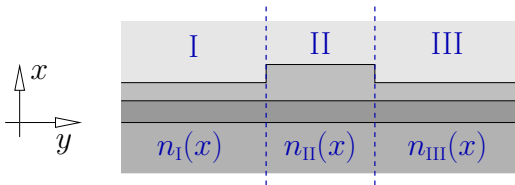
No analytical solutions :

- numerical mode solvers.
- approximations.

Effective index method



Effective index method



Outline:

(!)

- Divide into slices $\rho = \text{I, II, III}$: $n(x, y) = n_\rho(x)$, if $y \in \text{slice } \rho$.
- Compute polarized modes $X_\rho(x)$, β_ρ , $X_\rho'' + (k^2 n_\rho^2 - \beta_\rho^2)X_\rho = 0$, $N_\rho = \beta_\rho/k$.
- Consider a scalar mode equation for the principal component Ψ of the 3-D waveguide

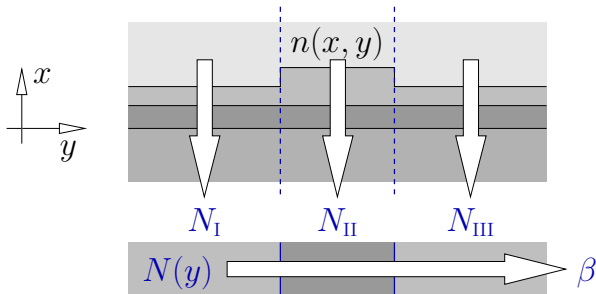
$$\partial_x^2 \Psi + \partial_y^2 \Psi + (k^2 n^2 - \beta^2) \Psi = 0, \quad \Psi = E_y \text{ (TE)}, \quad \Psi = H_y \text{ (TM)}.$$

- Ansatz: $\Psi(x, y) = X_\rho(x) Y(y)$, if $y \in \text{slice } \rho$; require continuity of Y and Y' .
- **Effective index profile:** $N(y) := N_\rho$, if $y \in \text{slice } \rho$.

↪ $Y'' + (k^2 N^2 - \beta^2)Y = 0,$

a 1-D mode equation for Y , β with the effective index profile N in place of the refractive indices.

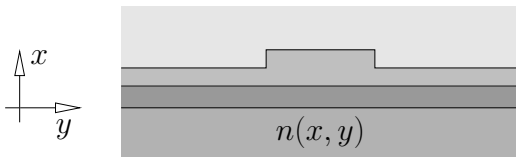
Effective index method, schematically



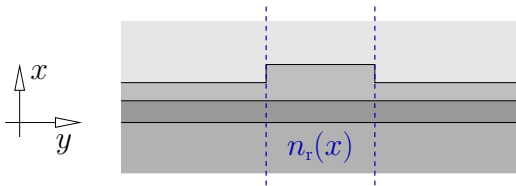
Remarks / issues:

- A popular, quite intuitive method.
- Frequently an (often informal) basis for discussion of waveguide properties.
- \leftrightarrow Relevance of the slab waveguide model.
- Manifold variants / ways of improvements exist.
- What if a slice does not support a guided slab mode?
- What about higher order modes?
- How to evaluate modal fields? What about other than principal components?
- ...

Variational effective index method



Variational effective index method



Outline:

(!)

- Identify a reference slice, refractive index profile $n_r(x)$.
- Compute polarized guided slab modes $(\bar{\mathbf{E}}, \bar{\mathbf{H}})_r$, β_r for the reference slice.
- For each reference slab mode: ...
- Choose an ansatz:

(VEIM)

$$\begin{pmatrix} E_x, E_y, E_z \\ H_x, H_y, H_z \end{pmatrix} (x, y, z) = \begin{pmatrix} 0, & \bar{E}_{r,y}(x)Y^{E_y}(y), & \bar{E}_{r,y}(x)Y^{E_z}(y) \\ \bar{H}_{r,x}(x)Y^{H_x}(y), & \bar{H}_{r,z}(x)Y^{H_y}(y), & \bar{H}_{r,z}(x)Y^{H_z}(y) \end{pmatrix} \quad (\text{TE})$$

$$\begin{pmatrix} E_x, E_y, E_z \\ H_x, H_y, H_z \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{E}_{r,x}(x)Y^{E_x}(y), & \bar{E}_{r,z}(x)Y^{E_y}(y), & \bar{E}_{r,z}(x)Y^{E_z}(y) \\ 0, & \bar{H}_{r,y}(x)Y^{H_y}(y), & \bar{H}_{r,y}(x)Y^{H_z}(y) \end{pmatrix} \quad (\text{TM})$$

↪ $Y^\cdot(y) = ?$

A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

- $$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad \beta \in \mathbb{R},$$
$$\bar{\mathbf{E}}, \bar{\mathbf{H}} \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.$$

- $$(\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{E}} = -i\omega\mu_0\bar{\mathbf{H}}, \quad (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{H}} = i\omega\epsilon_0\bar{\mathbf{E}},$$

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

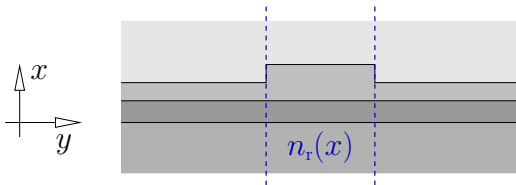
- $$\mathcal{B}(\mathbf{E}, \mathbf{H}) := \frac{\omega\epsilon_0\langle \mathbf{E}, \epsilon\mathbf{E} \rangle + \omega\mu_0\langle \mathbf{H}, \mathbf{H} \rangle + i\langle \mathbf{E}, \mathbf{C}\mathbf{H} \rangle - i\langle \mathbf{H}, \mathbf{C}\mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{R}\mathbf{H} \rangle - \langle \mathbf{H}, \mathbf{R}\mathbf{E} \rangle},$$

$$\langle \mathbf{F}, \mathbf{G} \rangle = \iint \mathbf{F}^* \cdot \mathbf{G} \, dx \, dy.$$

$$\mathcal{B}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta, \quad \left. \frac{d}{ds} \mathcal{B}(\bar{\mathbf{E}} + s \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + s \delta\bar{\mathbf{H}}) \right|_{s=0} = 0$$

at valid mode fields $\bar{\mathbf{E}}, \bar{\mathbf{H}}$, for arbitrary $\delta\bar{\mathbf{E}}, \delta\bar{\mathbf{H}}$.

Variational effective index method



Outline, continued:

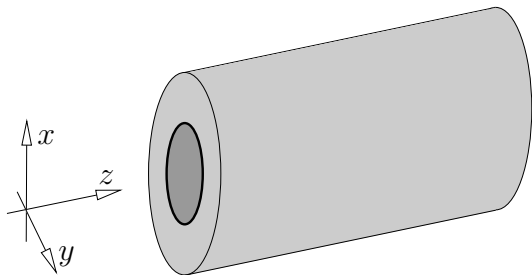
(!)

- Restrict \mathcal{B} to the VEIM ansatz, require stationarity with respect to the $\{Y\}$.

↪ 1-D mode (“-like”) equations for principal unknowns Y^{H_x} (TE) and Y^{E_x} (TM) with effective quantities in place of refractive indices, all other Y can be computed.

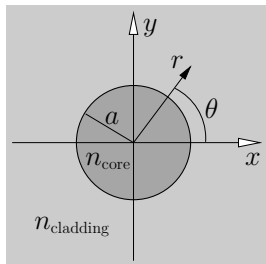


Optical fibers



[Optical Communication A-D]

Circular step index optical fibers



(FD)

Circular symmetry

↔ cylindrical coordinates r, θ, z .

$$\epsilon = n^2, \quad n(r) = \begin{cases} n_{\text{core}}, & r \leq a, \\ n_{\text{cladding}}, & r > a. \end{cases}$$

Circular and axial symmetry:

$$\left(\begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right) (r, \theta, z) = \left(\begin{matrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{matrix} \right) (r) e^{-il\theta - i\beta z}, \quad l \in \mathbb{Z}, \beta \in \mathbb{R}.$$

($E_r, E_\theta, E_z, H_r, H_\theta, H_z$)

Where $\partial\epsilon = 0$: $\Delta\psi + k^2 n^2 \psi = 0, \quad \psi \in \{E_r, \dots, H_z\}.$

$$\partial_r^2 \phi + \frac{1}{r} \partial_r \phi + (k^2 n^2 - \beta^2 - \frac{l^2}{r^2}) \phi = 0, \quad \phi \in \{\bar{E}_r, \dots, \bar{H}_z\}$$

(An ODE of Bessel type.)

& vectorial interface conditions at $r = a$. (Alternatively: Scalar theory, LP modes.)

(...)



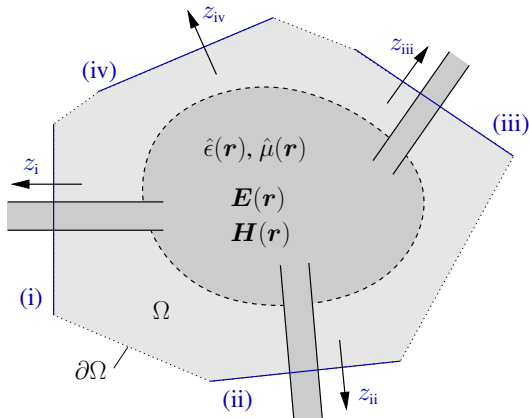
Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.



Scattering matrices, prerequisites

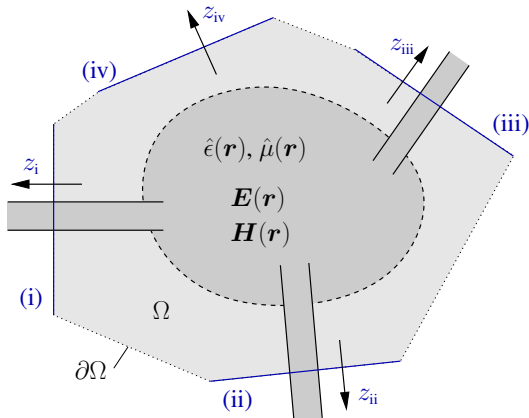
$\sim \exp(i\omega t)$ (FD)



- Passive, linear circuit.

Scattering matrices, prerequisites

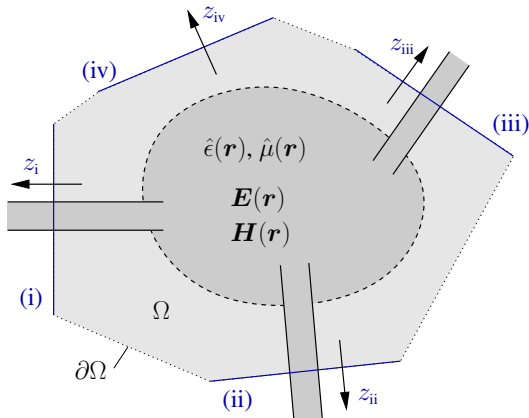
$\sim \exp(i\omega t)$ (FD)



- (Computational) domain of interest Ω , its boundary $\partial\Omega$.

Scattering matrices, prerequisites

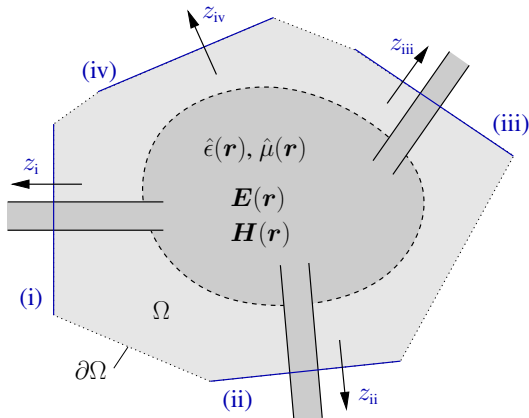
$\sim \exp(i\omega t)$ (FD)



- Connecting channels: lossless waveguides (or “half-spaces”).

Scattering matrices, prerequisites

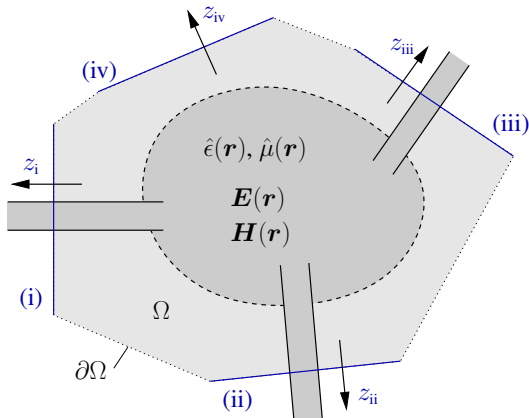
$\sim \exp(i\omega t)$ (FD)



- Physical **ports** $p = i, ii, \dots$: waveguide cross-section planes, local coordinates x_p, y_p, z_p ; local axis z_p oriented outwards of Ω .

Scattering matrices, prerequisites

$\sim \exp(i\omega t)$ (FD)

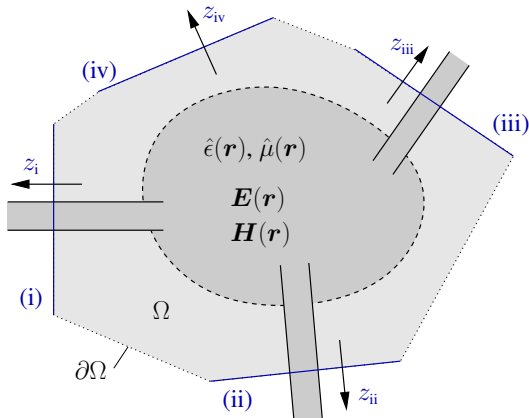


- Establish sets \mathcal{N}_p of *propagating* directional normal modes $\{\psi_{p,m}^d := (\mathbf{E}_{p,m}^d, \mathbf{H}_{p,m}^d), \beta_{p,m}; d = \text{f,b}\}$ on each port p .

(Restriction to propagating fields: a condition on port positioning / a model assumption.)

Scattering matrices, prerequisites

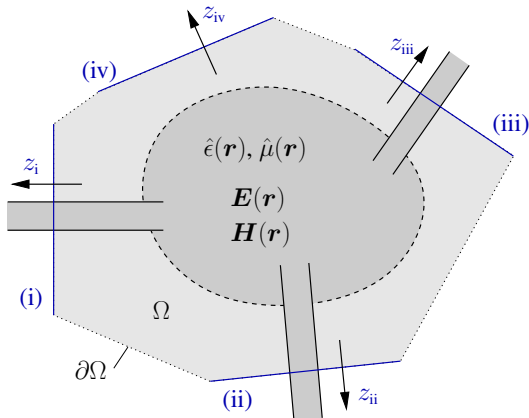
$\sim \exp(i\omega t)$ (FD)



- Ports & modes are such that all mode fields vanish
 - on all “other” port planes, and
 - on $\partial\Omega$ outside the ports.

Scattering matrices, prerequisites

$\sim \exp(i\omega t)$ (FD)



Field on port plane p and “outside”:

$$\begin{pmatrix} E \\ H \end{pmatrix} (x_p, y_p, z_p) = \sum_{m \in \mathcal{N}_p} F_{p,m} \psi_{p,m}^f(x_p, y_p) e^{-i\beta_{p,m}z_p} + B_{p,m} \psi_{p,m}^b(x_p, y_p) e^{i\beta_{p,m}z_p}.$$

Scattering matrices, prerequisites

$\sim \exp(i\omega t)$ (FD)

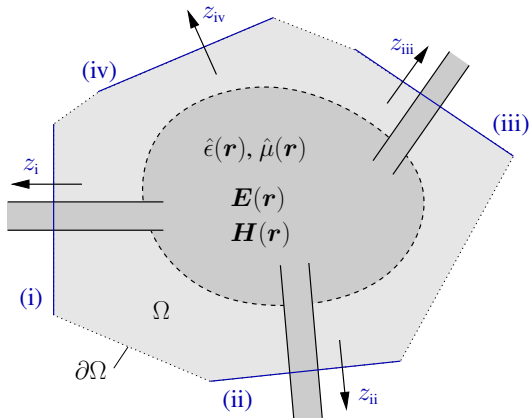
- Passive, linear circuit.
- (Computational) domain of interest Ω , its boundary $\partial\Omega$.
- Connecting channels: lossless waveguides (or “half-spaces”).
- Physical **ports** $p = \text{i, ii}, \dots$: waveguide cross-section planes, local coordinates x_p, y_p, z_p ; local axis z_p oriented outwards of Ω .
- Establish sets \mathcal{N}_p of *propagating* directional normal modes $\{\psi_{p,m}^d := (\mathbf{E}_{p,m}^d, \mathbf{H}_{p,m}^d), \beta_{p,m}; d = \text{f,b}\}$ on each port p .
(Restriction to propagating fields: a condition on port positioning / a model assumption.)
- Ports & modes are such that all mode fields vanish on all “other” port planes, and on $\partial\Omega$ outside the ports.

↪ Field on port plane p and “outside”:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x_p, y_p, z_p) = \sum_{m \in \mathcal{N}_p} F_{p,m} \psi_{p,m}^{\text{f}}(x_p, y_p) e^{-i\beta_{p,m} z_p} + B_{p,m} \psi_{p,m}^{\text{b}}(x_p, y_p) e^{i\beta_{p,m} z_p}.$$

Scattering matrices

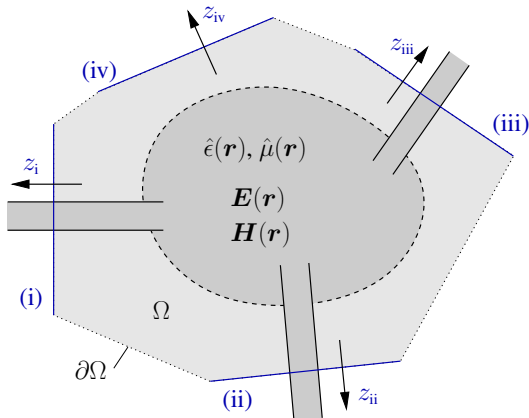
$\sim \exp(i\omega t)$ (FD)



- Merge all mode indices $\{m\}$ and port IDs $\{p\}$ into one set of mode identifiers $\{\nu\}$, $\mathcal{N} = \cup_p \mathcal{N}_p$.

Scattering matrices

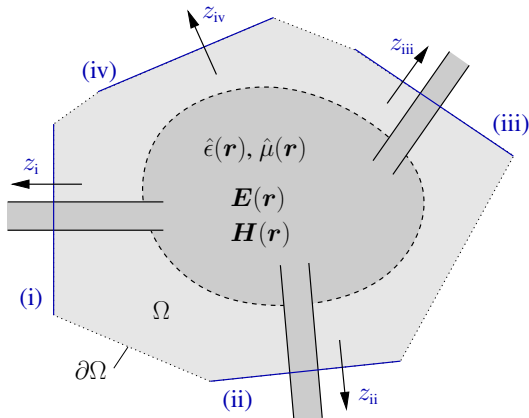
$\sim \exp(i\omega t)$ (FD)



- Assert that $\psi_{p,\cdot}(\mathbf{r}) = 0$ for all $\mathbf{r} \in \partial\Omega$, $\mathbf{r} \notin \text{port } p$.

Scattering matrices

$\sim \exp(i\omega t)$ (FD)

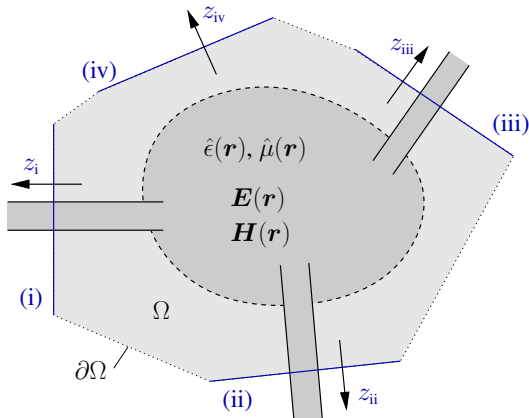


- Field on $\partial\Omega$:
$$\begin{pmatrix} E \\ H \end{pmatrix} = \sum_{\nu \in \mathcal{N}} \{F_{\nu} \psi_{\nu}^f + B_{\nu} \psi_{\nu}^b\}.$$

(Position arguments omitted.)

