Optical Waveguide Theory (E)

Manfred Hammer*

Theoretical Electrical Engineering
Paderborn University, Germany

Paderborn University — Summer Semester 2018

* *Theoretical Electrical Engineering, Paderborn University*
Warburger Straße 100, 33098 Paderborn, Germany
Phone: +49(0)5251/60-3560
E-mail: manfred.hammer@uni-paderborn.de

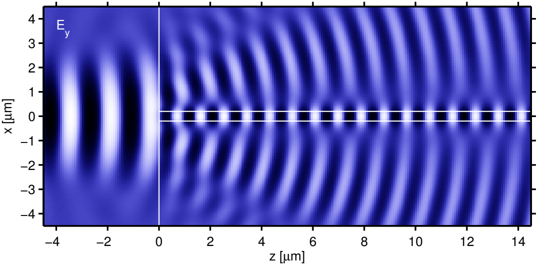
Course overview

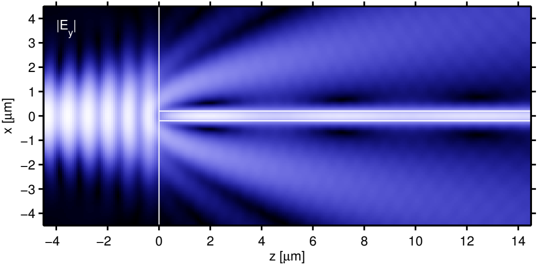
Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

Context: Relevance of guided modes

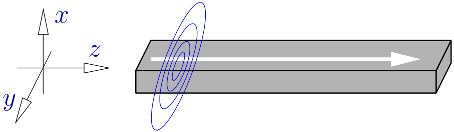
(2-D, TE)





Butt-coupling to a waveguide facet.

Waveguides: Mode problems

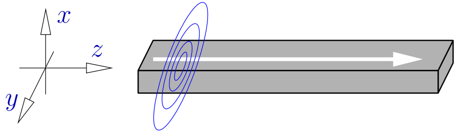


$\mu = 1,\ \epsilon = n^2,\ \sim \exp(\mathrm{i}\omega t)$ (FD)

$$\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H},$$
$$\nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}.$$

- Waveguide: a system that is homogeneous along its axis z,
  $\partial_z \epsilon = 0,\ \partial_z n = 0.$

Waveguides: Mode problems



$\mu = 1,\ \epsilon = n^2,\ \sim \exp(\mathrm{i}\omega t)$ (FD)

$$\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H},$$
$$\nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}.$$

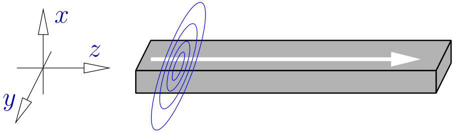
- Waveguide: a system that is homogeneous along its axis z,
  $\partial_z \epsilon = 0,\ \partial_z n = 0.$

- Look for solutions (modes) that vary harmonically with z:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z},$$

mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
propagation constant β,
effective index $n_{\mathrm{eff}} = \beta/k$.

Waveguides: Mode problems



$\mu = 1,\ \epsilon = n^2,\ \sim \exp(\mathrm{i}\omega t)$ (FD)

$$\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H},$$
$$\nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}.$$

- Waveguide: a system that is homogeneous along its axis z,
  $\partial_z \epsilon = 0,\ \partial_z n = 0.$

- Look for solutions (modes) that vary harmonically with z:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x, y)\ \mathrm{e}^{-\mathrm{i}\beta z},$$

mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
propagation constant β,
effective index $n_{\mathrm{eff}} = \beta/k$.

$$\partial_z \longrightarrow -\mathrm{i}\beta,$$

(& boundary conditions)

⇜ Eigenvalue problem with eigenvalue β, eigenfunction $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
"$\mathsf{M}(\beta)\ (\overrightarrow{\text{profile}}) = 0$".

Mode equations

(drop $^-$)

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\,\beta E_y \\ -\mathrm{i}\,\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\,\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\,\beta H_y \\ -\mathrm{i}\,\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\,\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

Mode equations

(drop $^-$)

