Optical Waveguide Theory (C)

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Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.
- Summary, concluding remarks.

. . . ?

"This concerns time harmonic fields . . . with angular frequency . . . , for vacuum wavenumber . . . , speed of light . . . , and wavelength . . . ."

"The problem is governed by the Maxwell curl equations in the frequency domain for the electric field . . . and magnetic field . . . , for (lossless) uncharged dielectric, nonmagnetic linear (isotropic) media with (piecewise constant) relative permittivity . . . :

. . . (.) "

[M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016)]

Maxwell equations, Fourier transform

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\mathrm{f}}+\dot{\boldsymbol{D}}$$

& $$\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)\ \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,\quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\frac{1}{\sqrt{2\pi}}\int\boldsymbol{F}(\boldsymbol{r},t)\ \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

Maxwell equations, Fourier transform

$$\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}} + \dot{\boldsymbol{D}}$$

$$\& \quad \boldsymbol{F}(\boldsymbol{r},t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)\ \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega, \quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega) = \frac{1}{\sqrt{2\pi}} \int \boldsymbol{F}(\boldsymbol{r},t)\ \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t$$

$$\boldsymbol{E}(\boldsymbol{r},t),\ \boldsymbol{D}(\boldsymbol{r},t),\ \boldsymbol{B}(\boldsymbol{r},t),\ \boldsymbol{H}(\boldsymbol{r},t),\ \rho_{\mathrm{f}}(\boldsymbol{r},t),\ \boldsymbol{J}_{\mathrm{f}}(\boldsymbol{r},t)$$

$$\leftrightsquigarrow \tilde{\boldsymbol{E}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{D}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{H}}(\boldsymbol{r},\omega),\ \tilde{\rho}_{\mathrm{f}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{J}}_{\mathrm{f}}(\boldsymbol{r},\omega),$$

$$\nabla \cdot \tilde{\boldsymbol{D}} = \tilde{\rho}_{\mathrm{f}}, \quad \nabla \times \tilde{\boldsymbol{E}} = -\mathrm{i}\omega\tilde{\boldsymbol{B}}, \quad \nabla \cdot \tilde{\boldsymbol{B}} = 0, \quad \nabla \times \tilde{\boldsymbol{H}} = \tilde{\boldsymbol{J}}_{\mathrm{f}} + \mathrm{i}\omega\tilde{\boldsymbol{D}}$$

(Caution: arbitrary choice of $\sim \exp(\pm\mathrm{i}\,\omega t)$!).

Maxwell equations, frequency domain

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}}=\tilde{\rho}_{\mathrm{f}},\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}=-\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}}=0,\quad \boldsymbol{\nabla}\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm{f}}+\mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

$$\boldsymbol{F}(\boldsymbol{r},t)\in\mathbb{R} \rightsquigarrow \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega)=(\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^{*}$$

Maxwell equations, frequency domain

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$$\boldsymbol{F}(\boldsymbol{r},t)\in\mathbb{R} \rightsquigarrow \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega)=(\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^*$$

"at frequency ω_0": $\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\delta(\omega-\omega_0)+\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\delta(\omega+\omega_0)$

Maxwell equations, frequency domain

$$\nabla\cdot\tilde{\boldsymbol{D}}=\tilde{\rho}_{\mathrm{f}},\quad \nabla\times\tilde{\boldsymbol{E}}=-\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \nabla\cdot\tilde{\boldsymbol{B}}=0,\quad \nabla\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm{f}}+\mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

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$$\hookrightarrow \boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{2}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}+\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\right\},$$

$$\boldsymbol{F}(\boldsymbol{r},t)=\mathrm{Re}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}\right\},$$

$$\text{“}\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{2}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}+\text{c.c.”}.$$

Maxwell equations, frequency domain

$$\nabla\cdot\tilde{\boldsymbol{D}}=\tilde{\rho}_{\mathrm{f}},\quad \nabla\times\tilde{\boldsymbol{E}}=-\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \nabla\cdot\tilde{\boldsymbol{B}}=0,\quad \nabla\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm{f}}+\mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

