Optical Waveguide Theory (B)

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Course overview

Optical waveguide theory

A Photonics / integrated optics; theory, motto; phenomena, introductory examples.

B Brush up on mathematical tools.

C Maxwell equations, different formulations, interfaces, energy and power flow.

D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.

E Normal modes of dielectric optical waveguides, mode interference.

F Examples for dielectric optical waveguides.

G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.

H Bent optical waveguides; whispering gallery resonances; circular microresonators.

I Coupled mode theory, perturbation theory.

J A touch of photonic crystals; a touch of plasmonics.

- Hybrid analytical / numerical coupled mode theory.
- Oblique semi-guided waves: 2-D integrated optics.

Vector calculus, keywords

Ingredients:

(here: Cartesian coordinates)

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- Time derivatives: $\dfrac{\partial \phi}{\partial t},\ \partial_t \phi,\ \dot{\phi},\ \cancel{\nabla_t \phi}.$

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- Laplacian: $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \boldsymbol{\nabla}^2,$

$$\Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_x^2\phi, \qquad \Delta\boldsymbol{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$$

Dirac delta

A linear functional
that extracts the value of a function at one point:

1-D: $$\int_a^b f(x)\,\delta(x-x_0)\,\mathrm{d}x = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x-x_0) = 0, \text{ if } x \neq x_0.$$

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3-D: $$\int_{\mathcal{V}} f(\boldsymbol{r})\,\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}\mathcal{V} = \begin{cases} f(\boldsymbol{r}_0), & \text{if } \boldsymbol{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise}; \end{cases}$$
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Implications: manifold.

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

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- $\delta(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k$.

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4-D: A field $\phi(\boldsymbol{r}, t)$:

$$\phi(\boldsymbol{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iint \tilde{\phi}(\boldsymbol{k}, \omega)\ \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 k\, \mathrm{d}\omega,$$

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Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y),\ldots,F(x,y)$.

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... a set of DEs in one unknown.

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... a DE in one unknown, with parameter k.

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(& boundary conditions, ...)

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

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- Functional: $$\begin{array}{rcl} \mathcal{L}: \quad U & \longrightarrow & \mathbb{R}, \mathbb{C}, \\ u & \longrightarrow & \mathcal{L}(u), \end{array}$$

  a map from a space U of functions to real / complex numbers.

A touch of variational calculus

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- Stationary functional: $$\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{L}(u + s\, v) \Big|_{s=0} = 0 \quad \text{for all } \; v,$$

  the variation of $\mathcal{L}$ at u vanishes for arbitrary directions v.

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- Restriction of a functional:

  . . . to a parametrized family of functions;

  $\leftrightsquigarrow$ extremization with respect to these parameters,
  $\leftrightsquigarrow$ approximations of stationary points of the functional.

Example:

$$U = \{u : [0,\pi] \to \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \to \mathbb{R}, \qquad \mathcal{L}(u) = \frac{\displaystyle\int_0^\pi (\partial_x u)^2 \mathrm{d}x}{\displaystyle\int_0^\pi u^2 \mathrm{d}x}.$$

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$$\partial_x^2 u = -\lambda u, \quad \lambda = \mathcal{L}(u),$$

$$u(0) = u(\pi) = 0.$$

$\downarrow$ Restrict $\mathcal{L}$, $\quad L(\boldsymbol{a}) = \mathcal{L}(u|_{\boldsymbol{a}})$.

L stationary at $\boldsymbol{a}$, $\quad \boldsymbol{\nabla}_{\boldsymbol{a}} L = 0.$ $\leftrightsquigarrow$ Approximate solution of DE / eigenproblem.

Upcoming

Next lectures:

- Maxwell equations, different formulations, interfaces, energy and power flow.
- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.