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\,\beta E_y \\ -\mathrm{i}\,\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\,\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\,\beta H_y \\ -\mathrm{i}\,\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\,\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express E_x, E_y, E_z, H_z through principal components H_x, H_y :

$$\partial_x^2 H_x + \epsilon\partial_y \frac{1}{\epsilon}\partial_y H_x + \partial_{xy} H_y - \epsilon\partial_y \frac{1}{\epsilon}\partial_x H_y + (k^2\epsilon - \beta^2) H_x = 0\,,$$

$$\epsilon\partial_x \frac{1}{\epsilon}\partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon\partial_x \frac{1}{\epsilon}\partial_y H_x + (k^2\epsilon - \beta^2) H_y = 0\,,$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{1}{\omega\epsilon_0\epsilon} \begin{pmatrix} \beta H_y - \beta^{-1}(\partial_{yx} H_x + \partial_y^2 H_y) \\ -\beta H_x + \beta^{-1}(\partial_{xy} H_y + \partial_x^2 H_x) \\ -\mathrm{i}\,(\partial_x H_y - \partial_y H_x) \end{pmatrix}, \quad \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ -\mathrm{i}\,\beta^{-1}(\partial_x H_x + \partial_y H_y) \end{pmatrix}.$$

(H_x, H_y are continuous for all x, y.)

Mode equations

(drop $^-$)

$$\hookrightarrow \quad \begin{pmatrix} \partial_y E_z + \mathrm{i}\,\beta E_y \\ -\mathrm{i}\,\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\,\beta H_y \\ -\mathrm{i}\,\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express H_x, H_y, H_z, E_z through principal components E_x, E_y :

$\hookrightarrow$ (. . .).

(E_x, E_y are discontinuous at specific interfaces.)

Mode equations

(drop $^{-}$)

$$\hookrightarrow \quad \begin{pmatrix} \partial_y E_z + \mathrm{i}\,\beta E_y \\ -\mathrm{i}\,\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\,\beta H_y \\ -\mathrm{i}\,\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express E_x, E_y, H_x, H_y through principal components E_z, H_z :

$\hookrightarrow$ (. . .).

(E_z, H_z are usually small components.)

Plane mode profiles

- Modes are eigenfunctions
  ↭ profiles are determined up to a complex constant only.

Plane mode profiles

- Modes are eigenfunctions
  $\leftrightsquigarrow$ profiles are determined up to a complex constant only.

- Propagating modes, $\beta \in \mathbb{R}$, lossless structures, $\epsilon \in \mathbb{R}$:

  $E_z := \mathrm{i}E_z'$, $H_z := \mathrm{i}H_z'$ $\rightsquigarrow$ real PDE for E_x, E_y, E_z', H_x, H_y, H_z' :

$$\begin{pmatrix} \partial_y E_z' + \beta E_y \\ -\beta E_x - \partial_x E_z' \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ -H_z' \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z' + \beta H_y \\ -\beta H_x - \partial_x H_z' \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ -E_z' \end{pmatrix};$$

  it is possible to choose a phase such that
  E_x, E_y, H_x, H_y are real,
  E_z, H_z are imaginary

  $\leftrightsquigarrow$ plane mode profiles.

(It makes sense to prepare real plots of mode profile components.)
(That requires a suitable adjustment of the global phase.)

Guided modes

- Guided modes: profiles located "around" the waveguide core

  $\rightsquigarrow$ discrete $\beta \in \mathbb{R}$, $\iint S_z \,\mathrm{d}x\,\mathrm{d}y < \infty$.

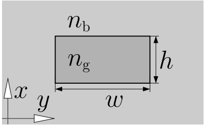
- In general: Hybrid modes, all six field components present.
  Planar-like waveguides $\rightsquigarrow$ adapt 2-D naming scheme;
  "TE-like" / "TM-like" modes.

  ($\leftrightarrow$ 5-component semivectorial approximations, plane $\perp$ x-axis:
  quasi-TE: tiny E_x, dominant E_y, small E_z; major H_x, small H_y, minor H_z,
  quasi-TM: tiny H_x, dominant H_y, small H_z; major E_x, small E_y, minor E_z.)

- Mode indices mostly relate to numbers of nodal lines in the dominant electric or magnetic field component.

(Naming schemes are highly context dependent.)

A rectangular strip waveguide, fundamental mode profiles



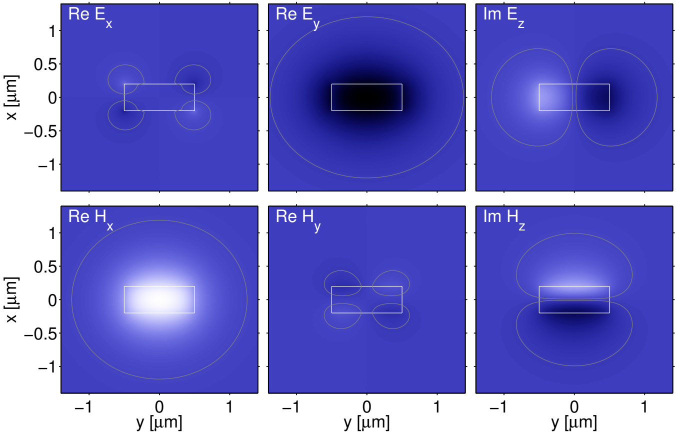
$\lambda = 1.55\ \mu m$,
$n_b = 1.45$,
$n_g = 1.99$,
$w = 1.0\ \mu m$,
$h = 0.4\ \mu m$;