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$$\text{“}\boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{2}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}+\mathrm{c.c.}\text{”}.$$

$$\hookrightarrow\ \bar{\boldsymbol{E}}(\boldsymbol{r}),\ \bar{\boldsymbol{D}}(\boldsymbol{r}),\ \bar{\boldsymbol{B}}(\boldsymbol{r}),\ \bar{\boldsymbol{H}}(\boldsymbol{r}),\ \tilde{\rho}_{\mathrm{f}}(\boldsymbol{r}),\ \bar{\boldsymbol{J}}_{\mathrm{f}}(\boldsymbol{r}),\ \sim\exp(\mathrm{i}\omega_0 t),$$

$$\nabla\cdot\bar{\boldsymbol{D}}=\bar{\rho}_{\mathrm{f}},\quad \nabla\times\bar{\boldsymbol{E}}=-\mathrm{i}\omega_0\bar{\boldsymbol{B}},\quad \nabla\cdot\bar{\boldsymbol{B}}=0,\quad \nabla\times\bar{\boldsymbol{H}}=\bar{\boldsymbol{J}}_{\mathrm{f}}+\mathrm{i}\omega_0\bar{\boldsymbol{D}}.$$

Caution: Decorations $\tilde{}$, $\bar{}$, $_0$ are ususally omitted; context determines interpretation of symbols.

Polarization

$\tilde{\boldsymbol{P}}$: density of electric dipole moment (bound charges).

$$\tilde{\boldsymbol{D}} = \epsilon_0 \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{P}}, \qquad [\tilde{\boldsymbol{D}}] = [\tilde{\boldsymbol{P}}] = \frac{\mathrm{As\,m}}{\mathrm{m}^3}, \ [\tilde{\boldsymbol{E}}] = \frac{\mathrm{V}}{\mathrm{m}},$$

- vacuum permittivity $\epsilon_0 = 8.854187817\ldots \cdot 10^{-12} \left[\frac{\mathrm{F}}{\mathrm{m}} = \frac{\mathrm{As}}{\mathrm{Vm}}\right]$.

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- Local dipoles induced by $\tilde{\boldsymbol{E}} \rightsquigarrow \tilde{\boldsymbol{P}}(\tilde{\boldsymbol{E}})$.
- Linear dielectrica:

$$\tilde{\boldsymbol{P}} = \epsilon_0 \hat{\chi}_\mathrm{e} \tilde{\boldsymbol{E}}, \qquad \hat{\chi}_\mathrm{e}: \text{ dielectric susceptibility, } [\hat{\chi}_\mathrm{e}] = \hat{1}.$$

$$\hookrightarrow \tilde{\boldsymbol{D}} = \epsilon_0(\hat{1} + \hat{\chi}_\mathrm{e})\tilde{\boldsymbol{E}} = \epsilon_0 \hat{\epsilon} \tilde{\boldsymbol{E}}, \qquad \hat{\epsilon}: \text{ relative permittivity, } [\hat{\epsilon}] = \hat{1}.$$

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$\hookrightarrow \tilde{\boldsymbol{D}} = \epsilon_0(\hat{1} + \hat{\chi}_\mathrm{e})\tilde{\boldsymbol{E}} = \epsilon_0 \hat{\epsilon} \tilde{\boldsymbol{E}}$, $\quad \hat{\epsilon}$: relative permittivity, $[\hat{\epsilon}] = \hat{1}$.

- $\hat{\chi}_\mathrm{e}(\boldsymbol{r}, \omega)$, $\hat{\epsilon}(\boldsymbol{r}, \omega)$ are determined in the frequency domain.
- Complications: $\mathrm{Im}\,\epsilon$, $\hat{\epsilon}(T)$, $\hat{\epsilon}(\boldsymbol{F})$, $\chi^{(2)}_{jkl} E_k E_l$, $\chi^{(3)}_{jklm} E_k E_l E_m$, ...
- Simpler cases: $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\epsilon} = \epsilon \hat{1}$.

Magnetization

$\tilde{\boldsymbol{M}}$: density of magnetic dipole moments (bound currents).

$$\tilde{\boldsymbol{H}} = \frac{1}{\mu_0}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{M}}, \qquad [\tilde{\boldsymbol{H}}] = [\boldsymbol{M}] = \frac{\mathrm{A\,m^2}}{\mathrm{m^3}}, \; [\tilde{\boldsymbol{B}}] = \mathrm{T} = \frac{\mathrm{Vs}}{\mathrm{m^2}},$$

vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{\mathrm{N}}{\mathrm{A^2}} = \frac{\mathrm{Vs}}{\mathrm{Am}} \right]$.