Scattering matrices

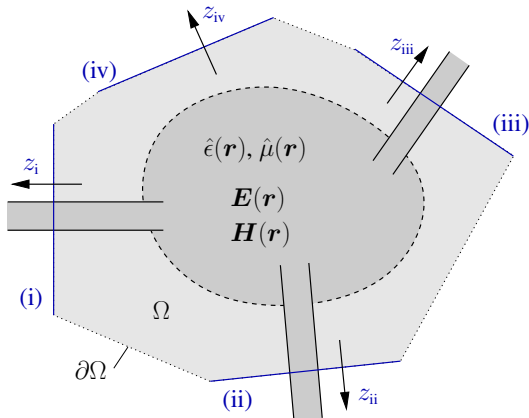
$\sim \exp(i\omega t)$ (FD)



- B_ν : \sim **incident modes**, traveling towards the interior of Ω .
 - F_ν : \sim **outgoing modes**, traveling towards the exterior of Ω .
- Combine into amplitude vectors \mathbf{B}, \mathbf{F} .

Scattering matrices

$\sim \exp(i\omega t)$ (FD)

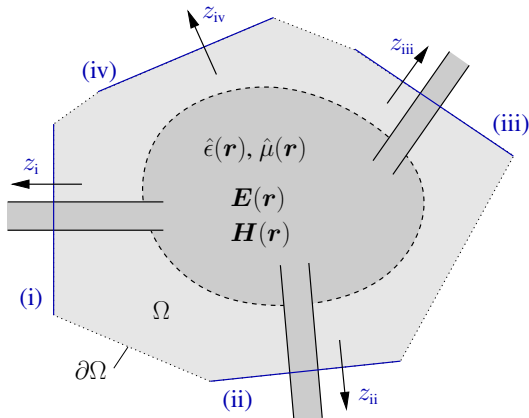


Linear circuit \longleftrightarrow linear dependence of \mathbf{F} on \mathbf{B} ,

Scattering matrix \mathbf{S} of the circuit: $\mathbf{F} = \mathbf{S}\mathbf{B}$, $\mathbf{S} = (S_{\nu\mu})$.

Scattering matrices

$\sim \exp(i\omega t)$ (FD)



- $S_{\nu\nu}$: $\sim (\nu, \mathbf{b}) \rightarrow (\nu, \mathbf{f})$, **reflection coefficient** for mode ν .
- $S_{\nu\mu}$: $\sim (\mu, \mathbf{b}) \rightarrow (\nu, \mathbf{f})$, **transmission coefficient** for modes μ, ν .

Scattering matrices

- Merge all mode indices $\{m\}$ and port IDs $\{p\}$ into one set of mode identifiers $\{\nu\}$, $\mathcal{N} = \cup_p \mathcal{N}_p$. $\sim \exp(i\omega t)$ (FD)
 - Assert that $\psi_{p,\cdot}(\mathbf{r}) = 0$ for all $\mathbf{r} \in \partial\Omega$, $\mathbf{r} \notin \text{port } p$.
 - Field on $\partial\Omega$:
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \sum_{\nu \in \mathcal{N}} \{F_\nu \psi_\nu^f + B_\nu \psi_\nu^b\}.$$
 (Position arguments omitted.)
 - B_ν : \sim incident modes, traveling towards the interior of Ω .
 F_ν : \sim outgoing modes, traveling towards the exterior of Ω .
Combine into amplitude vectors \mathbf{B}, \mathbf{F} .
-

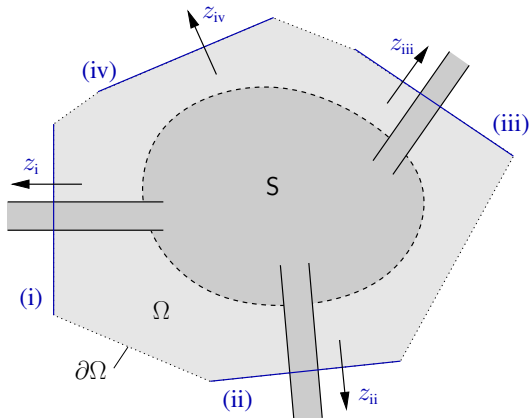
Linear circuit \longleftrightarrow linear dependence of \mathbf{F} on \mathbf{B} ,

Scattering matrix \mathbf{S} of the circuit: $\mathbf{F} = \mathbf{S}\mathbf{B}$, $\mathbf{S} = (S_{\nu\mu})$.

- $S_{\nu\nu}$: $\sim (\nu, b) \rightarrow (\nu, f)$, reflection coefficient for mode ν .
- $S_{\nu\mu}$: $\sim (\mu, b) \rightarrow (\nu, f)$, transmission coefficient for modes μ, ν .

PICs, OICs, scattering matrices, scenarios

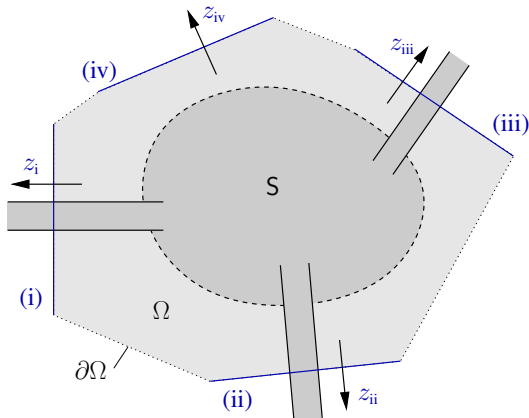
$\sim \exp(i\omega t)$ (FD)



- Scenario: Full matrix S , including guided and radiation modes, large $\dim S \leftrightarrow$ theoretical results.

PICs, OICs, scattering matrices, scenarios

$\sim \exp(i\omega t)$ (FD)

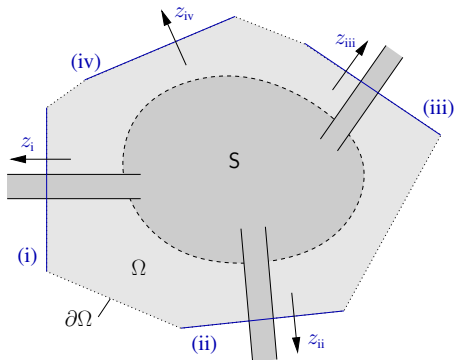


- Scenario: Restrict to a specific set of (guided) modes, or:
Only a small set of guided modes are relevant:
small $\dim \mathbf{S} = N \times N \leftrightarrow$ an N -port circuit, a $2N$ -pole.

(N : the total number of relevant modes, not the number of ports.)

Scattering matrices, port plane positions

$\sim \exp(i\omega t)$ (FD)



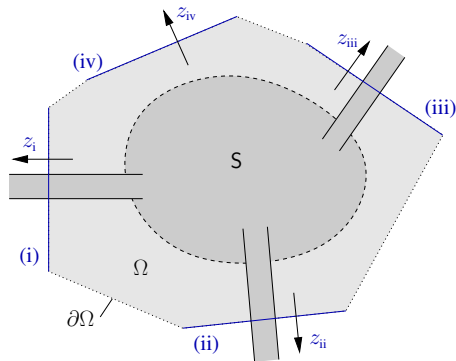
- Shift port plane of mode ν by Δz_ν : $F_\nu \rightarrow F'_\nu = F_\nu e^{-i\beta_\nu \Delta z_\nu}$,
 Shift port plane of mode μ by Δz_μ : $B_\mu \rightarrow B'_\mu = B_\mu e^{i\beta_\mu \Delta z_\mu}$,
 $\hookrightarrow F'_\nu = S'_{\nu\mu} B'_\mu$, $S'_{\nu\mu} = S_{\nu\mu} e^{-i(\beta_\nu \Delta z_\nu + \beta_\mu \Delta z_\mu)}$.

(Moving port planes \leftrightarrow Phase change in reflection/transmission coefficients.)

(Moving port planes \leftrightarrow No effect on reflectances/transmittances.)

Scattering matrices, port mode orthogonality

$\sim \exp(i\omega t)$ (FD)



- Orthogonality relations on port plane p :

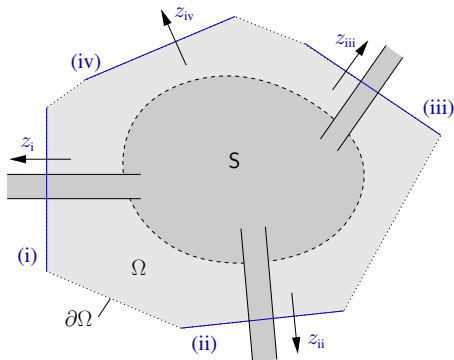
$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) = \frac{1}{4} \iint_p (E_{ax}^* H_{by} - E_{ay}^* H_{bx} + H_{ay}^* E_{bx} - H_{ax}^* E_{by}) \, dx_p \, dy_p$$

$$(\psi_{p,l}^d; \psi_{p,m}^r) = \pm \delta_{dr} \delta_{lm} P_{p,m}.$$

(Things restricted to propagating modes.)

Scattering matrices, port mode orthogonality

$\sim \exp(i\omega t)$ (FD)



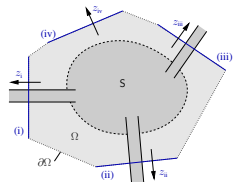
- Extend to the full boundary $\partial\Omega$:

$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) := \frac{1}{4} \int_{\partial\Omega} (\mathbf{E}_a^* \times \mathbf{H}_b + \mathbf{E}_b \times \mathbf{H}_a^*) \cdot d\mathbf{a}$$

$$\hookrightarrow (\psi_{p,l}^d; \psi_{q,m}^r) = \pm \delta_{dr} \delta_{pq} \delta_{lm} P_{p,m} \quad \text{or} \quad (\psi_\nu^d; \psi_\mu^r) = \pm \delta_{dr} \delta_{\nu\mu} P_\nu.$$

(Modes belonging to different ports are mutually orthogonal.)

Scattering matrices, power balance

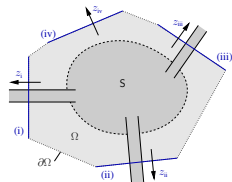


$\sim \exp(i\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_p \sum_{m \in \mathcal{N}_p} (|F_{p,m}|^2 - |B_{p,m}|^2) P_{p,m}$$

Scattering matrices, power balance

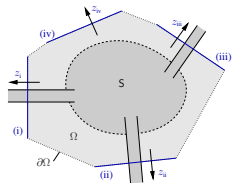


$\sim \exp(i\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$\begin{aligned}
 P &= \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_p \sum_{m \in \mathcal{N}_p} (|F_{p,m}|^2 - |B_{p,m}|^2) P_{p,m} \\
 &= \sum_{\nu \in \mathcal{N}} (|F_\nu|^2 - |B_\nu|^2) P_\nu,
 \end{aligned}$$

Scattering matrices, power balance



$\sim \exp(i\omega t)$ (FD)

- Net power outflow across the border of the circuit:

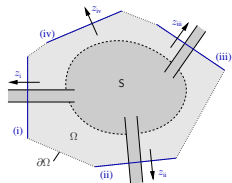
$$\begin{aligned}
 P &= \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_p \sum_{m \in \mathcal{N}_p} (|F_{p,m}|^2 - |B_{p,m}|^2) P_{p,m} \\
 &= \sum_{\nu \in \mathcal{N}} (|F_\nu|^2 - |B_\nu|^2) P_\nu,
 \end{aligned}$$

$|B_\mu|^2 P_\mu$: incident power carried by mode μ ,

$|F_\nu|^2 P_\nu$: outgoing power carried by mode ν ,

$$F_\nu = S_{\nu\mu} B_\mu.$$

Scattering matrices, power balance



$\sim \exp(i\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$\begin{aligned}
 P &= \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_p \sum_{m \in \mathcal{N}_p} (|F_{p,m}|^2 - |B_{p,m}|^2) P_{p,m} \\
 &= \sum_{\nu \in \mathcal{N}} (|F_\nu|^2 - |B_\nu|^2) P_\nu,
 \end{aligned}$$

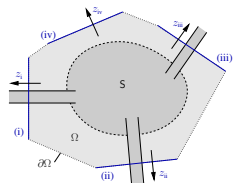
$|B_\mu|^2 P_\mu$: incident power carried by mode μ ,

$|F_\nu|^2 P_\nu$: outgoing power carried by mode ν , $F_\nu = S_{\nu\mu} B_\mu$.

$$|S_{\nu\mu}|^2 \frac{P_\nu}{P_\mu} = \frac{|F_\nu|^2 P_\nu}{|B_\mu|^2 P_\mu}, \quad \begin{array}{ll} \mu \neq \nu: & \text{power transmittance } \mu \rightarrow \nu, \\ \mu = \nu: & \text{power reflectance for mode } \nu. \end{array}$$

(Uniform normalized modes, $P_\nu = P_\mu$: transmittances are directly given by elements of the scattering matrix).

Scattering matrices, power balance



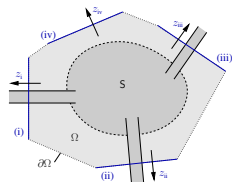
$\sim \exp(i\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = P_0 \left(\mathbf{B}^* \cdot (\mathbf{S}^\dagger \mathbf{S} - \mathbf{1}) \mathbf{B} \right),$$

uniform normalization, $P_\nu = P_0$ for all ν .

Scattering matrices, power balance



$\sim \exp(i\omega t)$ (FD)

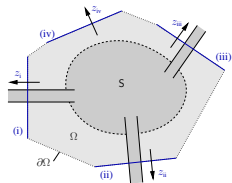
- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (E, H; E, H) = P_0 \left(\mathbf{B}^* \cdot (\mathbf{S}^\dagger \mathbf{S} - \mathbf{1}) \mathbf{B} \right),$$

uniform normalization, $P_\nu = P_0$ for all ν .

- Lossless circuit $\Leftrightarrow \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = 0 \Leftrightarrow \mathbf{S}^\dagger \mathbf{S} = \mathbf{1}$,
the scattering matrix of a lossless circuit is unitary.

Scattering matrices, power balance



$\sim \exp(i\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (E, H; E, H) = P_0 \left(\mathbf{B}^* \cdot (\mathbf{S}^\dagger \mathbf{S} - \mathbf{1}) \mathbf{B} \right),$$

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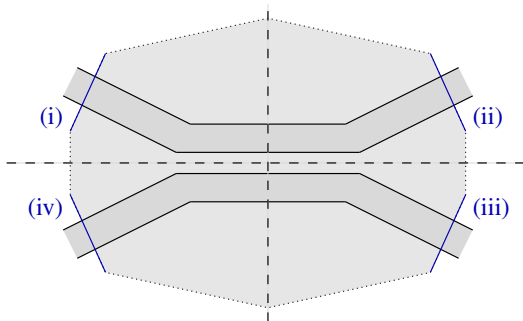
- Lossless circuit $\iff \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = 0 \iff \mathbf{S}^\dagger \mathbf{S} = \mathbf{1}$,
the scattering matrix of a lossless circuit is unitary.

- Lossy circuit $\iff \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} \leq 0 \rightsquigarrow \mathbf{B}^* \cdot \mathbf{S}^\dagger \mathbf{S} \mathbf{B} \leq \mathbf{B}^* \mathbf{B}$,

$$\sum_{\nu} |S_{\nu\mu}|^2 \leq 1 \text{ for all } \mu. \quad \text{(The sum of transmittances mode } \mu \text{ to all other modes } \nu \text{ is less than one.)}$$

(Interior lossy media, or radiative losses: outgoing propagating modes not taken into account.)

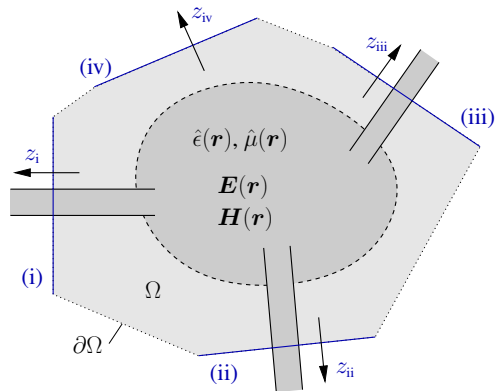
Scattering matrices, symmetry



Circuit with specific spatial symmetry
& symmetrical setting of the port planes

↪ respective symmetry in related coefficients of \mathbf{S} ,
symmetric power transmission properties.

Scattering matrices, reciprocity

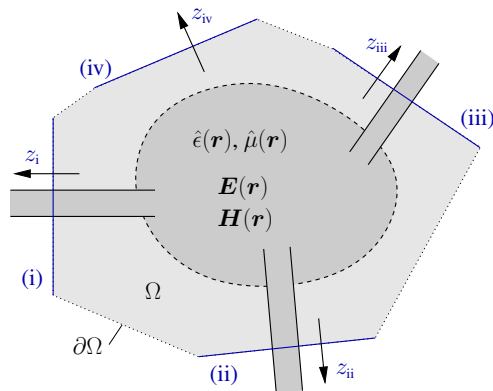


$\sim \exp(i\omega t)$ (FD)

Circuit properties
for reversed
wave propagation ?

$$S_{\nu\mu} \longleftrightarrow S_{\mu\nu} ?$$

Scattering matrices, reciprocity



$\sim \exp(i\omega t)$ (FD)

Circuit properties
for reversed
wave propagation ?

$$S_{\nu\mu} \longleftrightarrow S_{\mu\nu} ?$$

- $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ solve $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$, $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$.

$$\hookrightarrow \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) = 0, \quad \text{if } \hat{\epsilon} \text{ and } \hat{\mu} \text{ are symmetric.}$$

(i.e. if $\hat{\epsilon}^T = \hat{\epsilon}$, $\hat{\mu}^T = \hat{\mu}$.)

(Note: order of factors, no complex conjugates.)