(q-) $TE_{0\,0}$

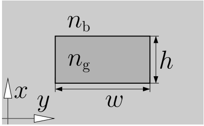
$x \in [-2, 2]\ \mu m$,
$y \in [-2, 2]\ \mu m$;

$n_{eff} = 1.63554$

[JCMwave].



A rectangular strip waveguide, fundamental mode profiles



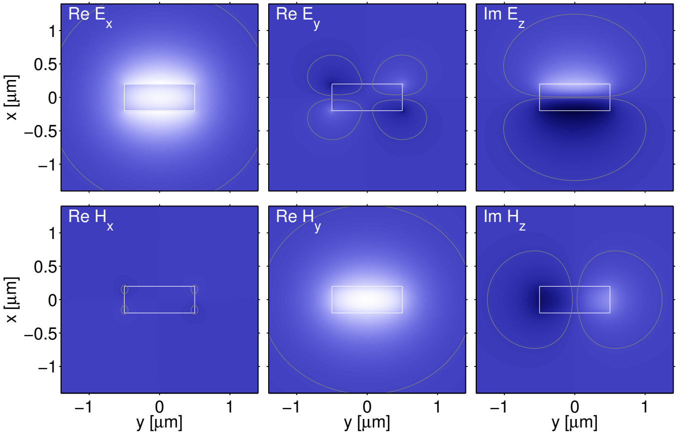
$\lambda = 1.55$ µm,
$n_b = 1.45$,
$n_g = 1.99$,
$w = 1.0$ µm,
$h = 0.4$ µm;

(q-) $TM_{0\,0}$

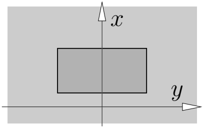
$x \in [-2, 2]$ µm,
$y \in [-2, 2]$ µm;

$n_{eff} = 1.56809$

[JCMwave].



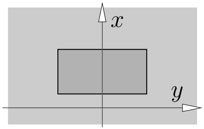
Symmetric waveguides



Waveguide with mirror symmetry $y \to -y$:

⇝ modes have a definite parity.

Symmetric waveguides

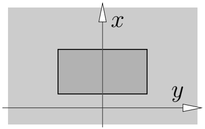


Waveguide with mirror symmetry $y \to -y$:

$\leftrightsquigarrow$ modes have a definite parity.

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Symmetric waveguides

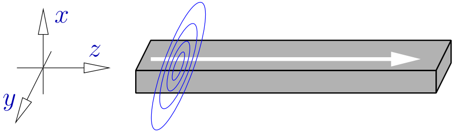


Waveguide with mirror symmetry $y \to -y$:
⇝ modes have a definite parity.

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

↪ Equal parity of H_x, E_y, H_z, reversed parity of E_x, H_y, E_z.

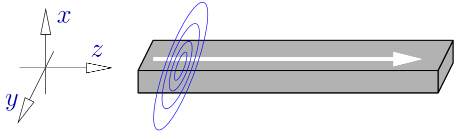
Directional modes



(FD) $\sim \exp(\mathrm{i}\,\omega t)$

Longitudinally homogeneous waveguide: mirror symmetry $z \to -z$.

Directional modes

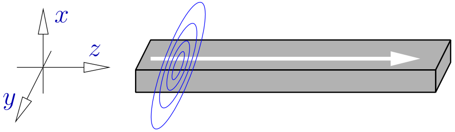


(FD) $\sim \exp(\mathrm{i}\,\omega t)$

Longitudinally homogeneous waveguide: mirror symmetry $z \to -z$.

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\,\beta E_y \\ -\mathrm{i}\,\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\,\beta H_y \\ -\mathrm{i}\,\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

Directional modes



(FD) $\sim \exp(\mathrm{i}\,\omega t)$

Longitudinally homogeneous waveguide: mirror symmetry $z \to -z$.