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- Local dipoles induced by $\tilde{\boldsymbol{H}}$ $\rightsquigarrow$ $\tilde{\boldsymbol{M}}(\tilde{\boldsymbol{H}})$.
- Linear magnetic media:

$$\tilde{\boldsymbol{M}} = \hat{\chi}_\mathrm{m}\tilde{\boldsymbol{H}}, \qquad \hat{\chi}_\mathrm{m}\text{: magnetic susceptibility, } [\hat{\chi}_\mathrm{m}] = \hat{1}.$$

$$\hookrightarrow\ \tilde{\boldsymbol{B}} = \mu_0(\hat{1} + \hat{\chi}_\mathrm{m})\tilde{\boldsymbol{H}} = \mu_0\hat{\mu}\tilde{\boldsymbol{H}}, \quad \hat{\mu}\text{: relative permeability, } [\hat{\mu}] = \hat{1}.$$

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$$\hookrightarrow \tilde{\boldsymbol{B}} = \mu_0(\hat{1} + \hat{\chi}_{\mathrm{m}})\tilde{\boldsymbol{H}} = \mu_0\hat{\mu}\tilde{\boldsymbol{H}},\ \hat{\mu}:\ \text{relative permeability},\ [\hat{\mu}] = \hat{1}.$$

- $\hat{\chi}_{\mathrm{m}}(\boldsymbol{r},\omega)$, $\hat{\mu}(\boldsymbol{r},\omega)$ are determined in the frequency domain.
- Complications: manifold.
- Traditional integrated optics (frequencies, media): $\hat{\mu}(\cancel{\boldsymbol{r}}) = \hat{1}$.

Maxwell equations, dispersion

(Material) dispersion: $\hat{\epsilon}(\boldsymbol{r}, \omega)$, $\hat{\mu}(\boldsymbol{r}, \omega)$ are frequency dependent.

$$\tilde{\boldsymbol{D}}(\boldsymbol{r}, \omega) = \epsilon_0 \hat{\epsilon}(\boldsymbol{r}, \omega)\tilde{\boldsymbol{E}}(\boldsymbol{r}, \omega), \quad \tilde{\boldsymbol{B}}(\boldsymbol{r}, \omega) = \mu_0 \hat{\mu}(\boldsymbol{r}, \omega)\tilde{\boldsymbol{H}}(\boldsymbol{r}, \omega)$$

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$$\hookrightarrow \quad \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \int \hat{\epsilon}_{\mathrm{TD}}(\boldsymbol{r},t-t')\boldsymbol{E}(\boldsymbol{r},t')\,\mathrm{d}t',$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \int \hat{\mu}_{\mathrm{TD}}(\boldsymbol{r},t-t')\boldsymbol{H}(\boldsymbol{r},t')\,\mathrm{d}t'.$$

Helmholtz equations

Linear dielectric media without free charges or currents,
time dependence $\sim \exp(\mathrm{i}\omega t)$, fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$,
material properties $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\mu}(\boldsymbol{r})$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega \boldsymbol{B}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega \boldsymbol{D},$$
$$\boldsymbol{D} = \epsilon_0 \hat{\epsilon}\, \boldsymbol{E}, \quad \boldsymbol{B} = \mu_0 \hat{\mu}\, \boldsymbol{H}.$$

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$$\boldsymbol{D}=\epsilon_0\hat{\epsilon}\,\boldsymbol{E},\quad \boldsymbol{B}=\mu_0\hat{\mu}\,\boldsymbol{H}.$$

$$\hookrightarrow\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H},\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E},\qquad \boldsymbol{\nabla}\cdot\hat{\epsilon}\boldsymbol{E}=0,\quad \boldsymbol{\nabla}\cdot\hat{\mu}\boldsymbol{H}=0.$$

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$$\boldsymbol{D}=\epsilon_0\hat{\epsilon}\boldsymbol{E},\quad \boldsymbol{B}=\mu_0\hat{\mu}\boldsymbol{H}.$$

$$\hookrightarrow\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H},\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E},\qquad \boldsymbol{\nabla}\cdot\hat{\epsilon}\boldsymbol{E}=0,\quad \boldsymbol{\nabla}\cdot\hat{\mu}\boldsymbol{H}=0.$$

$$\hookrightarrow\quad \boldsymbol{\nabla}\times(\hat{\mu}^{-1}\boldsymbol{\nabla}\times\boldsymbol{E})=\omega^2\epsilon_0\mu_0\hat{\epsilon}\boldsymbol{E}\quad\text{or}\quad \boldsymbol{\nabla}\times(\hat{\epsilon}^{-1}\boldsymbol{\nabla}\times\boldsymbol{H})=\omega^2\epsilon_0\mu_0\hat{\mu}\boldsymbol{H}.$$