Scattering matrices, reciprocity

- E_1, H_1 and E_2, H_2 solve $\nabla \times E = -i\omega\mu_0\hat{\mu}H$, $\nabla \times H = i\omega\epsilon_0\hat{\epsilon}E$
 $\hookrightarrow \nabla \cdot (E_1 \times H_2 + H_1 \times E_2) = 0$, if $\hat{\epsilon}$ and $\hat{\mu}$ are symmetric,

Scattering matrices, reciprocity

- $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ solve $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$, $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$

↪ $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) = 0$, if $\hat{\epsilon}$ and $\hat{\mu}$ are symmetric,

↪ $0 = \int_{\Omega} \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) d^3r = \int_{\partial\Omega} (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) \cdot d\mathbf{a}.$

Scattering matrices, reciprocity

- $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ solve $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$, $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$

↪ $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) = 0$, if $\hat{\epsilon}$ and $\hat{\mu}$ are symmetric,

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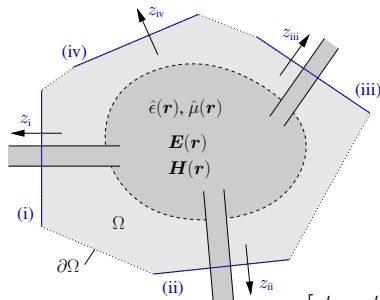
- Fields on $\partial\Omega$: $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_j = \sum_{\nu \in \mathcal{N}} \{F_{j,\nu}\psi_{\nu}^f + B_{j,\nu}\psi_{\nu}^b\}, \quad j = 1, 2,$

$$[\psi_a; \psi_b] := \int_{\partial\Omega} (\mathbf{E}_a \times \mathbf{H}_b + \mathbf{H}_a \times \mathbf{E}_b) \cdot d\mathbf{a},$$

↪ $0 = \sum_{\nu} \sum_{\mu} \left(F_{1,\nu}F_{2,\mu}[\psi_{\nu}^f; \psi_{\mu}^f] + F_{1,\nu}B_{2,\mu}[\psi_{\nu}^f; \psi_{\mu}^b] \right. \\ \left. + B_{1,\nu}F_{2,\mu}[\psi_{\nu}^b; \psi_{\mu}^f] + B_{1,\nu}B_{2,\mu}[\psi_{\nu}^b; \psi_{\mu}^b] \right).$

Scattering matrices, reciprocity

$\sim \exp(i\omega t)$ (FD)



$$[\psi_a; \psi_b] := \int_{\partial\Omega} (\mathbf{E}_a \times \mathbf{H}_b + \mathbf{H}_a \times \mathbf{E}_b) \cdot d\mathbf{a}.$$

- $[\psi_\nu; \psi_\mu] = 0$, if ν and μ relate to different ports.
- If ν and μ relate to the same port plane p :

$$[\psi_\nu^r; \psi_\mu^d] = \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d) dx_p dy_p.$$

Scattering matrices, reciprocity

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$$[\psi_\nu^r; \psi_\mu^d] = \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d) dx_p dy_p.$$

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p :

$$[\psi_\nu^r; \psi_\mu^d] = \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d) dx_p dy_p.$$

- Compare with the modal orthogonality relations on port plane p ,

$$(\psi_\nu^r; \psi_\mu^d) = \frac{1}{4} \iint_p ((E_{\nu x}^r)^* H_{\mu y}^d - (E_{\nu y}^r)^* H_{\mu x}^d + (H_{\nu y}^r)^* E_{\mu x}^d - (H_{\nu x}^r)^* E_{\mu y}^d) dx_p dy_p.$$

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p :

$$[\psi_\nu^r; \psi_\mu^d] = \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d) dx_p dy_p.$$

- Compare with the modal orthogonality relations on port plane p , for propagating modes with real transverse components:

$$(\psi_\nu^r; \psi_\mu^d) = \frac{1}{4} \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d + H_{\nu y}^r E_{\mu x}^d - H_{\nu x}^r E_{\mu y}^d) dx_p dy_p,$$

$$(\psi_\nu^f; \psi_\mu^f) = \delta_{\nu\mu} P_\nu, \quad (\psi_\nu^b; \psi_\mu^b) = -\delta_{\nu\mu} P_\nu, \quad (\psi_\nu^f; \psi_\mu^b) = (\psi_\nu^b; \psi_\mu^f) = 0.$$

Scattering matrices, reciprocity


- If ν and μ relate to the same port plane p :

$$[\psi_\nu^r; \psi_\mu^d] = \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d) dx_p dy_p.$$

- Compare with the modal orthogonality relations on port plane p , for propagating modes with real transverse components:

$$(\psi_\nu^r; \psi_\mu^d) = \frac{1}{4} \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d + H_{\nu y}^r E_{\mu x}^d - H_{\nu x}^r E_{\mu y}^d) dx_p dy_p,$$

$$(\psi_\nu^f; \psi_\mu^f) = \delta_{\nu\mu} P_\nu, \quad (\psi_\nu^b; \psi_\mu^b) = -\delta_{\nu\mu} P_\nu, \quad (\psi_\nu^f; \psi_\mu^b) = (\psi_\nu^b; \psi_\mu^f) = 0.$$

- 
$$\begin{aligned} \psi^f &= (E_x, E_y, iE_z, H_x, H_y, iH_z)^\top \\ \psi^b &= (E_x, E_y, -iE_z, -H_x, -H_y, iH_z)^\top. \end{aligned}$$
 (Real components).

Scattering matrices, reciprocity

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- $\begin{matrix} \psi^f = (E_x, E_y, iE_z, H_x, H_y, iH_z)^\top \\ \longleftrightarrow \psi^b = (E_x, E_y, -iE_z, -H_x, -H_y, iH_z)^\top. \end{matrix}$ (Real components).

$$[\psi_\nu^f; \psi_\mu^f] = [\psi_\nu^b; \psi_\mu^b] = 0, \quad [\psi_\nu^f; \psi_\mu^b] = -\delta_{\nu\mu} 4P_\nu, \quad [\psi_\nu^b; \psi_\mu^f] = \delta_{\nu\mu} 4P_\nu.$$

Scattering matrices, reciprocity

$$\hookrightarrow 0 = \sum_{\nu} 4P_{\nu} (B_{1,\nu} F_{2,\nu} - F_{1,\nu} B_{2,\nu}),$$

uniform normalization $P_{\nu} = P_0$,

$$\hookrightarrow 0 = \sum_{\nu} (B_{1,\nu} F_{2,\nu} - F_{1,\nu} B_{2,\nu}),$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot \mathbf{F}_2 - \mathbf{F}_1 \cdot \mathbf{B}_2,$$

$$\mathbf{F}_j = \mathbf{S} \mathbf{B}_j,$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot \mathbf{S} \mathbf{B}_2 - (\mathbf{S} \mathbf{B}_1) \cdot \mathbf{B}_2,$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot \mathbf{S} \mathbf{B}_2 - \mathbf{B}_1 \cdot \mathbf{S}^{\top} \mathbf{B}_2,$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot (\mathbf{S} - \mathbf{S}^{\top}) \mathbf{B}_2 \text{ for all } \mathbf{B}_1, \mathbf{B}_2.$$

$$\mathbf{S} = \mathbf{S}^{\top}, \quad S_{\nu\mu} = S_{\mu\nu} \text{ for all } \nu, \mu.$$

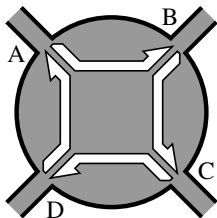
The scattering matrix of a *reciprocal circuit* is *symmetric*.

Reciprocal circuit: made of reciprocal media, with $\hat{\epsilon} = \hat{\epsilon}^{\top}$, $\hat{\mu} = \hat{\mu}^{\top}$.

Nonreciprocal devices



Isolator:
unidirectional transmission,
 $S_{BA} = 1, S_{AB} = 0.$



Circulator:
transmission cycle,
 $S_{BA} = 1, S_{CB} = 1, S_{DC} = 1, S_{AD} = 1,$
 $S_{..} = 0$ otherwise.

Required: nonreciprocal media with $\hat{\epsilon} \neq \hat{\epsilon}^T$,
↔ magnetooptic media, Faraday effect.

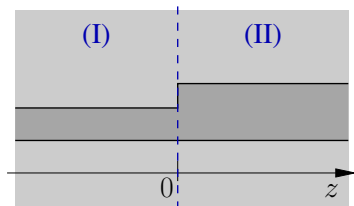
Nonreciprocal devices

What about, for example,

- a long, “adiabatic” Y-junction ?
- a junction between a single mode core and a wider multimode waveguide ?

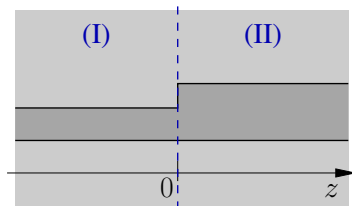


Waveguide discontinuities



Half-infinite waveguides (I), (II),
discontinuity at $z = 0$.

Waveguide discontinuities



Half-infinite waveguides (I), (II), discontinuity at $z = 0$.

- Expand into local normal modes $\{\psi_{s,m}^d, \beta_{s,m}\}$, $m \in \mathcal{N}_s$, $s = \text{I, II}$:

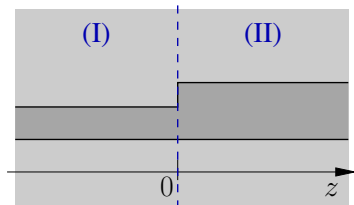
Transverse boundary conditions \longleftrightarrow discrete sets.

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_s(x, y, z) = \sum_{m \in \mathcal{N}_s} \left\{ f_{s,m} \psi_{s,m}^f(x, y) e^{-i\beta_{s,m}z} + b_{s,m} \psi_{s,m}^b(x, y) e^{+i\beta_{s,m}z} \right\},$$

$z < 0$: $s = \text{I}$, $f_{\text{I},m}$ given influx, $b_{\text{I},m}$ unknown,
 $z > 0$: $s = \text{II}$, $f_{\text{II},m}$ unknown, $b_{\text{II},m}$ given influx.

$\hookrightarrow (\mathbf{E}, \mathbf{H})_{\text{I,II}}$ are solutions for $z < 0$ and $z > 0$.

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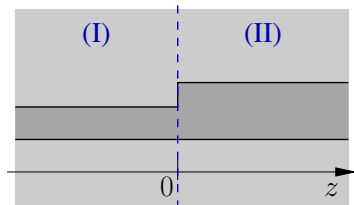
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- Continuity of the tangential components of \mathbf{E}, \mathbf{H} at the interface
 \longleftrightarrow formally equate expressions for $(\mathbf{E}, \mathbf{H})_{\text{I,II}}$ at $z = 0$.

(Only equality of E_x, E_y, H_x, H_y will be relevant.)

Waveguide discontinuities



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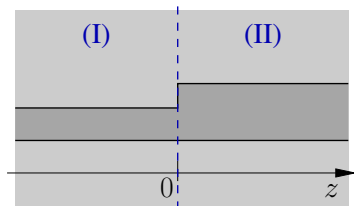
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(Only equality of E_x, E_y, H_x, H_y will be relevant.)

- Project on $\psi_{s,l}^d$ to extract coefficients ...

Waveguide discontinuities, scattering matrix



(Global coordinate $z \neq$ former local coordinate on port I)
(One variant of a projection procedure.)

- $(\psi_{I,l}^b; \cdot = \cdot), \quad l \in \mathcal{N}_I:$

$$\sum_{m \in \mathcal{N}_I} [f_{I,m}(\psi_{I,l}^b; \psi_{I,m}^f) + b_{I,m}(\psi_{I,l}^b; \psi_{I,m}^b)] = \sum_{m \in \mathcal{N}_{II}} [f_{II,m}(\psi_{I,l}^b; \psi_{II,m}^f) + b_{II,m}(\psi_{I,l}^b; \psi_{II,m}^b)],$$

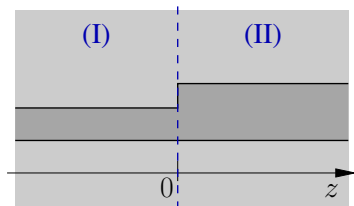
- $(\psi_{II,l}^f; \cdot = \cdot), \quad l \in \mathcal{N}_{II}:$

$$\sum_{m \in \mathcal{N}_I} [f_{I,m}(\psi_{II,l}^f; \psi_{I,m}^f) + b_{I,m}(\psi_{II,l}^f; \psi_{I,m}^b)] = \sum_{m \in \mathcal{N}_{II}} [f_{II,m}(\psi_{II,l}^f; \psi_{II,m}^f) + b_{II,m}(\psi_{II,l}^f; \psi_{II,m}^b)],$$

↪ ... ↪

$$\begin{pmatrix} \mathbf{b}_I \\ \mathbf{f}_{II} \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathbf{f}_I \\ \mathbf{b}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{I,I} & \mathbf{S}_{I,II} \\ \mathbf{S}_{II,I} & \mathbf{S}_{II,II} \end{pmatrix} \begin{pmatrix} \mathbf{f}_I \\ \mathbf{b}_{II} \end{pmatrix}.$$

Waveguide discontinuities, overlap model



Most simplified variant:
Unidirectional **overlap model**.

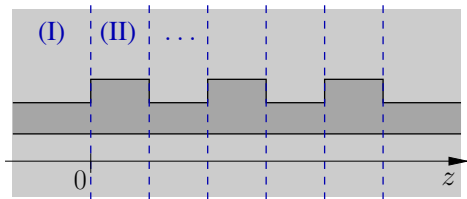
- (I): Incoming guided mode ψ_I , reflections & radiation neglected.
- (II): Outgoing guided modes $\psi_{II,m}$, radiation neglected.

- $f_I \psi_I \approx \sum_m f_{II,m} \psi_{II,m} \quad \text{at } z = 0.$

↪
$$f_{II,m} = \frac{(\psi_{II,m}; \psi_I)}{(\psi_{II,m}; \psi_{II,m})} f_I, \quad \text{or} \quad f_{II,m} = \frac{1}{P_{II,m}} (\psi_{II,m}; \psi_I) f_I.$$

(Transmission is given directly by the “overlaps” ↪ Relevance of the mode products $(\cdot; \cdot)$.)
(Cf. explicit expressions for overlaps of 2-D modes, involving only principal mode profile components.)

A sequence of waveguide discontinuities



- Divide into segments.
- Establish local normal mode expansions.
- Project on local modes.

↪ Linear system of equations for all local mode amplitudes.

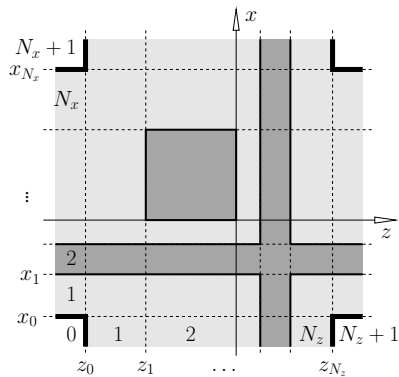
↪ Solve (...) $\rightsquigarrow \begin{pmatrix} E \\ H \end{pmatrix} (x, y, z).$

Bidirectional eigenmode propagation (BEP),
Eigenmode expansion method (EME),

...

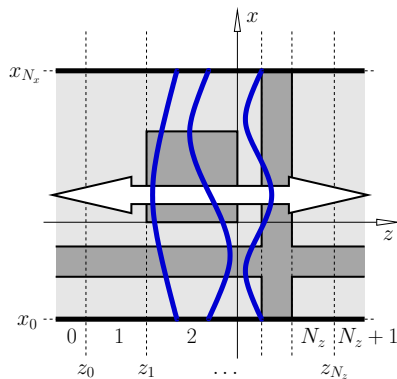
(Radiated outgoing fields: Open boundary conditions required (PMLs) \leftrightarrow Complex eigenmodes.)
(2-D: ok. 3-D: ?)

Rectangular 2-D circuits



- Divide into slices & layers.

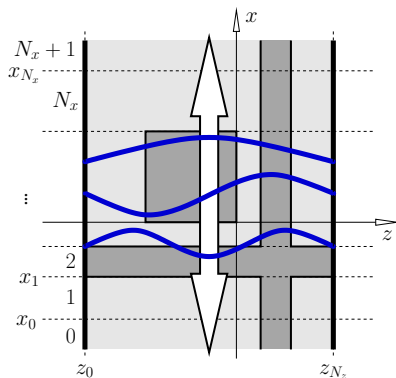
Rectangular 2-D circuits



- Divide into slices & layers.
- Establish local modes:
Propagation along $\pm z$,

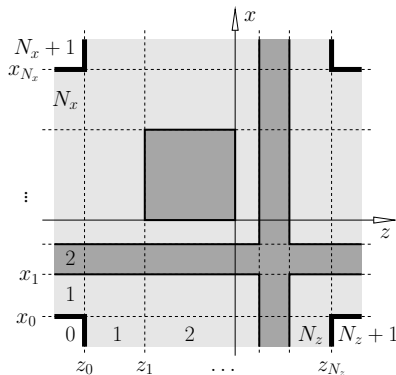
boundary conditions $\phi = 0$.

Rectangular 2-D circuits



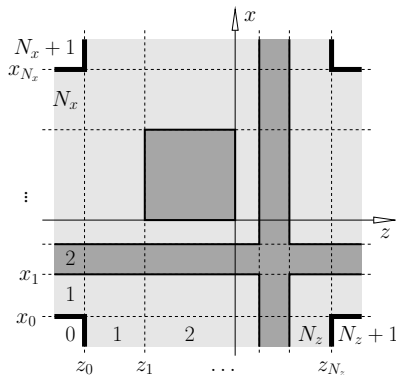
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Rectangular 2-D circuits

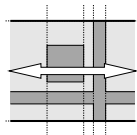


- Divide into slices & layers.
- Establish local modes:
 - Propagation along $\pm z$,
 - & Propagation along $\pm x$,
 - boundary conditions $\phi = 0$.
- Project at horizontal & vertical interfaces.

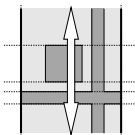
Rectangular 2-D circuits



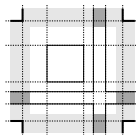
- Divide into slices & layers.
- Establish local modes:
 Propagation along $\pm z$,
 & Propagation along $\pm x$,
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horizontal BEP,

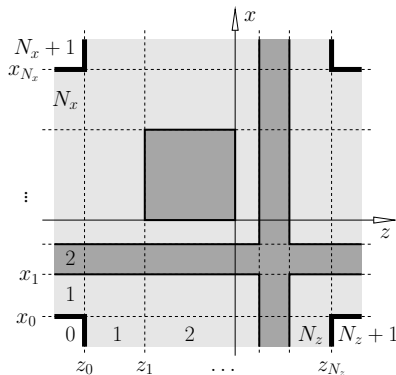


vertical BEP,



continuity at x_0, x_N, z_0, z_N .

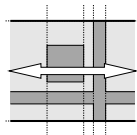
Rectangular 2-D circuits



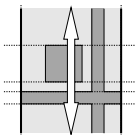
Quadridirectional Eigenmode Propagation (QUEP)



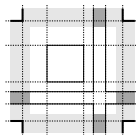
- Divide into slices & layers.
- Establish local modes:
 Propagation along $\pm z$,
 & Propagation along $\pm x$,
 boundary conditions $\phi = 0$.
- Project at horizontal
& vertical interfaces.



horizontal BEP,



vertical BEP,

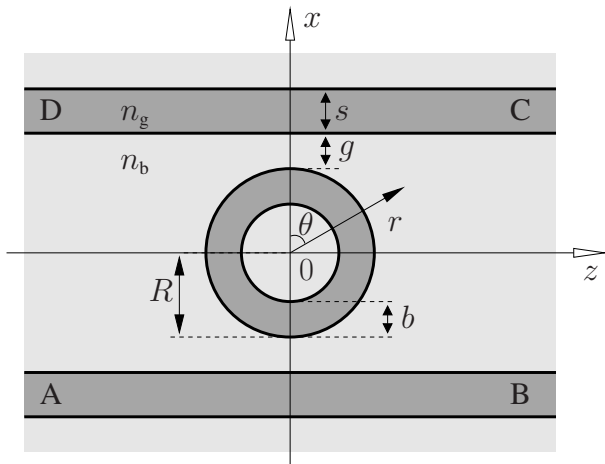


continuity at $x_0, x_{N_x}, z_0, z_{N_z}$.

Optical waveguide theory

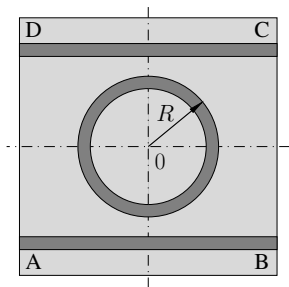
- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

Circular traveling wave resonators

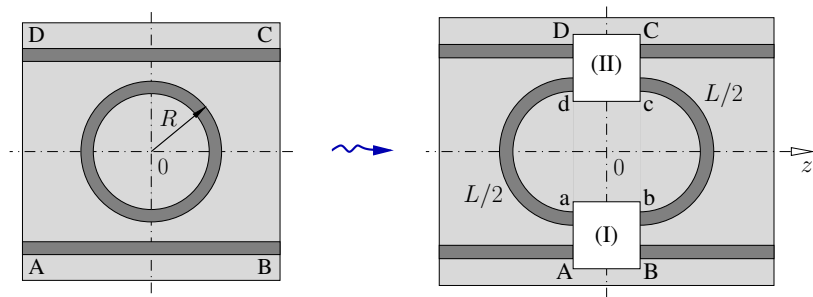


Integrated optical **micro-ring** or **micro-disk** resonators.