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\,\beta E_y \\ -\mathrm{i}\,\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\,\beta H_y \\ -\mathrm{i}\,\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

forward: $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f}} \\ \bar{\boldsymbol{H}}^{\mathrm{f}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\,\beta z}$, $\quad \bar{\boldsymbol{E}}^{\mathrm{f}} = (E_x, E_y, E_z)$, $\bar{\boldsymbol{H}}^{\mathrm{f}} = (H_x, H_y, H_z)$,

$\leftrightsquigarrow$

backward: $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{b}} \\ \bar{\boldsymbol{H}}^{\mathrm{b}} \end{pmatrix}(x,y)\ \mathrm{e}^{+\mathrm{i}\,\beta z}$, $\quad \bar{\boldsymbol{E}}^{\mathrm{b}} = (E_x, E_y, -E_z)$, $\bar{\boldsymbol{H}}^{\mathrm{b}} = (-H_x, -H_y, H_z)$.

Modal power

(FD) $\sim \exp(\mathrm{i}\,\omega t)$

- E.m. power density : $\boldsymbol{S} = \dfrac{1}{2}\mathrm{Re}\,(\boldsymbol{E}^* \times \boldsymbol{H})$.

Modal power

(FD) $\sim \exp(\mathrm{i}\,\omega t)$

- E.m. power density : $\boldsymbol{S} = \frac{1}{2}\mathrm{Re}\;(\boldsymbol{E}^* \times \boldsymbol{H})$.

- $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\;\mathrm{e}^{-\mathrm{i}\beta z}$,

$$\bar{\boldsymbol{E}} = a(\bar{E}_x, \bar{E}_y, \mathrm{i}\bar{E}_z{}'),$$
$$\bar{\boldsymbol{H}} = a(\bar{H}_x, \bar{H}_y, \mathrm{i}\bar{H}_z{}'),$$
$$a \in \mathbb{C},\; \bar{E}_x, \ldots, \bar{H}_z' \in \mathbb{R},$$
a guided mode, $\beta \in \mathbb{R}$.

Modal power

(FD) $\sim \exp(\mathrm{i}\,\omega t)$

- E.m. power density : $\boldsymbol{S} = \frac{1}{2}\mathrm{Re}\;(\boldsymbol{E}^* \times \boldsymbol{H})$.

- $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}$,

$$\bar{\boldsymbol{E}} = a(\bar{E}_x, \bar{E}_y, \mathrm{i}\bar{E}_z{}'),$$
$$\bar{\boldsymbol{H}} = a(\bar{H}_x, \bar{H}_y, \mathrm{i}\bar{H}_z{}'),$$
$$a \in \mathbb{C},\ \ \bar{E}_x, \ldots, \bar{H}_z' \in \mathbb{R},$$
a guided mode, $\beta \in \mathbb{R}$.

$$\hookrightarrow\quad \boldsymbol{S} = \frac{|a|^2}{2}\begin{pmatrix} 0 \\ 0 \\ \bar{E}_x\bar{H}_y - \bar{E}_y\bar{H}_x \end{pmatrix},$$

or $S_x = 0$, $S_y = 0$, $S_z = \frac{1}{2}\mathrm{Re}\;\left(E_x^* H_y - E_y^* H_x\right)$. $(S_z(x,y))$

Modal power

(FD) $\sim \exp(\mathrm{i}\,\omega t)$

- E.m. power density : $\boldsymbol{S} = \frac{1}{2}\mathrm{Re}\,(\boldsymbol{E}^* \times \boldsymbol{H})$.

- $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}$,

$$\bar{\boldsymbol{E}} = a(\bar{E}_x, \bar{E}_y, \mathrm{i}\bar{E}_z{}'),$$
$$\bar{\boldsymbol{H}} = a(\bar{H}_x, \bar{H}_y, \mathrm{i}\bar{H}_z{}'),$$
$$a \in \mathbb{C},\ \bar{E}_x, \ldots, \bar{H}_z' \in \mathbb{R},$$
a guided mode, $\beta \in \mathbb{R}$.

$$\hookrightarrow \quad \boldsymbol{S} = \frac{|a|^2}{2}\begin{pmatrix} 0 \\ 0 \\ \bar{E}_x\bar{H}_y - \bar{E}_y\bar{H}_x \end{pmatrix},$$

or $S_x = 0,\ S_y = 0,\ S_z = \frac{1}{2}\mathrm{Re}\,\left(E_x^* H_y - E_y^* H_x\right)$. $(S_z(x,y))$

- Power carried by the mode :

$$P = \iint S_z\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\iint \left(E_x^* H_y - E_y^* H_x + E_x H_y^* - E_y H_x^*\right)\,\mathrm{d}x\,\mathrm{d}y\,.$$