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Linear dielectric media without free charges or currents,
time dependence $\sim \exp(\mathrm{i}\omega t)$, fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$,
material properties $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\mu}(\boldsymbol{r})$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\mathrm{i}\omega\boldsymbol{B}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \mathrm{i}\omega\boldsymbol{D},$$
$$\boldsymbol{D} = \epsilon_0\hat{\epsilon}\,\boldsymbol{E}, \quad \boldsymbol{B} = \mu_0\hat{\mu}\,\boldsymbol{H}.$$

$$\hookrightarrow \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}, \qquad \boldsymbol{\nabla}\cdot\hat{\epsilon}\,\boldsymbol{E} = 0, \quad \boldsymbol{\nabla}\cdot\hat{\mu}\,\boldsymbol{H} = 0.$$

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Where $\hat{\epsilon} = \epsilon\hat{1}$, $\boldsymbol{\nabla}\epsilon = 0$, $\hat{\mu} = \mu\hat{1}$, $\boldsymbol{\nabla}\mu = 0$: (!)

$$\hookrightarrow \quad \Delta\boldsymbol{E} + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\,\boldsymbol{E} = 0 \quad \text{or} \quad \Delta\boldsymbol{H} + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\,\boldsymbol{H} = 0, \quad \mathrm{c} = \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$

Plane harmonic waves

Where $\hat{\epsilon} = \epsilon \hat{1}$, $\nabla \epsilon = 0$, $\hat{\mu} = \mu \hat{1}$, $\nabla \mu = 0$: (!)

Components of $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy

$$\Delta \psi + \frac{\omega^2}{c^2} \epsilon \mu \, \psi = 0.$$

$$\psi(\boldsymbol{r}, t) = \psi_0 \, e^{-i(\boldsymbol{k}_m \cdot \boldsymbol{r} - \omega t)}, \qquad -\boldsymbol{k}_m^2 + \frac{\omega^2}{c^2} \epsilon \mu = 0.$$

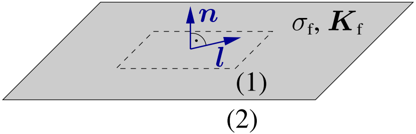
(Mixture of TD and FD expressions; ˜, ¯, Re , $1/2$, c.c. omitted; sloppy, but common.)

• Medium:	refractive index:	$n = \sqrt{\epsilon \mu}$
• Periodicity in time:	angular frequency:	ω,
	frequency:	$f = \omega/(2\pi)$,
	period:	$T = 1/f = 2\pi/\omega$,
• Spatial periodicity:	wave vector:	$\boldsymbol{k}_m$, $k_m = \lvert \boldsymbol{k}_m \rvert$,
	wavenumber:	$k_m = \omega/c_m = (\omega/c) n = k \, n$,
	vacuum wavenumber:	$k = \omega/c$,
	vacuum wavelength:	$\lambda = 2\pi/k = 2\pi c/\omega$,
	wavelength in the medium:	$\lambda_m = 2\pi/k_m = 2\pi/(kn) = \lambda/n$.
• Phase velocity:	speed of light in vacuum:	$c = 1/\sqrt{\epsilon_0 \mu_0} = \lambda f$,
	in the medium:	$c_m = c/n = \lambda_m f$.

(Use of symbols depends highly on context.)

Electromagnetic spectrum

Interface conditions

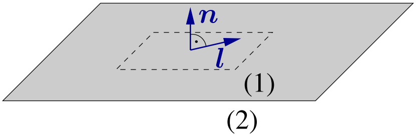


Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface charge density σ_f, surface current density $\boldsymbol{K}_\mathrm{f}$:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2)=\sigma_\mathrm{f},\qquad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$

$$\boldsymbol{n}\cdot(\boldsymbol{B}_1-\boldsymbol{B}_2)=0,\qquad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=\boldsymbol{l}\cdot(\boldsymbol{K}_\mathrm{f}\times\boldsymbol{n}).$$