Ringresonator: Abstract model

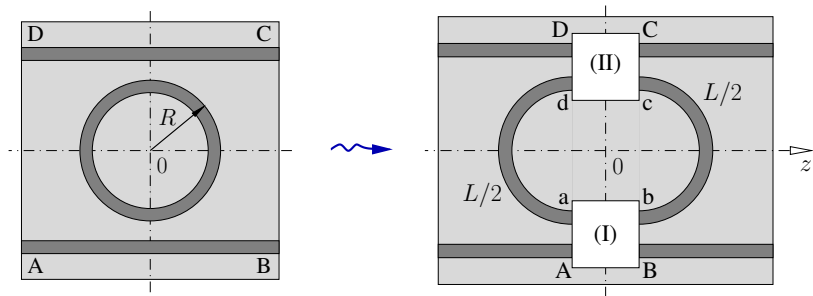


Ringresonator: Abstract model



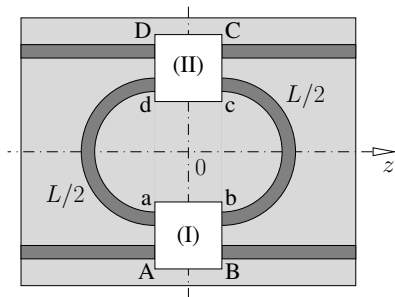
- Ringresonator \approx 2 couplers + 2 cavity segments

Ringresonator: Abstract model



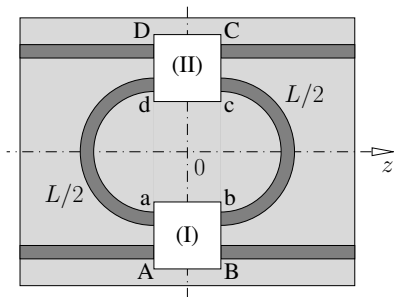
- Ringresonator \approx 2 couplers + 2 cavity segments
- CW description: $\mathbf{E}, \mathbf{H} \sim e^{i\omega t}$, $\omega = kc$, $k = 2\pi/\lambda$.

Couplers: Scattering matrices



- Uniform polarization, single mode waveguides.
- Linear, nonmagnetic (attenuating) elements.
- Backreflections are negligible.
- Interaction restricted to the couplers \leftrightarrow “port” definition.

Couplers: Scattering matrices



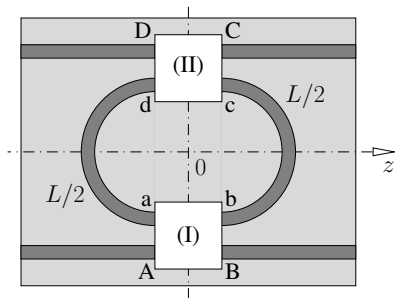
- Uniform polarization, single mode waveguides.
- Linear, nonmagnetic (attenuating) elements.
- Backreflections are negligible.
- Interaction restricted to the couplers \leftrightarrow “port” definition.

↪ Symmetric coupler scattering matrices :

$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \chi & \tau \\ \rho & \chi & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

$A_{\pm}, B_{\pm}, a_{\pm}, b_{\pm}$: Amplitudes of waves traveling in $\pm z$ -direction.

Coupler symmetries

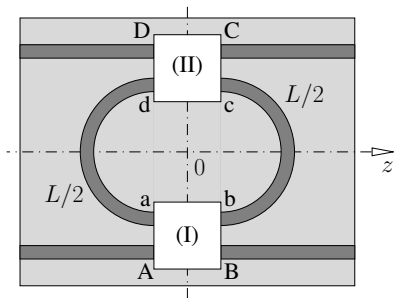


Symmetry $z \rightarrow -z$:

$$A_+ \rightarrow b_+ \stackrel{!}{=} B_- \rightarrow a_-$$

$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \chi & \tau \\ \rho & \chi & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

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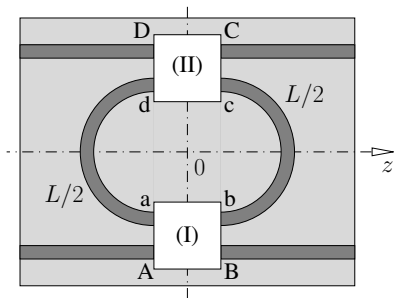


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Coupler symmetries



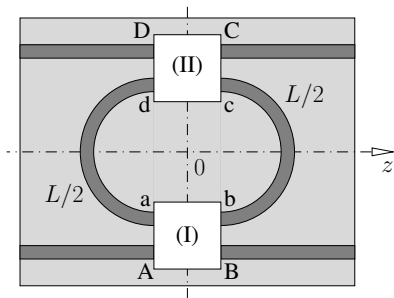
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$$\curvearrowright \begin{pmatrix} A_- \\ a_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} B_- \\ b_- \end{pmatrix}, \quad \begin{pmatrix} B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \end{pmatrix}.$$

Coupler symmetries



Symmetry $z \rightarrow -z$:

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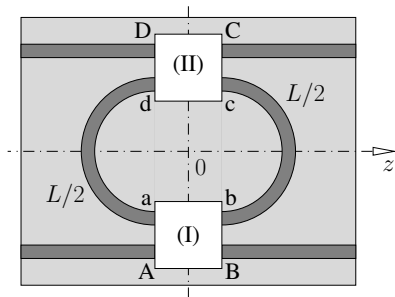
$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \kappa & \tau \\ \rho & \kappa & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} A_- \\ a_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} B_- \\ b_- \end{pmatrix}, \quad \begin{pmatrix} B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \end{pmatrix}.$$

Symmetry $x \rightarrow -x$, (I) = (II):

$$\hookrightarrow \begin{pmatrix} D_- \\ d_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} C_- \\ c_- \end{pmatrix}, \quad \begin{pmatrix} C_+ \\ c_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} D_+ \\ d_+ \end{pmatrix}.$$

Cavity segments



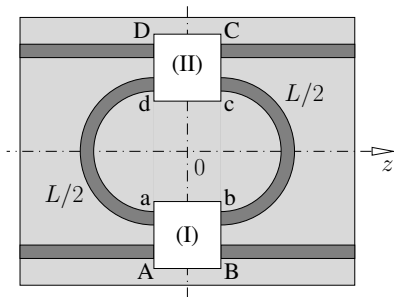
Field evolution $\sim e^{-i\gamma s}$
along the cavity core,
propagation distance s .

$$\gamma = \beta - i\alpha,$$

β : phase propagation constant,
 α : attenuation constant.

(\leftrightarrow bend modes, to come.)

Cavity segments



Field evolution $\sim e^{-i\gamma s}$
along the cavity core,
propagation distance s .

$$\gamma = \beta - i\alpha,$$

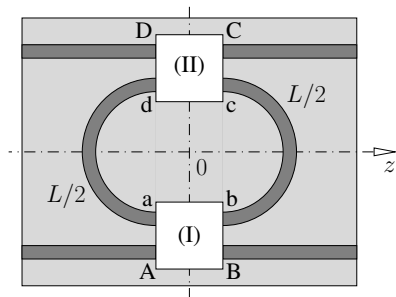
β : phase propagation constant,
 α : attenuation constant.

(\leftrightarrow bend modes, to come.)

Relations of amplitudes at the ends of the cavity segments :

$$\begin{aligned} c_- &= b_+ e^{-i\beta L/2} e^{-\alpha L/2}, & a_+ &= d_- e^{-i\beta L/2} e^{-\alpha L/2}, \\ b_- &= c_+ e^{-i\beta L/2} e^{-\alpha L/2}, & d_+ &= a_- e^{-i\beta L/2} e^{-\alpha L/2}. \end{aligned}$$

Output amplitudes

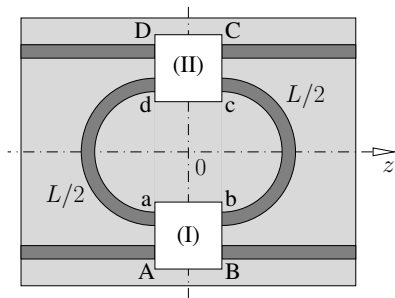


Coupler scattering matrices
+ Cavity field evolution
+ External input amplitudes

$$A_+ = \sqrt{P_{\text{in}}},$$

$$B_- = C_- = D_+ = 0$$

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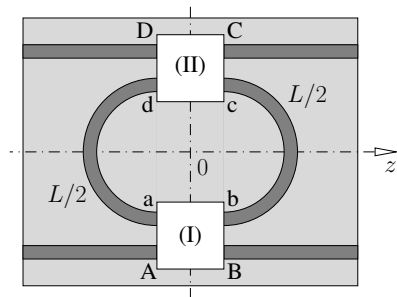
$$B_- = C_- = D_+ = 0$$

External output amplitudes :

$$A_- = 0, \quad C_+ = 0, \quad D_- = \frac{\kappa^2 p}{1 - \tau^2 p^2} A_+, \quad B_+ = \left(\rho + \frac{\kappa^2 \tau p^2}{1 - \tau^2 p^2} \right) A_+,$$

$$p = e^{-i\beta L/2} e^{-\alpha L/2}.$$

Power transfer



Power drop: $P_D = |D_-|^2$,

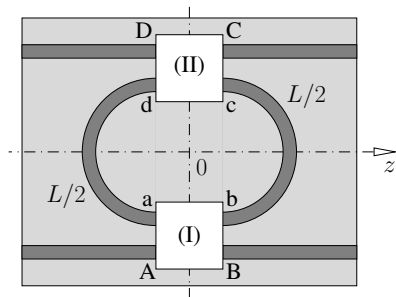
Transmission: $P_T = |B_+|^2$.

$$P_D = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$P_T = P_{\text{in}} \frac{|\rho|^2 (1 + |\tau|^2 d^2 e^{-2\alpha L} - 2|\tau|d e^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$\tau =: |\tau| e^{i\varphi}, \quad d e^{i\psi} := \tau - \kappa^2/\rho,$$

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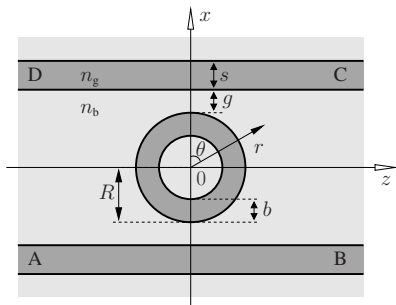
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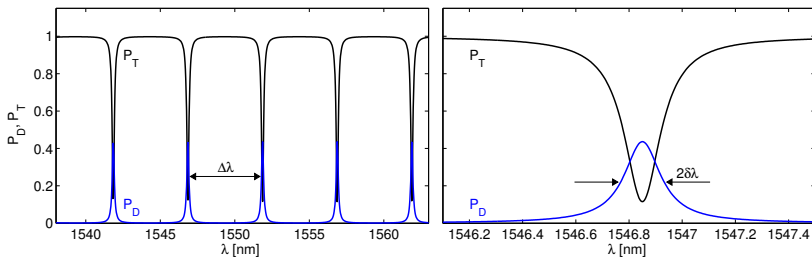
$$\tau =: |\tau| e^{i\varphi}, \quad d e^{i\psi} := \tau - \kappa^2/\rho, \quad L \neq 2\pi R.$$

Spectral response



$R = 50 \mu\text{m}$, $b = s = 1.0 \mu\text{m}$, $g = 0.9 \mu\text{m}$,
 $n_b = 1.45$, $n_g = 1.60$; 2-D, TE.

$\Delta\lambda = 5.0 \text{ nm}$, $2\delta\lambda = 0.17 \text{ nm}$,
 $F = 30$, $Q = 9400$, $P_{D,\text{res}} = 0.44$.



Resonances

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 $\beta L - 2\varphi = \beta L_{\text{cav}} - \phi, \quad \phi = 2\beta l + 2\varphi, \quad L_{\text{cav}} = 2\pi R, \quad \partial_\lambda \phi \approx 0.$

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$$\beta = \frac{2m\pi + \phi}{L_{\text{cav}}} =: \beta_m \quad \text{integer } m; \quad P_D|_{\beta=\beta_m} = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - |\tau|^2 e^{-\alpha L})^2}.$$

Free spectral range

- Resonance next to β_m :

$$\beta_{m-1} = \frac{2(m-1)\pi + \phi}{L_{\text{cav}}} = \beta_m - \frac{2\pi}{L_{\text{cav}}}$$

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q_j : waveguide parameters with dimension length,

$$\beta(a\lambda, aq_j) = \beta(\lambda, q_j)/a, \quad \partial_a |_{a=1}$$

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$$\text{FSR:} \quad \Delta \lambda = -\frac{2\pi}{L_{\text{cav}}} \left(\left. \frac{\partial \beta}{\partial \lambda} \right|_m \right)^{-1} \approx \frac{\lambda^2}{n_{\text{eff}} L_{\text{cav}}} \Big|_m, \quad n_{\text{eff}} = \beta/k.$$

(Free spectral range, the spectral distance (here: wavelength) between the drop peaks / the transmission dips.)

Spectral width of the resonances

- $$P_D = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L_{\text{cav}} - \phi)},$$
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- Expansion of cos-terms

↪
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FWHM:
$$2\delta\lambda = \frac{\lambda^2}{\pi L_{\text{cav}} n_{\text{eff}}} \bigg|_m \left(\frac{1}{|\tau|} e^{\alpha L/2} - |\tau| e^{-\alpha L/2} \right).$$

(Full width at half maximum of the spectral drop peaks / the transmission dips (wavelength).)

Finesse & Q-factor

Finesse :

$$F = \frac{\Delta\lambda}{2\delta\lambda} = \pi \frac{|\tau| e^{-\alpha L/2}}{1 - |\tau|^2 e^{-\alpha L}} .$$

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$$Q = \frac{\lambda}{2\delta\lambda} = \pi \frac{n_{\text{eff}} L_{\text{cav}}}{\lambda} \frac{|\tau| e^{-\alpha L/2}}{1 - |\tau|^2 e^{-\alpha L}} = \frac{n_{\text{eff}} L_{\text{cav}}}{\lambda} F .$$

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or

$$Q = kR n_{\text{eff}} F \quad \text{for} \quad L_{\text{cav}} = 2\pi R.$$

Performance versus coupling strength & losses

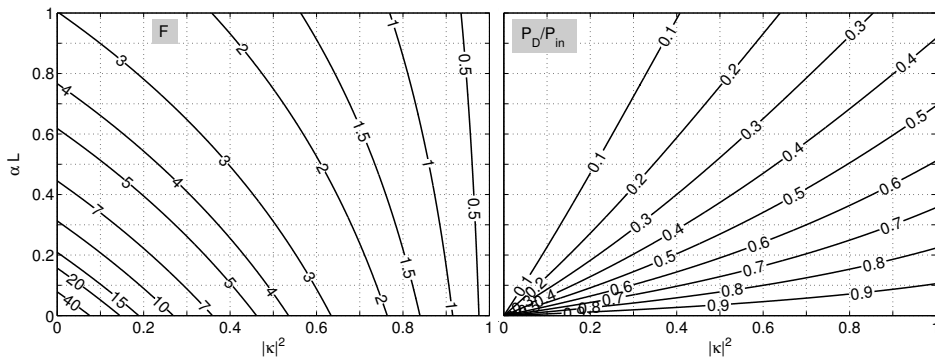
Assumption: Lossless coupler elements, $|\rho|^2 = |\tau|^2 = 1 - |\kappa|^2$.

$$F = \pi \frac{(\sqrt{1 - |\kappa|^2}) e^{-\alpha L/2}}{1 - (1 - |\kappa|^2) e^{-\alpha L}} , \quad P_D|_{\text{res}} = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - (1 - |\kappa|^2) e^{-\alpha L})^2} .$$

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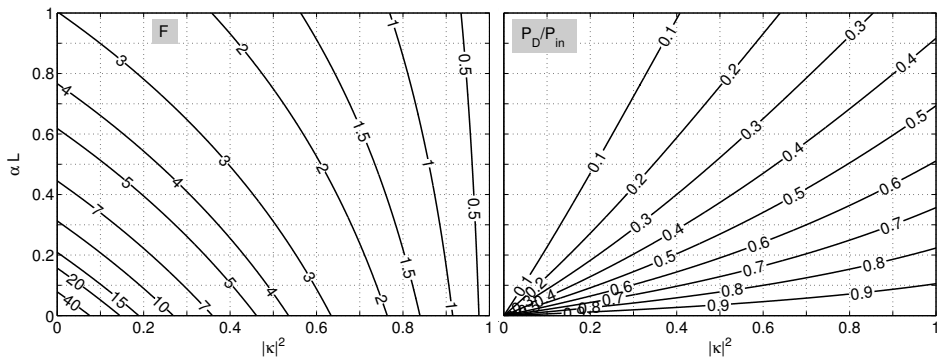
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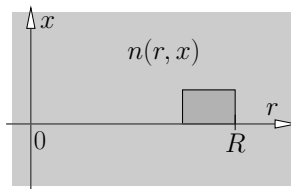
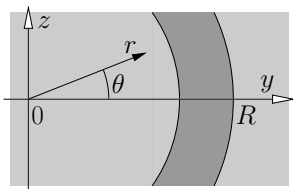
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$\alpha, \kappa = ?$

Modes of bent waveguides

$\sim \exp(i\omega t)$ (FD)



- Constant curvature \longleftrightarrow cylindrical coordinates r, θ, x .
- Bend radius R , $\partial_\theta \epsilon = 0$, $\partial_\theta n = 0$

$\hookrightarrow \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (r, \theta, x) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (r, x) e^{-i\gamma R\theta}, \quad \text{bend modes,}$

$\bar{\mathbf{E}}, \bar{\mathbf{H}}$: bend mode profile, components $\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x$,

$\gamma = \beta - i\alpha \in \mathbb{C}$: propagation constant,

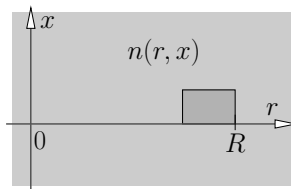
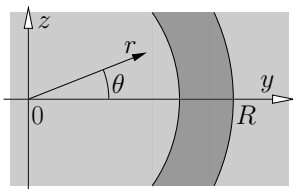
$\beta \in \mathbb{R}$: phase constant,

$\alpha \in \mathbb{R}$: attenuation constant.

(Exponent $i\gamma R\theta$: a convention, “propagation distance” $R\theta$.)

Modes of bent waveguides

$\sim \exp(i\omega t)$ (FD)



- Piecewise constant $n(r, x)$, $\psi \in \{\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x\}$,

$$\hookrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \psi = 0, \quad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),

& boundary conditions:

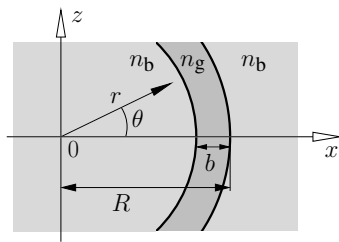
regularity at $r = 0$, outgoing waves at $x = \pm\infty$, $r = \infty$.

(or: normalizability versus x .)

Vectorial 3-D bend mode eigenvalue problem.

(Practical setting: computational domain $r_i < r < r_o$, $x_b < x < x_t$, PML boundary conditions / $\psi = 0$ at $r = r_i$.)

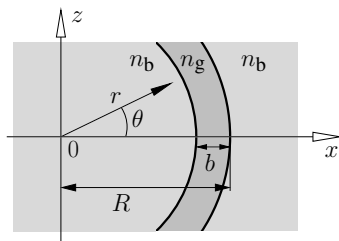
Modes of bent slab waveguides



$\sim \exp(i\omega t)$ (FD)

2-D TE/TM, cylind. coord. r, θ, y ,
 $\partial_y n = \partial_\theta n = 0$

Modes of bent slab waveguides



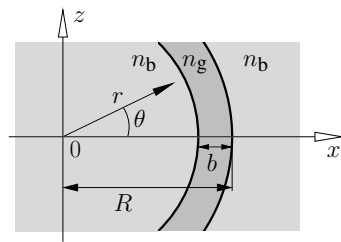
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$$\left(\begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right) (r, \theta) = \left(\begin{matrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{matrix} \right) (r) e^{-i\gamma R\theta},$$

bent slab mode $\{\bar{\mathbf{E}}, \bar{\mathbf{H}}, \gamma = \beta - i\alpha\}$.

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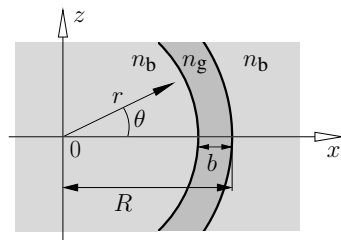
bent slab mode $\{\bar{\mathbf{E}}, \bar{\mathbf{H}}, \gamma = \beta - i\alpha\}$.