(backward mode, $E_x \to E_x,\ E_y \to E_y,\ H_x \to -H_x,\ H_y \to -H_y:\ P \to -P$)

Mode orthogonality

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z},$$

$$\begin{aligned} &\boldsymbol{\nabla} \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m, \\ &\boldsymbol{\nabla} \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_m, \\ &\beta_l \neq \beta_m, \ \text{if } l \neq m. \end{aligned}$$

Mode orthogonality

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z},$$

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E}_m &= -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m, \\ \boldsymbol{\nabla} \times \boldsymbol{H}_m &= \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_m, \\ \beta_l \neq \beta_m,\ &\text{if } l \neq m. \end{aligned}$$

- $$P_m = \frac{1}{4}\iint \left(E_{mx}^* H_{my} - E_{my}^* H_{mx} + E_{mx} H_{my}^* - E_{my} H_{mx}^*\right)\, \mathrm{d}x\, \mathrm{d}y\,.$$

Mode orthogonality

- A set of guided modes of the same waveguide (ϵ): $\beta \in \mathbb{R}$

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}, \qquad \begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{E}_m &= -\mathrm{i}\omega\mu_0\boldsymbol{H}_m, \\ \boldsymbol{\nabla}\times\boldsymbol{H}_m &= \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_m, \\ \beta_l \neq \beta_m,&\ \text{if } l \neq m. \end{aligned}$$

- $P_m = \dfrac{1}{4}\iint \left(E_{mx}^* H_{my} - E_{my}^* H_{mx} + E_{mx} H_{my}^* - E_{my} H_{mx}^*\right)\,\mathrm{d}x\,\mathrm{d}y\,.$

- $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0 \quad \text{for} \quad x, y \to \pm\infty.$

Mode orthogonality

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z},$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m,$$
$$\boldsymbol{\nabla} \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_m,$$
$$\beta_l \neq \beta_m, \text{ if } l \neq m.$$

- $$P_m = \frac{1}{4} \iint \left(E_{mx}^* H_{my} - E_{my}^* H_{mx} + E_{mx} H_{my}^* - E_{my} H_{mx}^*\right) \mathrm{d}x\,\mathrm{d}y\,.$$

- $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.

- $\boldsymbol{\nabla} \cdot (\boldsymbol{E}_l^* \times \boldsymbol{H}_m + \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = 0$ for all l, m

Mode orthogonality

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z},$$

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E}_m &= -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m, \\ \boldsymbol{\nabla} \times \boldsymbol{H}_m &= \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_m, \\ \beta_l \neq \beta_m, &\ \text{if } l \neq m. \end{aligned}$$

- $$P_m = \frac{1}{4}\iint \left(E^*_{mx}H_{my} - E^*_{my}H_{mx} + E_{mx}H^*_{my} - E_{my}H^*_{mx}\right)\,\mathrm{d}x\,\mathrm{d}y\,.$$

- $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0 \quad \text{for} \quad x,y \to \pm\infty.$

- $\boldsymbol{\nabla}\cdot(\boldsymbol{E}^*_l \times \boldsymbol{H}_m + \boldsymbol{E}_m \times \boldsymbol{H}^*_l) = 0$ for all l, m

↪ $$0 = \mathrm{i}(\beta_l - \beta_m)\left\{\iint \left(\bar{\boldsymbol{E}}^*_l \times \bar{\boldsymbol{H}}_m + \bar{\boldsymbol{E}}_m \times \bar{\boldsymbol{H}}^*_l\right)_z \mathrm{d}x\,\mathrm{d}y\right\}\,\mathrm{e}^{\mathrm{i}(\beta_l - \beta_m)z},$$

Mode orthogonality

- A set of guided modes of the same waveguide (ϵ): $\beta \in \mathbb{R}$

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z},$$

$$\nabla \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m, \quad \nabla \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_m, \quad \beta_l \neq \beta_m, \text{ if } l \neq m.$$

- $$P_m = \frac{1}{4}\iint \left(E^*_{mx}H_{my} - E^*_{my}H_{mx} + E_{mx}H^*_{my} - E_{my}H^*_{mx}\right)\,\mathrm{d}x\,\mathrm{d}y\,.$$
- $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0 \quad \text{for} \quad x, y \to \pm\infty.$
- $\nabla \cdot (\boldsymbol{E}^*_l \times \boldsymbol{H}_m + \boldsymbol{E}_m \times \boldsymbol{H}^*_l) = 0$ for all l, m

$$\hookrightarrow \quad 0 = \mathrm{i}(\beta_l - \beta_m)\left\{\iint \left(\bar{\boldsymbol{E}}^*_l \times \bar{\boldsymbol{H}}_m + \bar{\boldsymbol{E}}_m \times \bar{\boldsymbol{H}}^*_l\right)_z \mathrm{d}x\,\mathrm{d}y\right\}\mathrm{e}^{\mathrm{i}(\beta_l - \beta_m)z},$$

$$(\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2) := \frac{1}{4}\iint \left(E^*_{1x}H_{2y} - E^*_{1y}H_{2x} + H^*_{1y}E_{2x} - H^*_{1x}E_{2y}\right)\mathrm{d}x\,\mathrm{d}y$$