Interface conditions

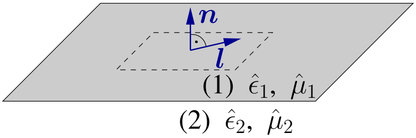


Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface without free charges or currents:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2)=0, \qquad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$

$$\boldsymbol{n}\cdot(\boldsymbol{B}_1-\boldsymbol{B}_2)=0, \qquad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=0\,.$$

Interface conditions



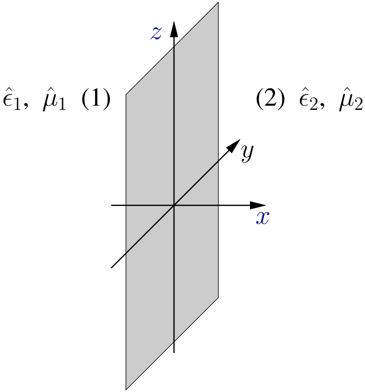
Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, linear media with permittivities $\hat{\epsilon}_1$, $\hat{\epsilon}_2$, and permeabilities $\hat{\mu}_1$, $\hat{\mu}_2$:

$$\boldsymbol{n} \cdot (\hat{\epsilon}_1 \boldsymbol{E}_1 - \hat{\epsilon}_2 \boldsymbol{E}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0,$$

$$\boldsymbol{n} \cdot (\hat{\mu}_1 \boldsymbol{H}_1 - \hat{\mu}_2 \boldsymbol{H}_2) = 0, \qquad \boldsymbol{l} \cdot (\boldsymbol{H}_1 - \boldsymbol{H}_2) = 0.$$

Reflection and transmission of plane waves at dielectric interfaces

(FD)



- $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}$,
  $\boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}$
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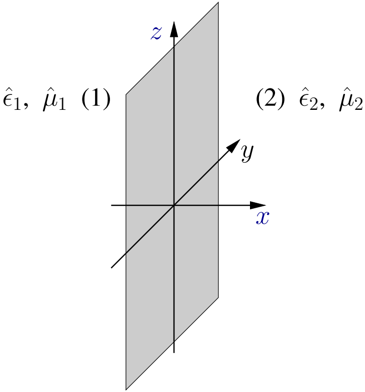

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1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

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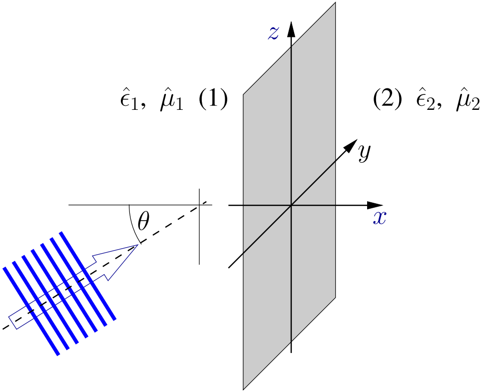

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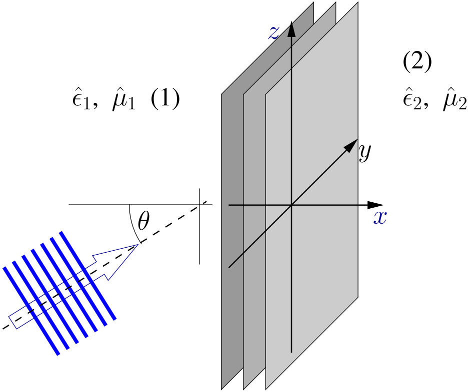
1-D problem for $\boldsymbol{E}',\ \boldsymbol{H}'$.

(incoming plane wave at angle θ)
(orient coordinates ($k_y = 0$), plane of incidence, distinguish polarizations)
(write ansatz functions for incoming, reflected, and transmitted waves)
(interface conditions determine the amplitudes)

Fresnel equations.

Dielectric multilayer structures

(FD)



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(. . . .)
(. . . .)
(. . . .)
(. . . .)

Reflectance and transmittance properties.