- Piecewise constant $n(r)$, $\phi = \bar{E}_y$ (TE), $\phi = \bar{H}_y$ (TM)

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \phi = 0,$$

(Bessel differential equation with (complex) order γR .)

Modes of bent slab waveguides



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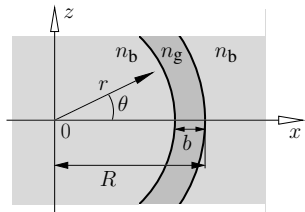
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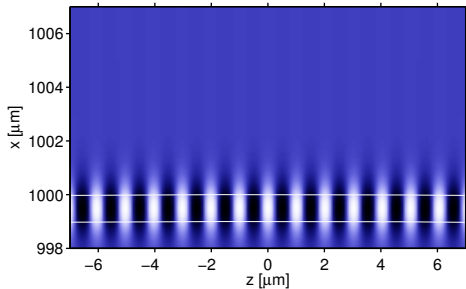
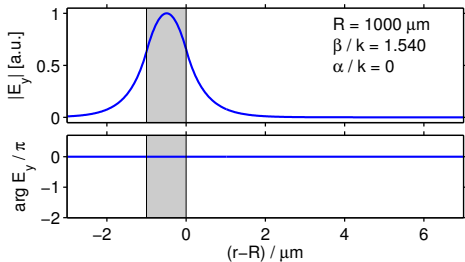
(Bessel differential equation with (complex) order γR .)

- Nonzero solutions,
- bounded at the origin, $\sim J_{\gamma R}(nkr)$ for $r < R - b$,
- outgoing exterior fields, $\sim H_{\gamma R}^{(2)}(nkr)$ for $r > R$, ($\sim \exp(i\omega t)$),
- continuity at interfaces: $\phi, \partial_r \phi$ (TE), $\phi, (\partial_r \phi)/n^2$ (TM).

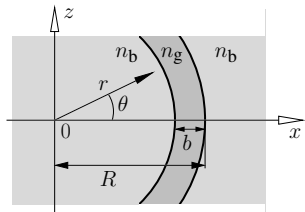
Bend modes, 2-D examples



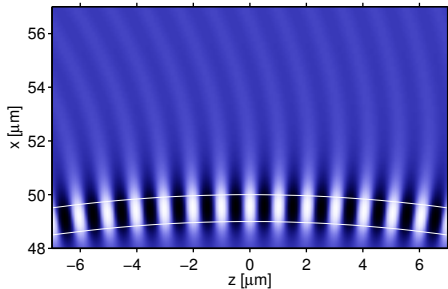
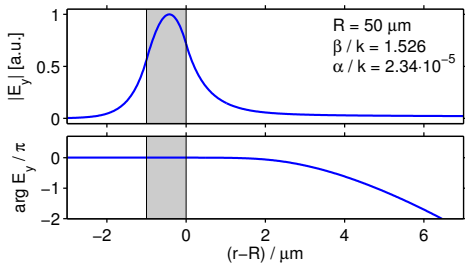
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 1000 \mu\text{m}$.



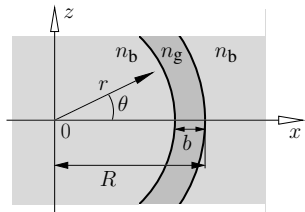
Bend modes, 2-D examples



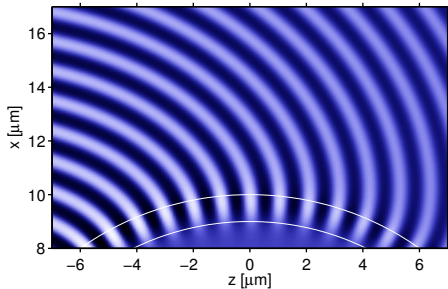
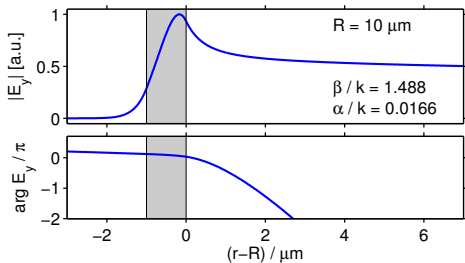
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 50 \mu\text{m}$.



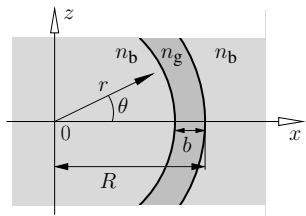
Bend modes, 2-D examples



2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 10 \mu\text{m}$.



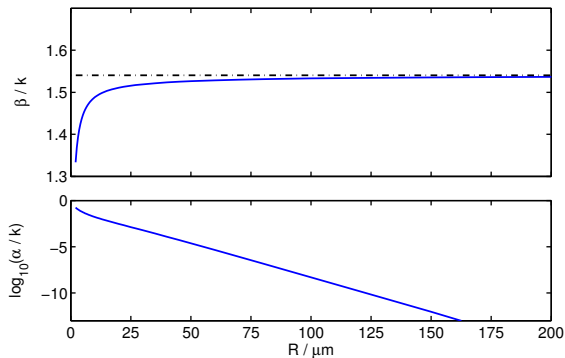
Propagation constant vs. bend radius



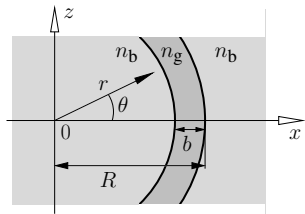
2-D, TE,

$n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,

$R \in [2, 200] \mu\text{m}$.

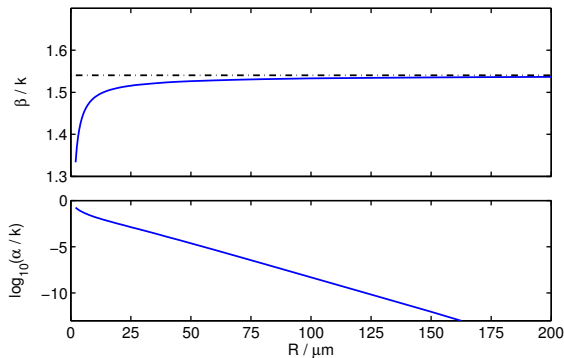


Propagation constant vs. bend radius



2-D, TE,

$n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R \in [2, 200] \mu\text{m}$.



Alternative definition :

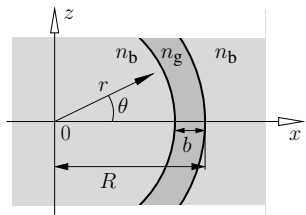
$$R' = R - b/2.$$

Identical physical fields

$$\gamma' R' = \gamma R,$$

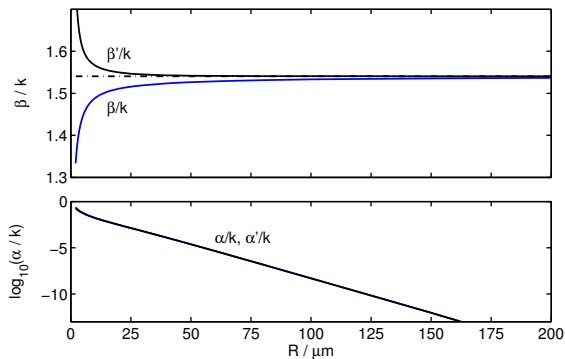
$$\gamma' = \gamma \frac{R}{R - b/2}.$$

Propagation constant vs. bend radius



2-D, TE,

$n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R \in [2, 200] \mu\text{m}$.



Alternative definition :

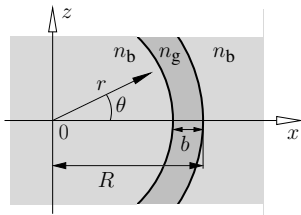
$$R' = R - b/2.$$

Identical physical fields

$$\gamma' R' = \gamma R,$$

$$\gamma' = \gamma \frac{R}{R - b/2}.$$

Power & orthogonality



2-D TE/TM bend modes:

- Power flow: $S_r \neq 0$, $S_r, S_\theta \sim e^{-2\alpha R\theta}$, $S_\theta \sim |\phi|^2/r$

$$\int_0^\infty S_\theta(r) dr < \infty \quad \longleftrightarrow \quad \text{power normalization.}$$

- Orthogonality of nondegenerate bend modes, product

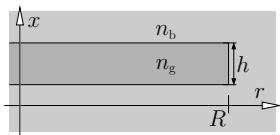
$$[\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2] = \int_0^\infty (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{e}_\theta dr.$$

(Here $[, ; ,]$ is complex valued.)

(Expressions $\sim \phi^2/r \longleftrightarrow$ convergence of the integrals.)

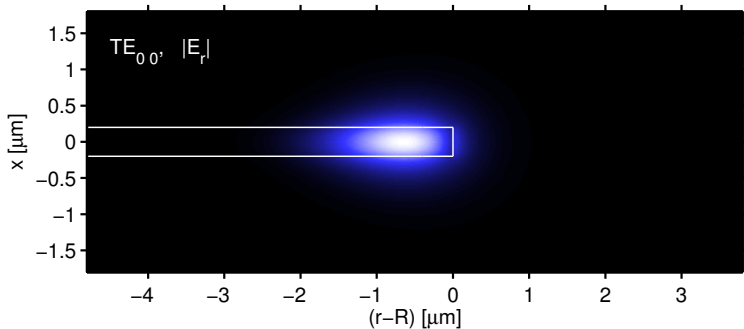
Bend modes supported by an angular disc segment

$$x \in [-3, 3] \mu\text{m},$$
$$(r - R) \in [-8, 4] \mu\text{m};$$

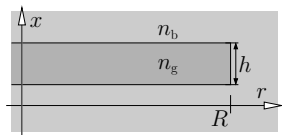


$$\lambda = 1.55 \mu\text{m},$$
$$n_b = 1.45,$$
$$n_g = 1.99,$$
$$h = 0.4 \mu\text{m},$$
$$R = 20 \mu\text{m};$$

$$\text{TE}_{00}$$
$$\beta/k = 1.634$$
$$\alpha/k = 3.1 \cdot 10^{-8}$$
$$[\text{JCMwave}].$$



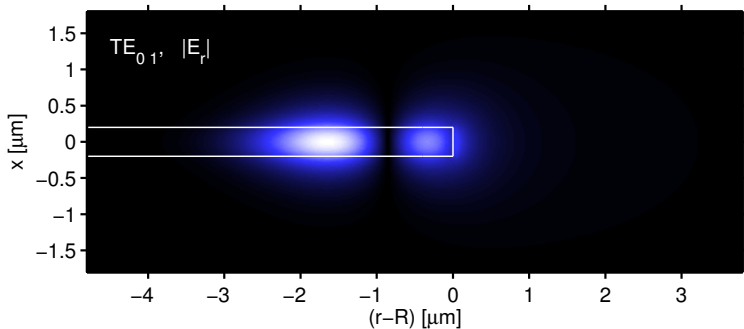
Bend modes supported by an angular disc segment



$\lambda = 1.55 \text{ } \mu\text{m},$
 $n_b = 1.45,$
 $n_g = 1.99,$
 $h = 0.4 \text{ } \mu\text{m},$
 $R = 20 \text{ } \mu\text{m};$

$x \in [-3, 3] \text{ } \mu\text{m},$
 $(r - R) \in [-8, 4] \text{ } \mu\text{m};$

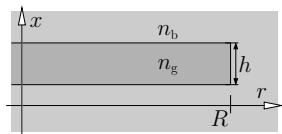
TE_{01}
 $\beta/k = 1.548$
 $\alpha/k = 1.5 \cdot 10^{-5}$
[\[JCMwave\]](#).



Bend modes supported by an angular disc segment

$$x \in [-3, 3] \text{ } \mu\text{m},$$

$$(r - R) \in [-8, 4] \text{ } \mu\text{m};$$



$$\lambda = 1.55 \text{ } \mu\text{m},$$

$$n_b = 1.45,$$

$$n_g = 1.99,$$

$$h = 0.4 \text{ } \mu\text{m},$$

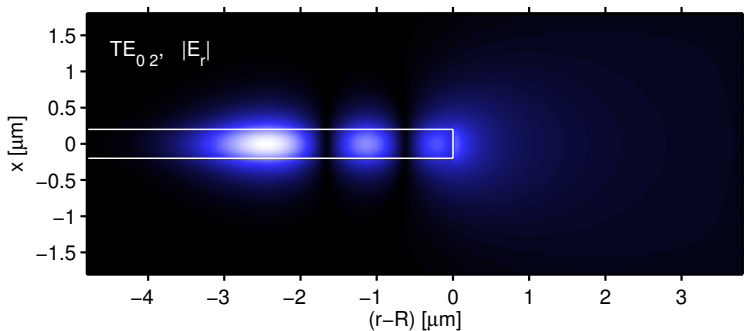
$$R = 20 \text{ } \mu\text{m};$$

$$\text{TE}_{02}$$

$$\beta/k = 1.480$$

$$\alpha/k = 4.0 \cdot 10^{-4}$$

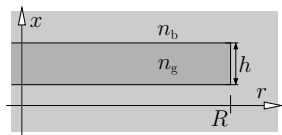
[JCMwave].



Bend modes supported by an angular disc segment

$$x \in [-3, 3] \text{ } \mu\text{m},$$

$$(r - R) \in [-8, 4] \text{ } \mu\text{m};$$



$$\lambda = 1.55 \text{ } \mu\text{m},$$

$$n_b = 1.45,$$

$$n_g = 1.99,$$

$$h = 0.4 \text{ } \mu\text{m},$$

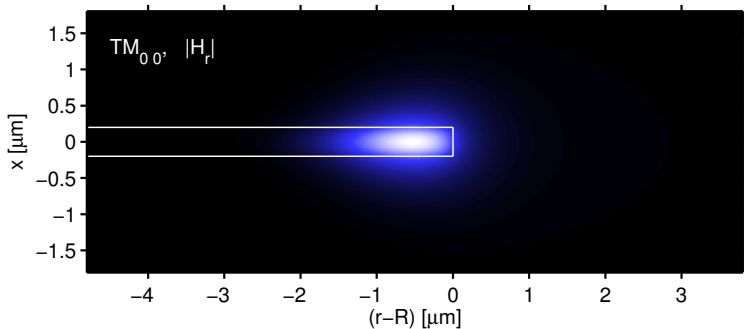
$$R = 20 \text{ } \mu\text{m};$$

$$\text{TM}_{00}$$

$$\beta/k = 1.551$$

$$\alpha/k = 2.4 \cdot 10^{-5}$$

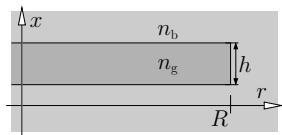
[JCMwave].



Bend modes supported by an angular disc segment

$$x \in [-3, 3] \text{ } \mu\text{m},$$

$$(r - R) \in [-8, 4] \text{ } \mu\text{m};$$



$$\lambda = 1.55 \text{ } \mu\text{m},$$

$$n_b = 1.45,$$

$$n_g = 1.99,$$

$$h = 0.4 \text{ } \mu\text{m},$$

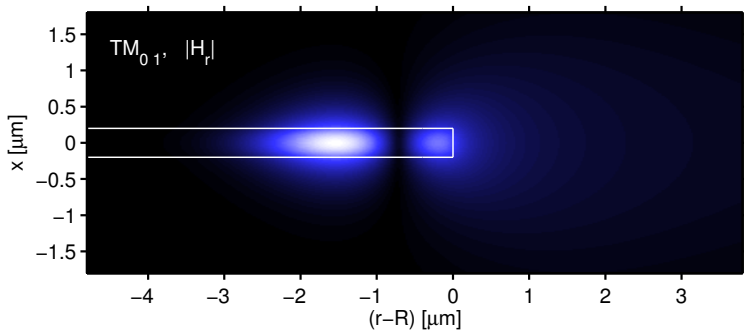
$$R = 20 \text{ } \mu\text{m};$$

$$\text{TM}_{01}$$

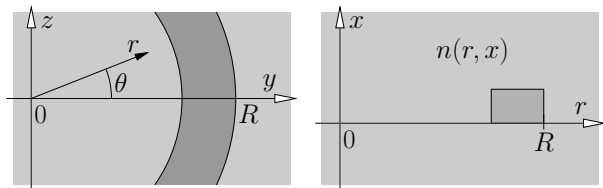
$$\beta/k = 1.468$$

$$\alpha/k = 7.6 \cdot 10^{-4}$$

[JCMwave].

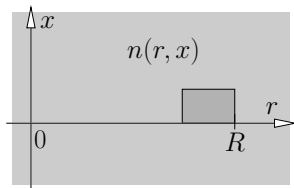
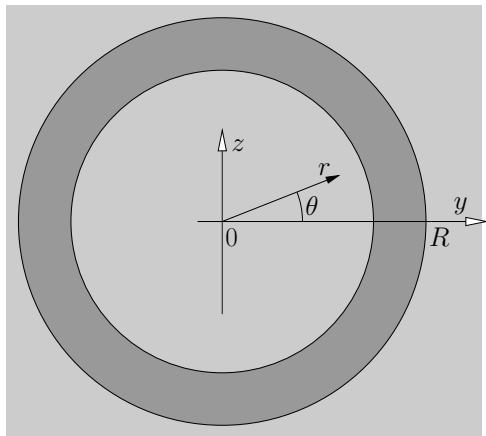


Circular microcavity



Bend modes

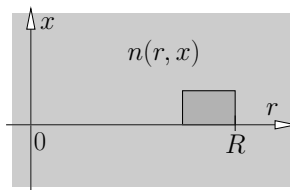
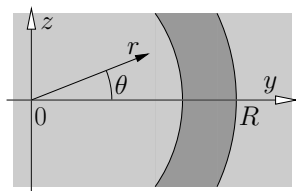
Circular microcavity



Bend modes \longleftrightarrow Whispering gallery resonances.

(Terms not always clearly distinguished.)

Whispering gallery resonances



(FD)

- Full cavity, $\theta \in [0, 2\pi]$:

Look for resonances in the form of **whispering gallery modes**

$$\left(\begin{matrix} E \\ H \end{matrix} \right) (r, \theta, x, t) = \left(\begin{matrix} \tilde{E} \\ \tilde{H} \end{matrix} \right) (r, x) e^{i\omega_c t - im\theta},$$

+c.c.

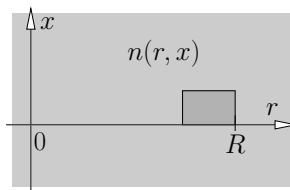
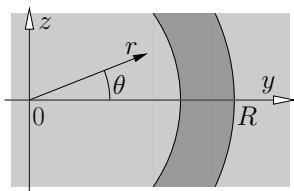
\bar{E}, \bar{H} : **WGM profile**, components $\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x$,

$m \in \mathbb{Z}$: **angular order**,

$\omega_c = \omega'_c + i\omega''_c \in \mathbb{C}$: **eigenfrequency**, $\omega'_c, \omega''_c \in \mathbb{R}$.

Q-factor $Q = \omega'_c / (2\omega''_c)$, resonance wavelength $\lambda_r = 2\pi c / \omega'_c$, outgoing radiation, FWHM: $2\delta\lambda = \lambda_r / Q$.

Whispering gallery resonances



(FD)

- Piecewise constant $n(r, x)$, $\psi \in \{\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x\}$,
(Dispersion ?)

$$\hookrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \left(\frac{\omega_c^2}{c^2} n^2 - \frac{m^2}{r^2} \right) \psi = 0, \quad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),

& boundary conditions:

regularity at $r = 0$, outgoing waves at $x = \pm\infty$, $r = \infty$.

(or: normalizability versus x .)

Vectorial eigenproblem for whispering gallery resonances.

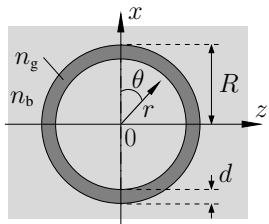
(Practical setting: computational domain $r_i < r < r_o$, $x_b < x < x_t$, PML boundary conditions / $\psi = 0$ at $r = r_i$.)

2-D whispering gallery resonances

... as discussed for the 2-D TE/TM bend modes.