Mode orthogonality

- A set of guided modes of the same waveguide (ϵ): $\beta \in \mathbb{R}$

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}, \qquad \begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{E}_m &= -\mathrm{i}\omega\mu_0\boldsymbol{H}_m, \\ \boldsymbol{\nabla}\times\boldsymbol{H}_m &= \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_m, \\ \beta_l \neq \beta_m, &\text{ if } l\neq m. \end{aligned}$$

- $P_m = \frac{1}{4}\iint \left(E^*_{mx}H_{my} - E^*_{my}H_{mx} + E_{mx}H^*_{my} - E_{my}H^*_{mx}\right)\,\mathrm{d}x\,\mathrm{d}y\,.$
- $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0 \quad \text{for} \quad x,y\to\pm\infty.$
- $\boldsymbol{\nabla}\cdot(\boldsymbol{E}^*_l\times\boldsymbol{H}_m + \boldsymbol{E}_m\times\boldsymbol{H}^*_l) = 0$ for all l, m

$$\hookrightarrow \quad 0 = \mathrm{i}(\beta_l-\beta_m)\left\{\iint \left(\bar{\boldsymbol{E}}^*_l\times\bar{\boldsymbol{H}}_m + \bar{\boldsymbol{E}}_m\times\bar{\boldsymbol{H}}^*_l\right)_z \mathrm{d}x\,\mathrm{d}y\right\}\mathrm{e}^{\mathrm{i}(\beta_l-\beta_m)z},$$

$$(\boldsymbol{E}_1,\boldsymbol{H}_1;\boldsymbol{E}_2,\boldsymbol{H}_2) := \frac{1}{4}\iint \left(E^*_{1x}H_{2y} - E^*_{1y}H_{2x} + H^*_{1y}E_{2x} - H^*_{1x}E_{2y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$(\boldsymbol{E}_l,\boldsymbol{H}_l;\boldsymbol{E}_m,\boldsymbol{H}_m) = \begin{cases} 0, & \text{if } l\neq m, \\ P_m, & \text{otherwise.} \end{cases}$$

(The modes are “power orthogonal”.)
(Statements hold for propagating guided modes.)
((., .; ., .) is frequently used for mode normalization.)

Power transport by a mode superposition

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}, \qquad P_m = (\boldsymbol{E}_m, \boldsymbol{H}_m; \boldsymbol{E}_m, \boldsymbol{H}_m)\,.$$

- Superposition with amplitudes $a_m \in \mathbb{C}$:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \sum_m a_m \begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \sum_m a_m \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}.$$

Power flow along the waveguide :

$$\begin{aligned} \iint S_z\,\mathrm{d}x\,\mathrm{d}y &= (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H}) \\ &= \sum_l \sum_m a_l^*\, a_m\, (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m) \\ &= \sum_m |a_m|^2 P_m\,. \end{aligned}$$

(Forward / backward modes: $P \gtrless 0$.)

Mode interference

$\beta \in \mathbb{R}$

- Two modes $m = 1, 2$:

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x, y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}.$$

- Superposition with amplitudes a_1, a_2:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) = a_1 \begin{pmatrix} \bar{\boldsymbol{E}}_1 \\ \bar{\boldsymbol{H}}_1 \end{pmatrix}(x, y)\ \mathrm{e}^{-\mathrm{i}\beta_1 z} + a_2 \begin{pmatrix} \bar{\boldsymbol{E}}_2 \\ \bar{\boldsymbol{H}}_2 \end{pmatrix}(x, y)\ \mathrm{e}^{-\mathrm{i}\beta_2 z}.$$

- Fix a position x, y and component F:

Omit (x, y).

$$F(z) = a_1\, \bar{F}_1\ \mathrm{e}^{-\mathrm{i}\beta_1 z} + a_2\, \bar{F}_2\ \mathrm{e}^{-\mathrm{i}\beta_2 z}, \qquad r\,\mathrm{e}^{-\mathrm{i}\phi} := a_1^* a_2 \bar{F}_1^* \bar{F}_2,$$