Energy of electromagnetic fields

(TD)

- Force on a particle with charge q, velocity $\boldsymbol{v}$, in a field $\boldsymbol{E}, \boldsymbol{B}$:
  $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

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- work for shifting the particle by $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$:
  $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\,\mathrm{d}t = q\boldsymbol{E} \cdot \boldsymbol{v}\,\mathrm{d}t$,

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For a charge density $\rho_\mathrm{f}(\boldsymbol{r}, t)$:

force density $\boldsymbol{f} = \rho_\mathrm{f}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

power density $\boldsymbol{f} \cdot \boldsymbol{v} = \rho_\mathrm{f}\boldsymbol{E} \cdot \boldsymbol{v} = \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}$,

total work per time unit done in $\mathcal{V}$: $\dfrac{\mathrm{d}W_\mathcal{V}}{\mathrm{d}t} = \displaystyle\int_\mathcal{V} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}\,\mathrm{d}\mathcal{V}$.

Power & energy density, Poynting theorem

(TD)

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} W_{\mathcal{V}}^{\mathrm{mech}} = \int_{\mathcal{V}} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E} \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}}) \,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \,\mathrm{d}\mathcal{V},$$

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- $\hat{\epsilon}^\dagger = \hat{\epsilon}, \ \hat{\epsilon} \cancel{(\omega)}, \ \boldsymbol{D} = \epsilon_0 \hat{\epsilon} \boldsymbol{E}, \ \hat{\mu}^\dagger = \hat{\mu}, \ \hat{\mu} \cancel{(\omega)}, \ \boldsymbol{B} = \mu_0 \hat{\mu} \boldsymbol{H}$ (!)

  $\rightsquigarrow \quad \dot{w} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$

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$$\rightsquigarrow \quad \dot{w} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$$

$\mathcal{V}$ arbitrary

$$\dot{w} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(W_\mathcal{V}^\mathrm{mech} + W_\mathcal{V}^\mathrm{field}\right) = -\oint_{\partial\mathcal{V}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}.$$

Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (. . .) linear media:

$$w = \frac{1}{2}(\epsilon_0 \boldsymbol{E} \cdot \hat{\epsilon}\boldsymbol{E} + \mu_0 \boldsymbol{H} \cdot \hat{\mu}\boldsymbol{H}), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \quad \dot{w} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = 0,$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\, \tilde{\boldsymbol{E}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{H}(\boldsymbol{r}, t) = \mathrm{Re}\, \tilde{\boldsymbol{H}}(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\omega t}$$

$\boldsymbol{S}$, w oscillate in time.

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$\hookrightarrow$ $\boldsymbol{S}$, w oscillate in time.

(FD)

Consider time-averaged quantities: $\bar{f}(t) = \dfrac{1}{T} \displaystyle\int_t^{t+T} f(t')\, \mathrm{d}t'$

(exercise)

$$\hookrightarrow \quad \overline{w} = \frac{1}{4} \mathrm{Re} \left(\epsilon_0 \tilde{\boldsymbol{E}}^* \cdot \hat{\epsilon}\, \tilde{\boldsymbol{E}} + \mu_0 \tilde{\boldsymbol{H}}^* \cdot \hat{\mu}\, \tilde{\boldsymbol{H}}\right), \quad \overline{\boldsymbol{S}} = \frac{1}{2} \mathrm{Re} \left(\tilde{\boldsymbol{E}}^* \times \tilde{\boldsymbol{H}}\right).$$

$$\overline{\dot{w}} = \dot{\overline{w}} = 0, \quad \overline{\boldsymbol{\nabla} \cdot \boldsymbol{S}} = \boldsymbol{\nabla} \cdot \overline{\boldsymbol{S}} \quad \rightsquigarrow \quad \boldsymbol{\nabla} \cdot \overline{\boldsymbol{S}} = 0, \quad \oint_{\mathcal{V}} \overline{\boldsymbol{S}} \cdot \mathrm{d}\boldsymbol{a} = 0;$$

"power balance", conservation of energy.

Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law $\quad \boldsymbol{J}_{\mathrm{f}} = \sigma \boldsymbol{E}, \quad \sigma$: conductivity of the material.