(WGMs: Bessel differential equation of integer order.)
(Notation: $\text{WGM}(\rho, m)$ — mode of radial order ρ and angular order m .)

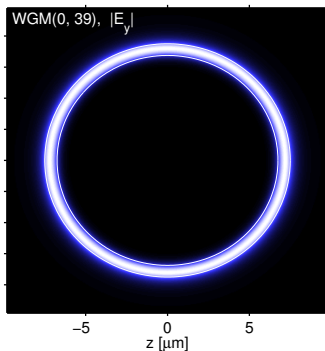
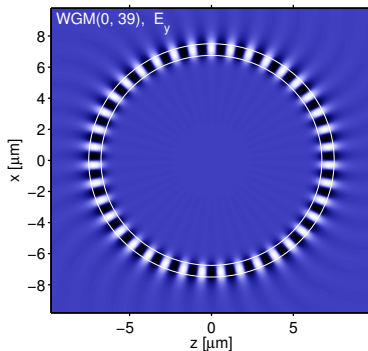
2-D whispering gallery resonances



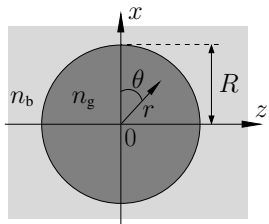
TE, $R = 7.5 \mu\text{m}$, $d = 0.75 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.5637 \mu\text{m}$, $Q = 1.1 \cdot 10^5$, $2\delta\lambda = 1.4 \cdot 10^{-5} \mu\text{m}$.



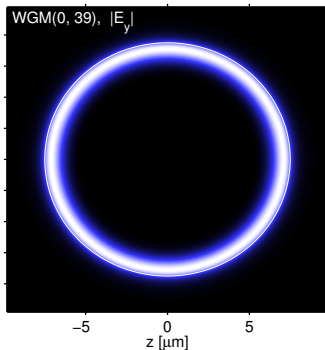
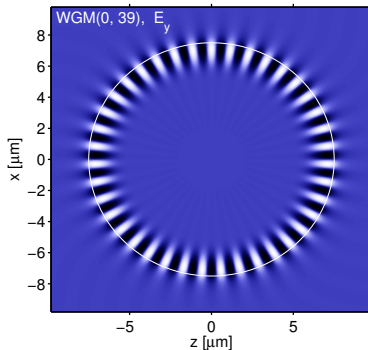
2-D whispering gallery resonances



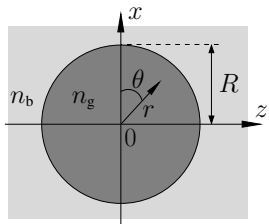
TE, $R = 7.5 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.6025 \mu\text{m}$, $Q = 5.7 \cdot 10^5$, $2\delta\lambda = 2.8 \cdot 10^{-6} \mu\text{m}$.



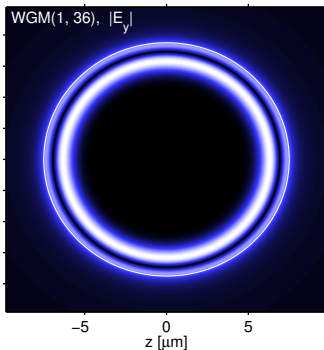
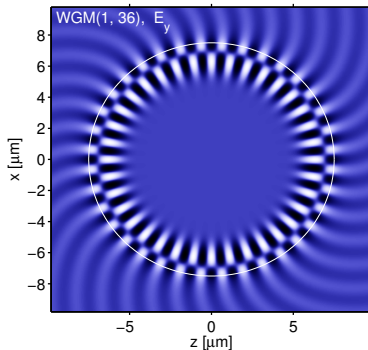
2-D whispering gallery resonances



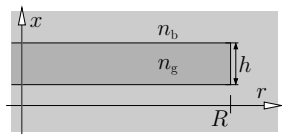
TE, $R = 7.5 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(1, 36):

$\lambda_r = 1.5367 \mu\text{m}$, $Q = 2.2 \cdot 10^3$, $2\delta\lambda = 7.0 \cdot 10^{-4} \mu\text{m}$.



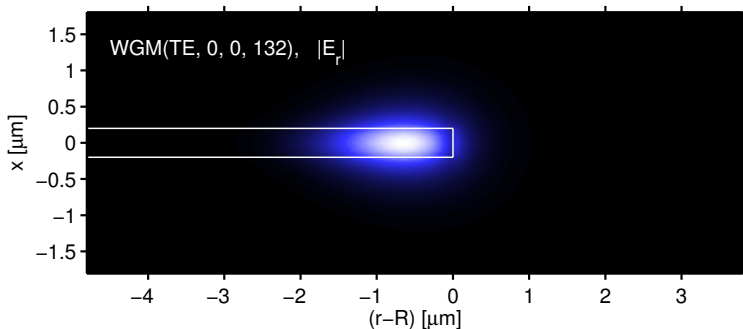
WGMs supported by a circular slab disc



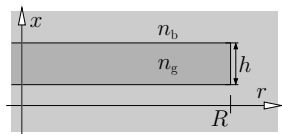
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TE, 0, 0, 132)
 $\lambda_r = 1.555 \mu\text{m}$
 $Q = 6.9 \cdot 10^6$
[JCMwave].



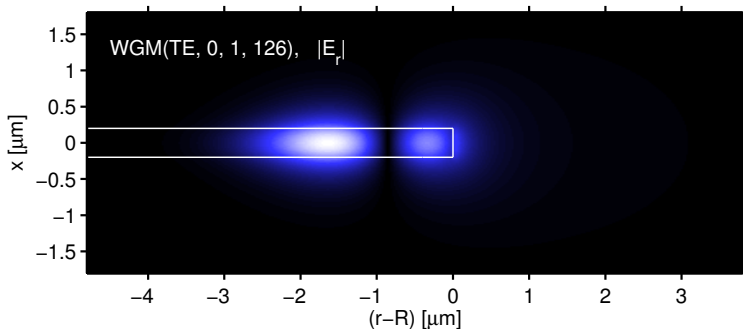
WGMs supported by a circular slab disc



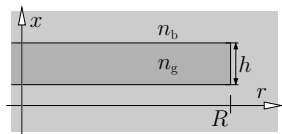
$$\begin{aligned}\lambda &= 1.55 \text{ } \mu\text{m}, \\ n_b &= 1.45, \\ n_g &= 1.99, \\ h &= 0.4 \text{ } \mu\text{m}, \\ R &= 20 \text{ } \mu\text{m};\end{aligned}$$

$$\begin{aligned}x &\in [-3, 3] \text{ } \mu\text{m}, \\ (r - R) &\in [-8, 4] \text{ } \mu\text{m};\end{aligned}$$

$$\begin{aligned}\text{WGM}(\text{TE}, 0, 1, 126) \\ \lambda_r &= 1.545 \text{ } \mu\text{m} \\ Q &= 1.7 \cdot 10^4 \\ [\text{JCMwave}].\end{aligned}$$



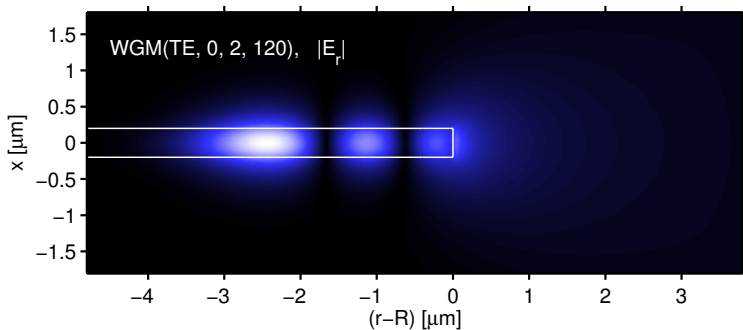
WGMs supported by a circular slab disc



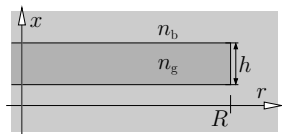
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TE, 0, 2, 120)
 $\lambda_r = 1.550 \mu\text{m}$
 $Q = 5.7 \cdot 10^2$
[JCMwave].



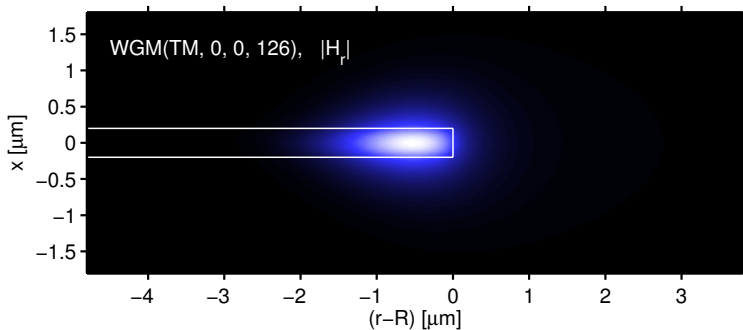
WGMs supported by a circular slab disc



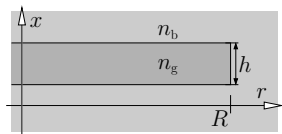
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TM, 0, 0, 126)
 $\lambda_r = 1.547 \mu\text{m}$
 $Q = 1.0 \cdot 10^4$
[JCMwave].



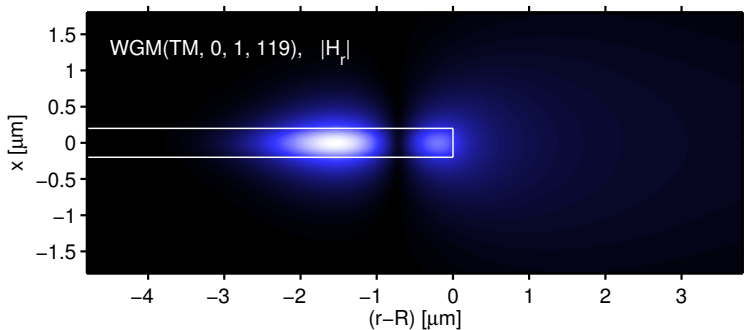
WGMs supported by a circular slab disc



$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;

WGM(TM, 0, 1, 119)
 $\lambda_r = 1.550 \mu\text{m}$
 $Q = 3.0 \cdot 10^2$
[JCMwave].



Bend modes versus whispering gallery resonances

(Field supported by a full circular cavity.)

(Incompatible models, in principle.)


[BWG] $\omega \in \mathbb{R}$ given, $\gamma = \beta - i\alpha \in \mathbb{C}$ eigenvalue,

$$\Phi(r, \theta, t) = \phi(r) e^{i\omega t - i\beta R\theta} e^{-\alpha R\theta}.$$

[WGM] $\omega_c = \omega_c + i\omega_c'' \in \mathbb{C}$ eigenvalue, $m \in \mathbb{Z}$ given,

$$\Psi(r, \theta, t) = \psi(r) e^{i\omega_c' t - im\theta} e^{-\omega_c'' t}.$$

Look at a resonant low-loss configuration:

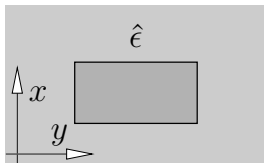
- Translate $\omega \approx \omega_c'$, $m \approx \beta R$.
- Equate the power loss during one time period $T = 2\pi/\omega \approx 2\pi/\omega_c'$
 $\beta/\alpha \approx \omega_c'/\omega_c'' = 2Q$.

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

Perturbations of single modes

$\sim \exp(i\omega t)$ (FD)



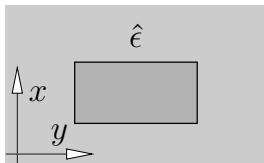
$\lambda, \hat{\epsilon}(x, y)$



$\beta,$
 \bar{E}, \bar{H}

Perturbations of single modes

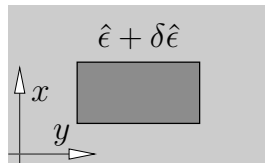
$\sim \exp(i\omega t)$ (FD)



$$\lambda, \hat{\epsilon}(x, y)$$



$$\beta, \\ \bar{\mathbf{E}}, \bar{\mathbf{H}}$$



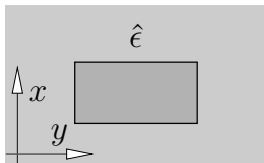
$$\lambda, \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$$



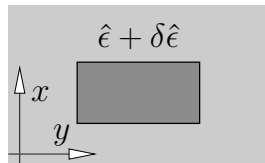
$$\beta + \delta\beta, \\ \bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}$$

Perturbations of single modes

$\sim \exp(i\omega t)$ (FD)



?



$\lambda, \hat{\epsilon}(x, y)$
 $\beta,$
 $\bar{\mathbf{E}}, \bar{\mathbf{H}}$



$\lambda, \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$
 $\beta + \delta\beta,$
 $\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}$

A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

- $$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad \beta \in \mathbb{R},$$

$$\bar{\mathbf{E}}, \bar{\mathbf{H}} \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.$$

- $$(\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{E}} = -i\omega\mu_0\bar{\mathbf{H}}, \quad (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{H}} = i\omega\epsilon_0\hat{\mathbf{E}},$$

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

- $$\mathcal{B}_{\hat{\mathbf{e}}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) := \frac{\omega\epsilon_0\langle\bar{\mathbf{E}}, \hat{\mathbf{E}}\rangle + \omega\mu_0\langle\bar{\mathbf{H}}, \bar{\mathbf{H}}\rangle + i\langle\bar{\mathbf{E}}, \mathbf{C}\bar{\mathbf{H}}\rangle - i\langle\bar{\mathbf{H}}, \mathbf{C}\bar{\mathbf{E}}\rangle}{\langle\bar{\mathbf{E}}, \mathbf{R}\bar{\mathbf{H}}\rangle - \langle\bar{\mathbf{H}}, \mathbf{R}\bar{\mathbf{E}}\rangle},$$

$$\langle\bar{\mathbf{F}}, \bar{\mathbf{G}}\rangle = \iint \bar{\mathbf{F}}^* \cdot \bar{\mathbf{G}} \, dx \, dy.$$

$$\mathcal{B}_{\hat{\mathbf{e}}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta \quad (*), \quad \left. \frac{d}{ds} \mathcal{B}_{\hat{\mathbf{e}}}(\bar{\mathbf{E}} + s\bar{\mathbf{F}}, \bar{\mathbf{H}} + s\bar{\mathbf{G}}) \right|_{s=0} = 0 \quad (**)$$

at valid mode fields $\bar{\mathbf{E}}, \bar{\mathbf{H}}$, for arbitrary $\bar{\mathbf{F}}, \bar{\mathbf{G}}$.

(*): “arbitrary” $\hat{\mathbf{e}}$.


(**): Hermitian $\hat{\mathbf{e}}$.

Perturbations of single modes


- Available: Mode $\beta, \bar{\mathbf{E}}, \bar{\mathbf{H}}$ for parameters $\lambda, \hat{\epsilon}$; ($\hat{\epsilon} = \hat{\epsilon}^\dagger$)
 $\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$, $\mathcal{B}_{\hat{\epsilon}}$ stationary at $\bar{\mathbf{E}}, \bar{\mathbf{H}}$.

- Investigate parameters $\lambda, \hat{\epsilon} + \delta\hat{\epsilon}$, for a “small” change $\delta\hat{\epsilon}$:

$$\mathcal{B}_{\hat{\epsilon}+\delta\hat{\epsilon}}(\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}) = \beta + \delta\beta$$

 ... $\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}) \approx \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$

 ... $\delta(\cdot)\delta(\cdot)$

 $\delta\beta = \frac{\omega\epsilon_0 \langle \bar{\mathbf{E}}, \delta\hat{\epsilon} \bar{\mathbf{E}} \rangle}{\langle \bar{\mathbf{E}}, \mathbf{R}\bar{\mathbf{H}} \rangle - \langle \bar{\mathbf{H}}, \mathbf{R}\bar{\mathbf{E}} \rangle}, \quad \text{or} \quad \delta\beta = \frac{\omega\epsilon_0 \iint \bar{\mathbf{E}}^* \cdot \delta\hat{\epsilon} \bar{\mathbf{E}} \, dx \, dy}{2 \operatorname{Re} \iint (\bar{\mathbf{E}}_x^* \bar{\mathbf{H}}_y - \bar{\mathbf{E}}_y^* \bar{\mathbf{H}}_x) \, dx \, dy}.$

(Valid for *small* perturbations: The original mode profiles are good approximations of the true fields in the modified structure.)

Small uniform change in refractive index



- $n \longrightarrow n + \delta n$ on \square , $n, \delta n$ constant on \square

$$\hookrightarrow \beta \longrightarrow \beta + \delta\beta, \quad \delta\beta = \frac{\omega\epsilon_0 n \iint_{\square} |\bar{\mathbf{E}}|^2 dx dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta n.$$

($\delta\epsilon = 2n\delta n$.)
(Plausible: $\delta\beta \sim \delta n$, $\delta\beta \sim |\bar{\mathbf{E}}|^2|_{\square}$.)

Small attenuation



- $n \longrightarrow n - i n''$ on \square , n, n'' constant on \square , $n, n'' \in \mathbb{R}$

$$\beta \longrightarrow \beta + \delta\beta, \quad \delta\beta = \frac{-i\omega\epsilon_0 n \iint_{\square} |\bar{\mathbf{E}}|^2 dx dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} n''.$$

$$(\delta\epsilon = -i2nn'')$$

(Different attenuation for each mode.)

(Damping, power, plane wave: $\sim \exp(-2kn''z)$, mode: $\not\sim \exp(-2kn''z)$.)

Small anisotropy



- $\epsilon \hat{1} \longrightarrow \epsilon \hat{1} + \delta \epsilon$ on \square , $\epsilon, \delta \epsilon$ constant on \square

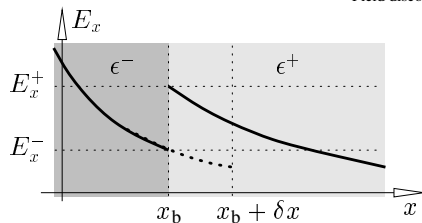
$$\beta \longrightarrow \beta + \delta \beta, \quad \delta \beta = \frac{\omega \epsilon_0 \iint_{\square} \bar{\mathbf{E}}^* \cdot \delta \epsilon \bar{\mathbf{E}} \, dx \, dy}{2 \operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) \, dx \, dy}.$$

(Phase shifts due to anisotropic contributions to the permittivity.)

(Polarization coupling might occur for modes with “close” propagation constants \longleftrightarrow CMT.)

Small displacements of dielectric interfaces

Interface displacement \longleftrightarrow Locally *strong* thin layer perturbation.
 Field discontinuity \rightsquigarrow Previous expressions are not directly applicable.



- $\epsilon^- \neq \epsilon^+$,
 shift of interface
 $x_b \rightarrow x_b + \delta x$.