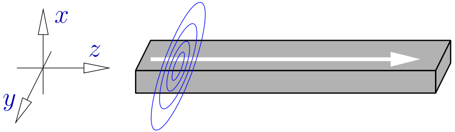
$$\hookrightarrow \quad |F|^2(z) = |a_1|^2 |\bar{F}_1|^2 + |a_2|^2 |\bar{F}_2|^2 + 2r\cos((\beta_1 - \beta_2)z + \phi).$$

Periodic beating pattern with half-beat-length $L_\mathrm{c} = \dfrac{\pi}{|\beta_1 - \beta_2|}$.

(Supermodes ▶) (Evanescent coupling ▶)

("Coupling length" L_c.)

Polarization of a guided wave field



Unidirectional guided waves in a "long" dielectric channel that supports fundamental TE- and TM-like modes only:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = a_{\mathrm{TE}} \begin{pmatrix} \bar{\boldsymbol{E}}_{\mathrm{TE}} \\ \bar{\boldsymbol{H}}_{\mathrm{TE}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{\mathrm{TE}} z} + a_{\mathrm{TM}} \begin{pmatrix} \bar{\boldsymbol{E}}_{\mathrm{TM}} \\ \bar{\boldsymbol{H}}_{\mathrm{TM}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{\mathrm{TM}} z},$$

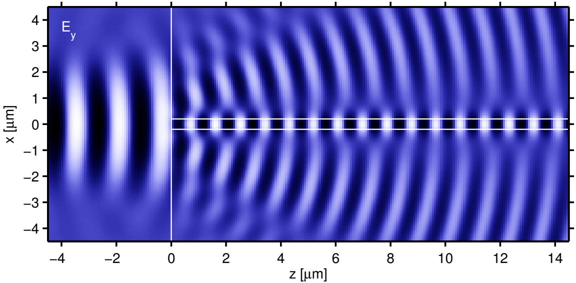
amplitudes $a_{\mathrm{TE}}, a_{\mathrm{TM}} \in \mathbb{C}$.

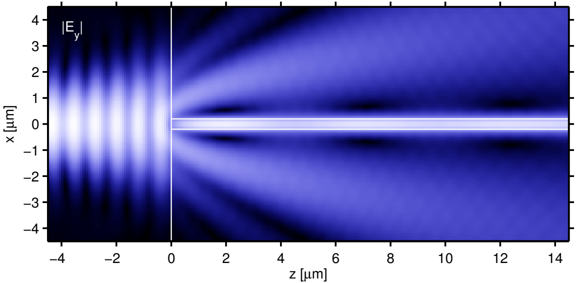
- $E_{\mathrm{TE}\,z} \neq 0,\ E_{\mathrm{TM}\,z} \neq 0$.
- $\bar{\boldsymbol{E}}_{\mathrm{TE}}(x,y) \neq \bar{\boldsymbol{E}}_{\mathrm{TM}}(x,y)$.
- At (x,y): adjust $\boldsymbol{E}/|\boldsymbol{E}|$ via $a_{\mathrm{TE}}, a_{\mathrm{TM}}$.
- $a_{\mathrm{TE}}, a_{\mathrm{TM}}$ fixed: $(\boldsymbol{E}/|\boldsymbol{E}|)(x,y)$ varies.

"Polarization" frequently indicates the presence of only one mode.

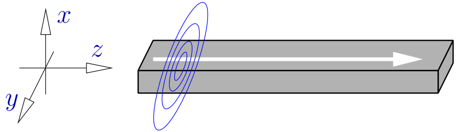
What about non-guided fields?

(2-D, TE)





Normal modes: real mode problems



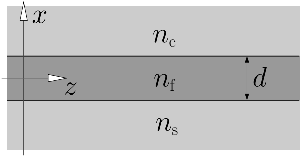
- lossless waveguide, $\epsilon \in \mathbb{R}$,
- "real" boundary conditions at x, y "far away" from the core,
- "real" vectorial mode equations:

$$\partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x = 0\,,$$

$$\epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y = 0\,,$$

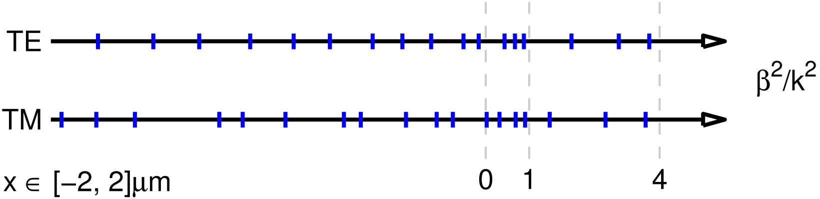
↪ real principal components $H_x(x, y), H_y(x, y)$, $\beta^2 \in \mathbb{R}$.