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$$\dot{\rho}_{\mathrm{f}} = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_{\mathrm{f}}, \quad \rho_{\mathrm{f}}(\boldsymbol{r}, t) = \rho_{\mathrm{f}}(\boldsymbol{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0)\right),$$

assume $\rho_{\mathrm{f}}(\boldsymbol{r}, t_0) = 0 \rightsquigarrow \rho_{\mathrm{f}}(\boldsymbol{r}, t) = 0 \quad \forall t.$

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$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \mu \dot{\boldsymbol{H}}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{H} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon_0 \epsilon \dot{\boldsymbol{E}}.$$

Telegrapher equation

$$\nabla\cdot\boldsymbol{E}=0,\quad \nabla\times\boldsymbol{E}=-\mu_0\mu\dot{\boldsymbol{H}},\quad \nabla\cdot\boldsymbol{H}=0,\quad \nabla\times\boldsymbol{H}=\sigma\boldsymbol{E}+\epsilon_0\epsilon\dot{\boldsymbol{E}}$$

$$\Delta\boldsymbol{E}-\epsilon_0\mu_0\epsilon\mu\ddot{\boldsymbol{E}}-\mu_0\mu\sigma\dot{\boldsymbol{E}}=0,\quad \Delta\boldsymbol{H}-\epsilon_0\mu_0\epsilon\mu\ddot{\boldsymbol{H}}-\mu_0\mu\sigma\dot{\boldsymbol{H}}=0,$$

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Telegrapher equation.

Frequency domain: $\boldsymbol{E}(\boldsymbol{r},t)=\tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}$,

$$\Delta\tilde{\boldsymbol{E}}+\left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu-\mathrm{i}\omega\mu_0\mu\sigma\right)\tilde{\boldsymbol{E}}=0.$$

Nonconducting media $\sigma=0$, $\Delta\tilde{\boldsymbol{E}}+\left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\right)\tilde{\boldsymbol{E}}=0.$

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Nonconducting media $\sigma=0$, $\Delta\tilde{\boldsymbol{E}}+\left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\right)\tilde{\boldsymbol{E}}=0.$

Define $\bar{\epsilon}$ such that $\frac{\omega^2}{\mathrm{c}^2}\bar{\epsilon}\mu=\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu-\mathrm{i}\omega\mu_0\mu\sigma$, i.e. $\bar{\epsilon}=\epsilon-\mathrm{i}\frac{\sigma}{\epsilon_0\omega}$

$$\hookrightarrow\quad \Delta\tilde{\boldsymbol{E}}+k^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}}=0,$$ Helmholtz equation, $\bar{\epsilon}\in\mathbb{C}$, $\qquad k=\frac{\omega}{\mathrm{c}}$.

Wave attenuation

$$\Delta\tilde{\boldsymbol{E}} + k^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}} = 0, \quad \bar{\epsilon} \in \mathbb{C}$$

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solutions $\sim \mathrm{e}^{-\mathrm{i}\left(k\bar{n}z - \omega t\right)}$

with refractive index $\bar{n} = n' - \mathrm{i}n'' = \sqrt{\bar{\epsilon}\mu} \in \mathbb{C}$, (!)

Wave attenuation

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$$\mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)} = \mathrm{e}^{-\mathrm{i}(kn'z - \omega t)}\,\mathrm{e}^{-kn''z},$$

damped plane wave solutions

$\leftrightsquigarrow$ sign of n'', choice of $\exp(\pm\mathrm{i}\omega)$.

Issues:

- penetration depth,
- $\boldsymbol{S}$ and w decay with z,
- still transverse waves,
- $\boldsymbol{E}$, $\boldsymbol{H}$ no longer in phase,
- notions of wavenumber, wavelength, phase velocity $\in \mathbb{C}$.

A typical setting:

- "uncharged dielectric medium": $\cancel{q_{\mathrm{f}}}$, $\cancel{\boldsymbol{J}_{\mathrm{f}}}$.
- "linear medium": $\boldsymbol{D} = \epsilon_0 \hat{\epsilon} \boldsymbol{E}$, $\boldsymbol{B} = \mu_0 \hat{\mu} \boldsymbol{H}$.
- "isotropic medium": $\hat{\epsilon} = \epsilon \, \hat{1}$, $\hat{\mu} = \mu \, \hat{1}$.
- "nonmagnetic medium": $\hat{\mu} = \hat{1}$.
- "lossless medium": $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\mu}^\dagger = \hat{\mu}$, $(\epsilon, \mu \in \mathbb{R})$.
- "piecewise constant" $\rightarrow$ "dependent on position".
- "electric and magnetic field": eliminate $\boldsymbol{D}$ and $\boldsymbol{B}$, retain $\boldsymbol{E}$ and $\boldsymbol{H}$.
- "governed by the curl equations": divergence eqns. are satisfied.
- "frequency domain, time harmonic fields, frequency, wavelength": . . . as discussed.

Upcoming

Next lectures:

- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.