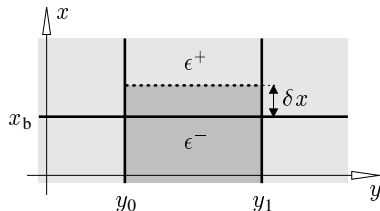
- Reposition discontinuity in field: $E_x \rightarrow E_x + \delta E_x$,

$$\delta E_x(x, y) = \begin{cases} \frac{\epsilon^+ - \epsilon^-}{\epsilon^-} E_x(x, y), & \text{for } x_b < x < x_b + \delta x, \\ 0, & \text{otherwise.} \end{cases}$$

- Use functional with locally modified field

$\hookrightarrow \dots$ (omitted) $\dots \rightsquigarrow$

Small displacements of dielectric interfaces

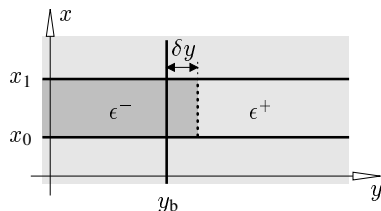


- Displacement of the interface at x_b between y_0 and y_1 by δx :

$$\beta \longrightarrow \beta + \delta\beta,$$

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{y_0}^{y_1} \left(\frac{1}{\epsilon^- \epsilon^+} |\epsilon \bar{E}_x|^2 + |\bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x_b, y) dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta x.$$

Small displacements of dielectric interfaces

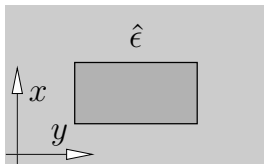


- Displacement of the interface at y_b between x_0 and x_1 by δy :

$$\beta \longrightarrow \beta + \delta\beta,$$

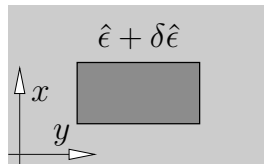
$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} \left(|\bar{E}_x|^2 + \frac{1}{\epsilon^- \epsilon^+} |\epsilon^- \bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x, y_b) dx}{\text{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta y.$$

Perturbations of single modes



$$\lambda, \hat{\epsilon}(x, y)$$

$$\beta,$$
$$\bar{\mathbf{E}}, \bar{\mathbf{H}}$$



$$\lambda, \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$$

$$\beta + \delta\beta,$$
$$\approx \bar{\mathbf{E}}, \approx \bar{\mathbf{H}}$$

- View $\frac{\delta\beta}{\delta p}$ as $\frac{\partial\beta}{\partial p}$: slope of the dispersion curves β vs. p .
- Depending on the parametrization, change of a parameter value might require several perturbations.
- First order theory: In case of multiple perturbations, add the effects of the individual expressions.
- Estimation of fabrication tolerances: The phase shifts $\delta\beta$ enter into respective scattering matrix models.
- Wavelength shifts . . . ?

Small shift of frequency or vacuum wavelength

(*) : Explicit frequency dependence of \mathcal{B} & dependence through $\hat{\epsilon}$.

(**) : Frequency dependence of $\bar{\mathbf{E}}, \bar{\mathbf{H}}$.

$$\beta(\omega) = \mathcal{B}_{\hat{\epsilon}}(\omega; \bar{\mathbf{E}}(\omega), \bar{\mathbf{H}}(\omega))$$

$$\begin{aligned} \hookrightarrow \frac{\partial \beta}{\partial \omega} &= \frac{\partial \mathcal{B}_{\hat{\epsilon}}}{\partial \omega} \quad (*) + \frac{\partial}{\partial s} \mathcal{B}_{\hat{\epsilon}} \left(\omega; \bar{\mathbf{E}} + s \frac{\partial \bar{\mathbf{E}}}{\partial \omega}, \bar{\mathbf{H}} \right) \bigg|_{s=0} \quad (**) \\ &\quad + \frac{\partial}{\partial s} \mathcal{B}_{\hat{\epsilon}} \left(\omega; \bar{\mathbf{E}}, \bar{\mathbf{H}} + s \frac{\partial \bar{\mathbf{H}}}{\partial \omega} \right) \bigg|_{s=0} \quad (**) \\ &= \frac{\partial \mathcal{B}_{\hat{\epsilon}}}{\partial \omega}, \quad \text{(Stationarity of } \mathcal{B} \text{ at } \bar{\mathbf{E}}, \bar{\mathbf{H}}.) \end{aligned}$$

$$\hookrightarrow \frac{\partial \beta}{\partial \omega} = \frac{\iint \left(\epsilon_0 \bar{\mathbf{E}}^* \cdot \frac{\partial(\omega \hat{\epsilon})}{\partial \omega} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2 \right) dx dy}{2 \operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy}.$$

Small shift of frequency or vacuum wavelength

If dispersion can be neglected, $\partial_\omega \hat{\epsilon} = 0$:

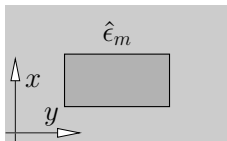
$$\hookrightarrow \frac{\partial \beta}{\partial \omega} = \frac{\iint \left(\epsilon_0 \bar{\mathbf{E}}^* \cdot \hat{\epsilon} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2 \right) dx dy}{2 \operatorname{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x \right) dx dy},$$

$$\hookrightarrow \frac{\partial \beta}{\partial \lambda} = -\frac{\pi c}{\lambda^2} \frac{\iint \left(\epsilon_0 \bar{\mathbf{E}}^* \cdot \hat{\epsilon} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2 \right) dx dy}{\operatorname{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x \right) dx dy}.$$

$(\omega = 2\pi c / \lambda \quad \longleftrightarrow \quad \partial_\lambda \omega = -2\pi c / \lambda^2)$
(Compare with expression based on homogeneity, H, 12.)

Coupled mode theory (CMT)

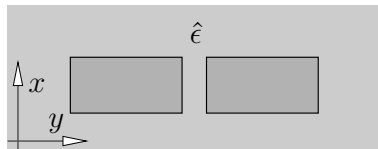
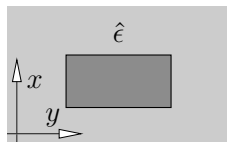
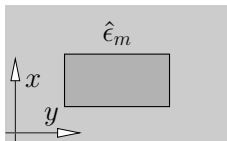
$\sim \exp(i\omega t)$ (FD)



$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$

Coupled mode theory (CMT)

$\sim \exp(i\omega t)$ (FD)

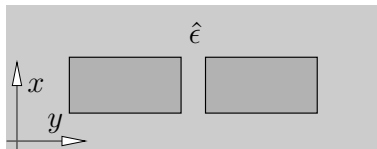
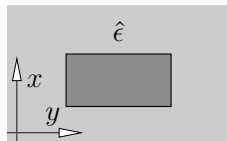
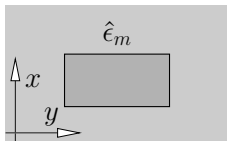


$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z)$$

Coupled mode theory (CMT)

$\sim \exp(i\omega t)$ (FD)



?

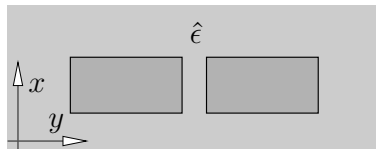
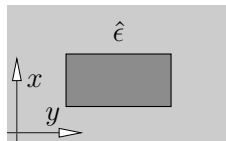
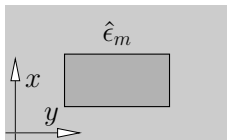
$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$



$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z)$$

Coupled mode theory (CMT)

$\sim \exp(i\omega t)$ (FD)



?

$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$



$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z)$$

(Next: One of many variants of approaches to CMT.)

(Propagation & interaction of basis fields along a common propagation coordinate.)

[D.G. Hall, B.J. Thompson, *Selected papers on Coupled-Mode Theory in Guided-Wave Optics*, SPIE Milestone series MS 84 (1993)]

(Codirectional coupling (here), versus contradirectional coupling, coupling to radiation modes, nonlinear coupling.)

(Hybrid variant (HCMT): separate lecture.)

Coupled mode theory (CMT)

- Investigate a permittivity $\hat{\epsilon}$, look for fields \mathbf{E}, \mathbf{H} with

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}.$$

($\hat{\epsilon}(x, y, z)$, in general.)

- Available: A set of fields $\{\mathbf{E}_m, \mathbf{H}_m\}$ for permittivities $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$;

$$\nabla \times \mathbf{E}_m = -i\omega\mu_0\mathbf{H}_m, \quad \nabla \times \mathbf{H}_m = i\omega\epsilon_0\hat{\epsilon}_m\mathbf{E}_m.$$

(Not necessarily “modes”.)

Coupled mode theory (CMT)

- Investigate a permittivity $\hat{\epsilon}$, look for fields \mathbf{E}, \mathbf{H} with

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}.$$

($\hat{\epsilon}(x, y, z)$, in general.)

- Available: A set of fields $\{\mathbf{E}_m, \mathbf{H}_m\}$ for permittivities $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$;

$$\nabla \times \mathbf{E}_m = -i\omega\mu_0\mathbf{H}_m, \quad \nabla \times \mathbf{H}_m = i\omega\epsilon_0\hat{\epsilon}_m\mathbf{E}_m.$$

(Not necessarily “modes”.)

- Assume that (\mathbf{E}, \mathbf{H}) can be well approximated by

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) \approx \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z),$$

C_m : unknown amplitudes, common propagation coordinate z .

(Choose $\hat{\epsilon}_m$ as close as possible to $\hat{\epsilon}$.)

Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”.)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for \mathbf{E}, \mathbf{H} .)

...

($\iint dx dy$, assume $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.)

...

(Apply identity $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}_m$.)

...

($\iint dx dy$, assume $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.)

...

(Manipulate, arrange terms, tidy up.)

Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”).

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for \mathbf{E}, \mathbf{H} .)



...

($\iint dx dy$, assume $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.)



...

(Apply identity $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}_m$.)



...

($\iint dx dy$, assume $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.)



...

(Manipulate, arrange terms, tidy up.)

$$\sum_m o_{lm} \partial_z C_m = -i \sum_m k_{lm} C_m \quad \forall l, \quad \text{coupled mode equations.}$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”.)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for \mathbf{E}, \mathbf{H} .)



...

($\iint dx dy$, assume $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.)



...

(Apply identity $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}_m$.)



...

($\iint dx dy$, assume $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.)



...

(Manipulate, arrange terms, tidy up.)

$$\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K}\mathbf{C},$$

coupled mode equations.

$$\mathbf{C} = (C_m), \mathbf{O} = (o_{lm}), \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

Coupled mode theory (CMT)

(Variational derivation of CMT equations.)

$$\mathcal{F}(\mathbf{E}, \mathbf{H}) = \iiint \left\{ \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E}^* \cdot (\nabla \times \mathbf{H}) + i\omega\mu_0 \mathbf{H}^* \cdot \mathbf{H} + i\omega\epsilon_0 \mathbf{E}^* \cdot \hat{\epsilon} \mathbf{E} \right\} dx dy dz,$$

$$\delta \mathcal{F} = 0 \quad \forall \delta \mathbf{E}, \delta \mathbf{H} \quad \longleftrightarrow \quad \nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0 \hat{\epsilon} \mathbf{E}.$$

(Restrict \mathcal{F} to the CMT ansatz for $\mathbf{E}, \mathbf{H} \rightsquigarrow \mathcal{F}_c(\mathbf{C})$, require $\delta \mathcal{F}_c = 0 \quad \forall \delta \mathbf{C}$.)

$$\begin{aligned} & \dots \\ & (\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}, \quad \iint dx dy, \quad \mathbf{E}_m, \mathbf{H}_m \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.) \\ & \dots \end{aligned}$$

(Manipulate, arrange terms, tidy up.)

$$\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K} \mathbf{C},$$

coupled mode equations.

$$\mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

Coupled mode equations

...

$$\curvearrowright \quad \mathbf{O} \partial_z \mathbf{C} = -i \mathbf{K} \mathbf{C}, \quad \mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z \, dx \, dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega \epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m \, dx \, dy.$$

- A set of coupled *ordinary* linear differential equations, of first order. (Here.)
 - o_{lm} : **power coupling coefficients** (field overlaps).
 - k_{lm} : **coupling coefficients**. (No reason to assume $o_{lm} = \delta_{lm}$, in general.)
 - z -dependence of $\hat{\epsilon}, \hat{\epsilon}_m, \mathbf{E}_m, \mathbf{H}_m \rightsquigarrow o_{lm}(z), k_{lm}(z), \mathbf{O}(z), \mathbf{K}(z)$. (Compare the bend-straight couplers, Lecture H.)
-

... to be solved by numerical procedures.

(In general.)

CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

basis: **guided modes** $\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z},$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y).$$

$$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$$

↪ ...

$$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$$

↪ ...

(Symmetrize coefficients.)

CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

basis: **guided modes** $\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z},$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y).$$

$$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$$

↪ ...

$$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$$

↪ ...

(Symmetrize coefficients.)

$$\sum_m \sigma_{lm} \partial_z c_m = -i \sum_m (b_{lm} + \kappa_{lm}) c_m \quad \forall l,$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z dx dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m dx dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

$$\text{basis: guided modes} \quad \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z},$$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y).$$

$$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$$

↪ ...

$$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$$

↪ ...

(Symmetrize coefficients.)

$$\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q}) \mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}),$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z dx dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m dx dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

Longitudinally constant structures, coupled mode equations

...

$$(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$$

↪ $\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}).$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z \, dx \, dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega \epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \bar{\mathbf{E}}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$
$$\delta \hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- A set of coupled *ordinary* linear differential equations, of first order (Here.)
- σ_{lm} : **power coupling coefficients** (field overlaps).
- κ_{lm} : **coupling coefficients**. (No reason to assume $\sigma_{lm} = \delta_{lm}$, in general.)
- $\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0 \rightsquigarrow \partial_z \sigma_{lm} = \partial_z b_{lm} = \partial_z \kappa_{lm} = 0.$

(ODEs with **constant coefficients**.)

... quasi-analytical solutions.

Longitudinally constant structures, coupled mode equations

...

$$(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$$

↪ $\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}).$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z dx dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega \epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \bar{\mathbf{E}}_m dx dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta \hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- $\sigma_{ml}^* = \sigma_{lm}, \quad b_{ml}^* = b_{lm}; \quad \kappa_{ml}^* = \kappa_{lm}, \quad \text{if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m,$
 $\mathbf{S}^\dagger = \mathbf{S}, \quad \mathbf{B}^\dagger = \mathbf{B}; \quad \mathbf{Q}^\dagger = \mathbf{Q}, \quad \text{if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m.$

- Power: $P = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_{l,m} c_l^* (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) c_m = \mathbf{c}^* \cdot \mathbf{S} \mathbf{c}$

↪ $\partial_z P = i \mathbf{c}^* \cdot ((\mathbf{B} + \mathbf{Q})^\dagger - (\mathbf{B} + \mathbf{Q})) \mathbf{c}, \quad \partial_z P = 0 \quad \text{for } \mathbf{B}^\dagger = \mathbf{B}, \quad \mathbf{Q}^\dagger = \mathbf{Q}.$

(For lossless waveguides the scheme is power conservative.)

Longitudinally constant structures, formal solution

$$\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c},$$

$$\partial_z \mathbf{S} = \partial_z \mathbf{B} = \partial_z \mathbf{Q} = 0.$$

$$\text{Ansatz: } \mathbf{c}(z) = \mathbf{a} e^{-ibz},$$

\mathbf{a}, b constants.



$$(\mathbf{B} + \mathbf{Q})\mathbf{a} = b \mathbf{S}\mathbf{a},$$

a generalized eigenvalue problem.

(Dimension: number of basis modes included.)

Solutions: $\{\mathbf{a}, b\}$,



“supermodes”

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \left(\sum_m a_m \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) \right) e^{-ibz}.$$

(Superpositions of the original mode profiles with constant coefficients.)

(As many supermodes as there are basis modes.)

(Formalism can be continued: power/orthogonality of supermodes . . .)

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\mathbf{E}_1, \mathbf{H}_1)$, $(\mathbf{E}_2, \mathbf{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)

(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0.$$

(Orthogonal modes, uniform normalization $P_m = P_0$.)

(Or: apply inverse of \mathbf{S} to CM equations, continue with redefined expressions for β_m, κ_{lm} .)

$$\begin{aligned} \hookrightarrow \begin{pmatrix} \partial_z c_1 \\ \partial_z c_2 \end{pmatrix} &= -i \begin{pmatrix} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, & \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ & & \kappa &= \kappa_{12}/P_0. \end{aligned}$$

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\mathbf{E}_1, \mathbf{H}_1)$, $(\mathbf{E}_2, \mathbf{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)

(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0.$$

(Orthogonal modes, uniform normalization $P_m = P_0$.)

(Or: apply inverse of \mathbf{S} to CM equations, continue with redefined expressions for β_m, κ_{lm} .)

$$\begin{aligned} \left(\begin{array}{c} \partial_z c_1 \\ \partial_z c_2 \end{array} \right) &= -i \left(\begin{array}{cc} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{array} \right) \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right), & \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ & & \kappa &= \kappa_{12}/P_0. \end{aligned}$$

...

$$\left(\begin{array}{c} c_1 \\ c_2 \end{array} \right)(z) = e^{-i \frac{(\beta'_1 + \beta'_2)}{2} z} \left(\begin{array}{cc} \cos \rho z - i \frac{\Delta \beta'}{2\rho} \sin \rho z & -i \frac{\kappa}{\rho} \sin \rho z \\ -i \frac{\kappa^*}{\rho} \sin \rho z & \cos \rho z + i \frac{\Delta \beta'}{2\rho} \sin \rho z \end{array} \right) \left(\begin{array}{c} c_{10} \\ c_{20} \end{array} \right),$$

$$\Delta \beta' = \beta'_1 - \beta'_2, \quad \rho = \sqrt{\left(\frac{\Delta \beta'}{2} \right)^2 + |\kappa|^2}.$$

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\mathbf{E}_1, \mathbf{H}_1)$, $(\mathbf{E}_2, \mathbf{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)
(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0. \quad (\text{Orthogonal modes, uniform normalization } P_m = P_0.)$$

(Or: apply inverse of \mathbf{S} to CM equations, continue with redefined expressions for β_m , κ_{lm} .)

$$\begin{aligned} \hookrightarrow \begin{pmatrix} \partial_z c_1 \\ \partial_z c_2 \end{pmatrix} = -i \begin{pmatrix} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \begin{aligned} \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ \kappa &= \kappa_{12}/P_0. \end{aligned} \end{aligned}$$

$$\bullet \quad c_{20} = 0 \quad \rightsquigarrow \quad \left| \frac{c_2(z)}{c_1(0)} \right|^2 = \eta_{\max} \sin^2(\rho z), \quad \eta_{\max} = \frac{|\kappa|^2}{|\kappa|^2 + (\Delta\beta'/2)^2}.$$

• **Maximum conversion** η_{\max} at $z = L_c$ with $\rho L_c = \pi/2$,

$$\text{coupling length } L_c = \frac{\pi}{\sqrt{(\Delta\beta')^2 + 4|\kappa|^2}}, \quad (\text{Conversion length, half-beat length.})$$

• In case of **phase matching** $\Delta\beta' = \beta'_1 - \beta'_2 = 0$: $\eta_{\max} = 1$, $L_c = \frac{\pi}{2|\kappa|}$.

(Here the *phase-shifted* propagation constants are relevant.)

(Small interaction (small maximum conversion) for out-of-phase modes, i.e. for $|\Delta\beta'|^2 \gg |\kappa|^2$.)

Longitudinally constant structures, one “coupled” mode

CMT with one basis mode: $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = c_1(z) \begin{pmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{H}}_1 \end{pmatrix}(x, y)$

↪ $\partial_z c_1 = -i \frac{b_{11} + \kappa_{11}}{\sigma_{11}} c_1,$

$$\frac{b_{11}}{\sigma_{11}} = \beta_1, \quad \frac{\kappa_{11}}{\sigma_{11}} = \frac{\omega \epsilon_0 \iint \bar{\mathbf{E}}_1^* \cdot (\hat{\epsilon} - \hat{\epsilon}_1) \bar{\mathbf{E}}_1 \, dx \, dy}{2 \operatorname{Re} \iint (\bar{E}_{1x}^* \bar{H}_{1y} - \bar{E}_{1y}^* \bar{H}_{1x}) \, dx \, dy} =: \delta \beta_1,$$

↪ $\partial_z c_1 = -i(\beta_1 + \delta \beta_1) c_1,$

↪ $c_1(z) = c_1(0) e^{-i(\beta_1 + \delta \beta_1)z}.$

↔ Theory of single mode perturbations.

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

A touch of photonic crystals

“Photonic crystals”: ?

Keywords:

- A branch of photonics.
- Optics involving structures with (1-D, 2-D, 3-D) **spatial periodicity**.
- 1-D periodicity: Multilayer stacks / coatings, gratings, corrugated waveguides.
- 2-D periodicity: Corrugated dielectric slabs, membranes, gratings.
- 3-D periodicity: Bulk photonic crystals.
- “Molding the flow of light” \longleftrightarrow tunability, degrees of freedom in design.
- Defect cavities & defect waveguides in photonic crystals.
- Phenomena & fundamental research.
- Photonic crystal fibers.

Context of this lecture:

- Problems of general classical electromagnetics & methods as discussed; different emphasis.
- Periodicity: **Restrict computations to unit cells**.

Structures with spatial periodicity

$\sim \exp(i\omega t)$ (FD)

Infinite system with periodic permittivity:

$$\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r}) \quad \text{for all lattice vectors } \mathbf{g}.$$

➡ Consider Floquet-Bloch waves

(Floquet: 1-D, context of mechanics;
Bloch: context of solid state physics.)

$$\begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r}) = U_{\mathbf{k}}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}},$$

\mathbf{k} : wavevector of the FB wave,

$U_{\mathbf{k}}$: a periodic function, $U_{\mathbf{k}}(\mathbf{r} + \mathbf{g}) = U_{\mathbf{k}}(\mathbf{r})$.

(A plane wave, modulated by a periodic function.)

{FB waves}: A complete basis for the periodic system.

(Bloch theorem: any solution can be written as a superposition of FB waves.)

(Background: Hilbert space theory, self-adjoint operators; familiar from Quantum theory.)

(Hermitian Hamiltonian and translation operators commute; Bloch waves are a simultaneous eigenbasis of these operators.)


(Required: Hermitian “Hamiltonian” \longleftrightarrow Hermitian $\hat{\epsilon}$.)

($U_{\mathbf{k}} = ?$, but $U_{\mathbf{k}}$ satisfies different equations than $E, H \dots$)

Structures with spatial periodicity

\mathbf{g} : a lattice vector, such that $\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r})$

$\sim \exp(i\omega t)$ (FD)


$$\begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r} + \mathbf{g}) = \begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{g}}. \quad (\text{QPBC})$$

(. . . if \mathbf{g} connects the boundaries of a unit cell.)


FB-wave eigenproblem:

Given a wavevector \mathbf{k} , look for frequencies $\omega \in \mathbb{R}$, such that there exist nonzero solutions (\mathbf{E}, \mathbf{H}) on a **unit cell domain**, with quasi-periodic boundary conditions (QPBC).

Structures with spatial periodicity

\mathbf{g} : a lattice vector, such that $\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r})$

$\sim \exp(i\omega t)$ (FD)

 $\begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r} + \mathbf{g}) = \begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{g}}. \quad (\text{QPBC})$

(... if \mathbf{g} connects the boundaries of a unit cell.)

FB-wave eigenproblem:

Given a wavevector \mathbf{k} , look for frequencies $\omega \in \mathbb{R}$, such that there exist nonzero solutions (\mathbf{E}, \mathbf{H}) on a **unit cell domain**, with quasi-periodic boundary conditions (QPBC).


- Outcome:

$\exists \omega$ with $(\mathbf{E}, \mathbf{H}) \neq 0$: $(\mathbf{k}, \omega) \in$ a frequency **band**, or

$\nexists \omega$ with $(\mathbf{E}, \mathbf{H}) \neq 0$: $\omega \in$ a **bandgap** region.

 “Bandstructure” calculations.

- QPBC for \mathbf{k} are the same as for $\mathbf{k} + \mathbf{K}$, if $\mathbf{K} \cdot \mathbf{g} = m2\pi$, $m \in \mathbb{Z}$.

 Restrict \mathbf{k} to the **first Brillouin zone**.


(Exclude $\mathbf{k} + \mathbf{K} \forall \mathbf{g}, m$)

(\mathbf{K} : A vector of the **reciprocal lattice**.)

Structures with spatial periodicity

\mathbf{g} : a lattice vector, such that $\epsilon(\mathbf{r} + \mathbf{g}) = \epsilon(\mathbf{r})$

$\sim \exp(i\omega t)$ (FD)

 $\begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r} + \mathbf{g}) = \begin{pmatrix} E \\ H \end{pmatrix}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{g}}. \quad (\text{QPBC})$

(... if \mathbf{g} connects the boundaries of a unit cell.)

FB-wave eigenproblem:

Given a wavevector \mathbf{k} , look for frequencies $\omega \in \mathbb{R}$, such that there exist nonzero solutions (\mathbf{E}, \mathbf{H}) on a **unit cell domain**, with quasi-periodic boundary conditions (QPBC).

(Include this in the list of computational problems of lecture D.)

(Bandstructure calculations: Information on infinite periodic structures.)

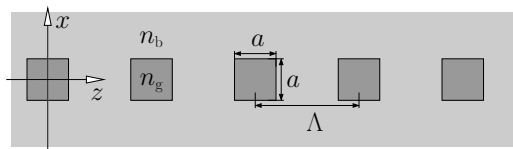
(Calculations on a (small) unit cell domain, typically computationally cheap.)

(Finite structures, (most) defects, external excitation, etc.: scattering solvers (FD, TD)

or resonance solvers required, on the full system domain.)

A sequence of dielectric rods

$\sim \exp(i\omega t)$ (FD)



$$a = 0.4 \mu\text{m}, \Lambda = 1 \mu\text{m}, \\ n_b = 1.0, n_g = \sqrt{12}.$$


[Joannopoulos, Johnson, Winn, Meade, *Photonic Crystals: Molding the Flow of Light*, 2nd edition, Princeton, 2008.]

- 1-D periodicity, $\epsilon(x, z) = \epsilon(x, z + \Lambda)$.
- 2-D TE setting, $E_y(x, z) = ?$, $(\partial_x^2 + \partial_z^2 + k^2\epsilon)E_y = 0$. (*)
- Look for FB waves $E_y(x, z) = u(x, z) e^{-i\beta z}$.
(β : the FB wavenumber, $u(x, z) = u(x, z + \Lambda) \forall z$)
- $E_y(x, z + \Lambda) = u(x, z + \Lambda) e^{-i\beta(z + \Lambda)} = E_y(x, z) e^{-i\beta\Lambda}$
 Restrict (*) to $z \in [0, \Lambda]$ with boundary conditions
 $E_y(x, \Lambda) = e^{-i\beta\Lambda} E_y(x, 0)$, $\partial_z E_y(x, \Lambda) = e^{-i\beta\Lambda} \partial_z E_y(x, 0)$.
- Brillouin zone: $K\Lambda = \pm m 2\pi \rightsquigarrow \beta \in [-\pi/\Lambda, \pi/\Lambda]$.





(BEP simulations (Lecture G.24), ω given, β determined from an eigenvalue problem.)
 (Shaded region: above the “light line”, $\omega^2 n_b^2 / c^2 > k_z^2$, potentially leaky solutions.)

Defect waveguides

(At a frequency in the bandgap of a photonic crystal: \exists “forbidden” regions  The waves travel elsewhere . . .)

Line defects in a square lattice of dielectric rods,
excitation through conventional waveguides, 2-D QUEP simulations.

- A straight defect waveguide. 
- 90° corner in a defect waveguide. 

A touch of plasmonics

“Plasmonics”: ?

Keywords:

- A branch of photonics.
- Optics involving metals and metal surfaces.
- Interaction between the electromagnetic field and free electrons in the metal / at the surface.
- Strong field confinement, “beyond the diffraction limit”.
- “Strong” local fields, near field enhancement (nonlinearity).
- “Small” structures: Nano
- Applications: Sensing, focusing (“antennas”, microscopy), communication (short-range), chemistry, art.

Context of this lecture:

- Problems of general classical electromagnetics & methods as discussed; different emphasis.
- Presence of metals: complex (negative) permittivity, strong [dispersion](#), [losses](#); some concepts do not apply.
- Among the phenomena not encountered so far: [Surface plasmon polaritons](#) (SPPs).

Surface plasmon polaritons

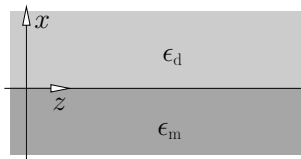
(Surface waves,

“plasmon”: oscillations of the free electron plasma,

“polariton”: strong interaction of the optical e.m. field with polarizable matter; here discussed merely as . . .)

Optical waves confined at a metal / dielectric interface.

(. . . accepting the permittivities as given, disregarding any processes in the metal or dielectric that lead to this permittivity.)



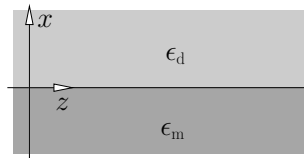
$x > 0$: dielectric, $\epsilon_d = n_d^2 \in \mathbb{R}$.

$x < 0$: metal, $\epsilon_m \in \mathbb{C}$.

(Coordinates in line with the previous discussion in this lecture, but different from literature “standard”).

Surface plasmon polaritons

$\sim \exp(i\omega t)$ (FD)



$x > 0$: dielectric, $\epsilon_d = n_d^2 \in \mathbb{R}$.

$x < 0$: metal, $\epsilon_m \in \mathbb{C}$.

2-D TE/TM waves.

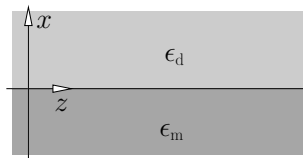
- Look for fields $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix}(x) e^{-i\gamma z}$,
 $\gamma = \beta - i\alpha \in \mathbb{C}$, $\beta, \alpha \geq 0$.
- Principal component $\phi = \bar{E}_y$ (TE) and $\phi = \bar{H}_y$ (TM),
 continuity of ϕ , $\eta \partial_x \phi$ at the interface, $\eta = 1$ (TE), $\eta = 1/\epsilon$ (TM),

$$\partial_x^2 \phi + (k^2 \epsilon - \gamma^2) \phi = 0 \quad \text{for } x < 0 \text{ and } x > 0.$$
- Ansatz:

$$\phi(x) = \begin{cases} \phi_0 e^{-ik_d x}, & x > 0, \\ \phi_0 e^{ik_m x}, & x < 0, \end{cases} \quad \begin{aligned} k_d &= \chi_d - i\kappa_d, & \kappa_d &> 0, \\ k_m &= \chi_m - i\kappa_m, & \kappa_m &> 0. \end{aligned}$$

Surface plasmon polaritons

$\sim \exp(i\omega t)$ (FD)



$x > 0$: dielectric, $\epsilon_d = n_d^2 \in \mathbb{R}$.

$x < 0$: metal, $\epsilon_m \in \mathbb{C}$.

- $x > 0$: $k^2 \epsilon_d - k_d^2 - \gamma^2 = 0$,
 $x < 0$: $k^2 \epsilon_m - k_m^2 - \gamma^2 = 0$.

- $x = 0$: Continuity of ϕ .

(Ansatz.)

$x = 0$: Continuity of $\eta \partial_x \phi \rightsquigarrow -k_d \eta_d = k_m \eta_m$.

(TE): $-k_d = k_m \rightsquigarrow$ No TE solution.

(Required: $\kappa_d > 0$ & $\kappa_m > 0$.)

(TM): $-\frac{k_d}{\epsilon_d} = \frac{k_m}{\epsilon_m}$.

(OK, if $\text{Re } \epsilon_m < 0$.)

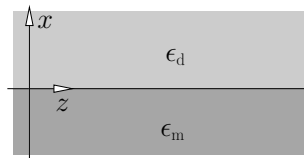
(No solution for an interface between pure dielectrics.)

$\hookrightarrow \gamma = \frac{\omega}{c} \sqrt{\frac{\epsilon_d \epsilon_m}{\epsilon_d + \epsilon_m}}$, the dispersion equation for SPPs.

(Note that, in general, $\epsilon_m(\omega)$.)

Surface plasmon polaritons

$\sim \exp(i\omega t)$ (FD)



$x > 0$: dielectric, $\epsilon_d = n_d^2 \in \mathbb{R}$.

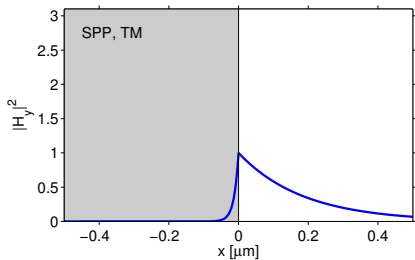
$x < 0$: metal, $\epsilon_m \in \mathbb{C}$.

Characteristic lengths:

- $x > 0$: $|\phi(x)|^2 \sim e^{-2\kappa_d x} \rightsquigarrow d_d = \frac{1}{2\kappa_d}$.
(Penetration depth, dielectric.)
- $x < 0$: $|\phi(x)|^2 \sim e^{2\kappa_m x} \rightsquigarrow d_m = \frac{1}{2\kappa_m}$.
(Penetration depth, metal.)
- $|E.|^2 \sim e^{-2\alpha z} \rightsquigarrow L_p = \frac{1}{2\alpha}$, the SPP propagation length.

Field profiles

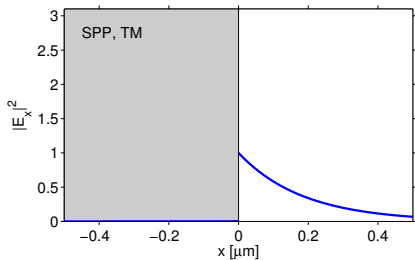
SPP, Ag/air, $\lambda = 0.633 \mu\text{m}$,
 $\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16 \mu\text{m}$,
 $\beta/k = 1.036$,
 $d_d = 190 \text{ nm}$,
 $d_m = 12 \text{ nm}$.

Field profiles

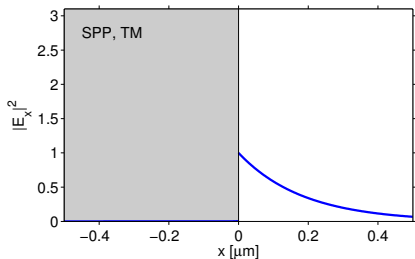
SPP, Ag/air, $\lambda = 0.633 \mu\text{m}$,
 $\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16 \mu\text{m}$,
 $\beta/k = 1.036$,
 $d_d = 190 \text{ nm}$,
 $d_m = 12 \text{ nm}$.

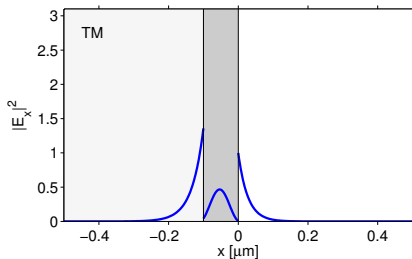
Field profiles

SPP, Ag/air, $\lambda = 0.633 \mu\text{m}$,
 $\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16 \mu\text{m}$,
 $\beta/k = 1.036$,
 $d_d = 190 \text{ nm}$,
 $d_m = 12 \text{ nm}$.

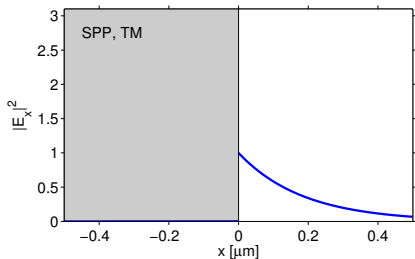
$\text{SiO}_2/\text{Si}(100 \text{ nm})/\text{air}$, $\lambda = 0.633 \mu\text{m}$,
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$,
 $n_{\text{eff}} = 2.106$,
 $d_{\text{air}} = 27 \text{ nm}$.

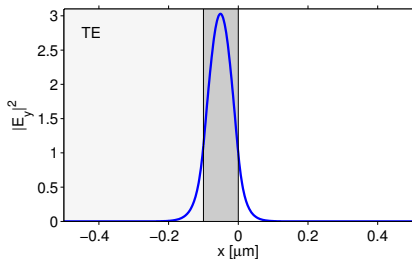
Field profiles

SPP, Ag/air, $\lambda = 0.633 \mu\text{m}$,
 $\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16 \mu\text{m}$,
 $\beta/k = 1.036$,
 $d_d = 190 \text{ nm}$,
 $d_m = 12 \text{ nm}$.

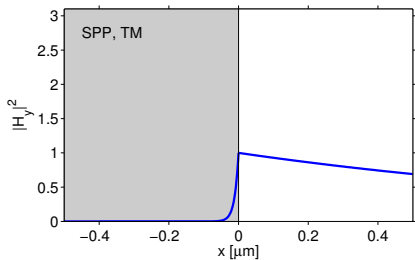
$\text{SiO}_2/\text{Si}(100 \text{ nm})/\text{air}$, $\lambda = 0.633 \mu\text{m}$,
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$,
 $n_{\text{eff}} = 2.883$,
 $d_{\text{air}} = 19 \text{ nm}$.

Field profiles

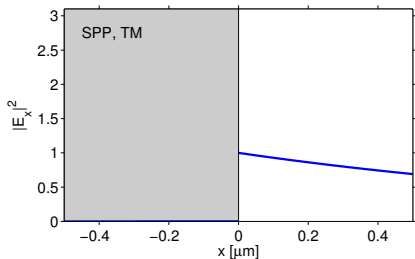
SPP, Ag/air, $\lambda = 1.550 \mu\text{m}$,
 $\epsilon_m = -121 - 4.4i$, $\epsilon_d = 1.0$



$L_p = 812 \mu\text{m}$,
 $\beta/k = 1.0042$,
 $d_d = 1350 \text{ nm}$,
 $d_m = 11 \text{ nm}$.

Field profiles

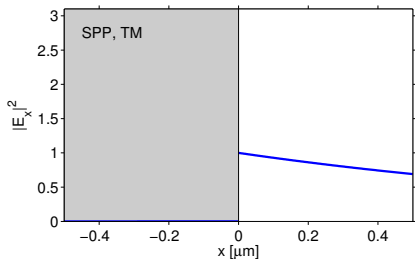
SPP, Ag/air, $\lambda = 1.550 \mu\text{m}$,
 $\epsilon_m = -121 - 4.4i$, $\epsilon_d = 1.0$



$L_p = 812 \mu\text{m}$,
 $\beta/k = 1.0042$,
 $d_d = 1350 \text{ nm}$,
 $d_m = 11 \text{ nm}$.

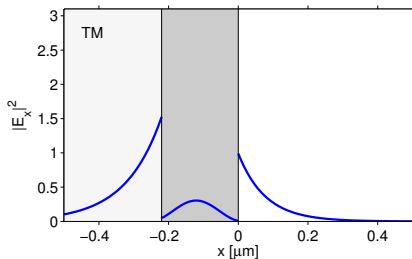
Field profiles

SPP, Ag/air, $\lambda = 1.550 \mu\text{m}$,
 $\epsilon_m = -121 - 4.4i$, $\epsilon_d = 1.0$



$L_p = 812 \mu\text{m}$,
 $\beta/k = 1.0042$,
 $d_d = 1350 \text{ nm}$,
 $d_m = 11 \text{ nm}$.

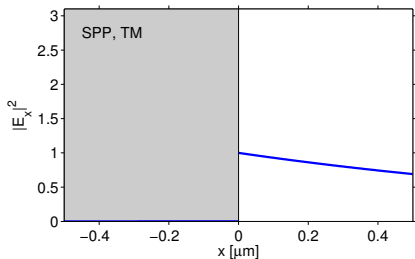
$\text{SiO}_2/\text{Si}(220 \text{ nm})/\text{air}$, $\lambda = 1.550 \mu\text{m}$,
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$,
 $n_{\text{eff}} = 1.874$,
 $d_{\text{air}} = 78 \text{ nm}$.

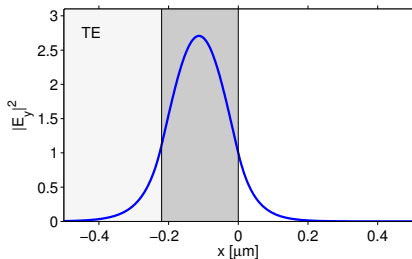
Field profiles

SPP, Ag/air, $\lambda = 1.550 \mu\text{m}$,
 $\epsilon_m = -121 - 4.4i$, $\epsilon_d = 1.0$



$L_p = 812 \mu\text{m}$,
 $\beta/k = 1.0042$,
 $d_d = 1350 \text{ nm}$,
 $d_m = 11 \text{ nm}$.

$\text{SiO}_2/\text{Si}(220 \text{ nm})/\text{air}$, $\lambda = 1.550 \mu\text{m}$,
 $\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$,
 $n_{\text{eff}} = 2.805$,
 $d_{\text{air}} = 47 \text{ nm}$.

Upcoming

Next lecture:

- — .