2-D slab waveguide, normal mode spectrum

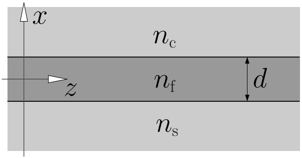


$n_s = n_c = 1.0, \; n_f = 2.0,$
$d = 1.3\,\mu\text{m}, \; \lambda = 1.55\,\mu\text{m},$

$E_y = 0, H_y = 0$ at $x = \pm 2\,\mu\text{m}.$

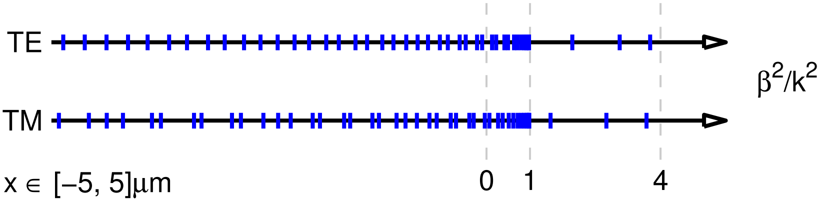


- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: guided modes.
- $0 < \beta^2/k^2 < n_s^2$: propagating radiation modes.
- $\beta^2/k^2 < 0$: evanescent radiation modes.



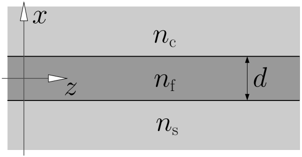
$n_s = n_c = 1.0,\ \ n_f = 2.0,$
$d = 1.3\ \mu\text{m},\ \ \lambda = 1.55\ \mu\text{m},$

$E_y = 0,\ H_y = 0$ at $x = \pm 5\ \mu\text{m}.$



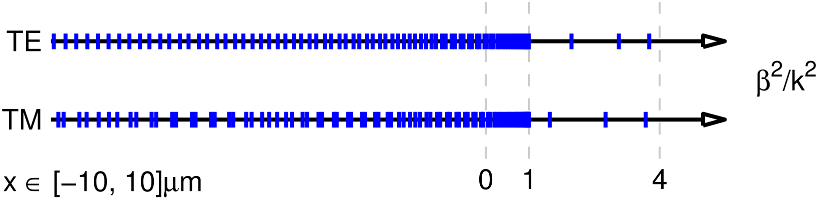
- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: guided modes.
- $0 < \beta^2/k^2 < n_s^2$: propagating radiation modes.
- $\beta^2/k^2 < 0$: evanescent radiation modes.

2-D slab waveguide, normal mode spectrum



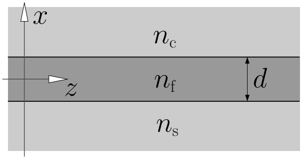
$n_s = n_c = 1.0,\ n_f = 2.0,$
$d = 1.3\ \mu m,\ \lambda = 1.55\ \mu m,$

$E_y = 0,\ H_y = 0$ at $x = \pm 10\ \mu m.$



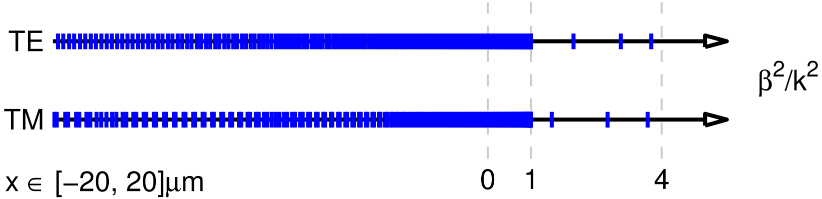
- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: guided modes.
- $0 < \beta^2/k^2 < n_s^2$: propagating radiation modes.
- $\beta^2/k^2 < 0$: evanescent radiation modes.

2-D slab waveguide, normal mode spectrum



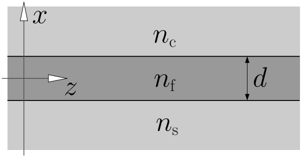
$n_s = n_c = 1.0,\ n_f = 2.0,$
$d = 1.3\ \mu m,\ \lambda = 1.55\ \mu m,$

$E_y = 0, H_y = 0$ at $x = \pm 20\ \mu m.$



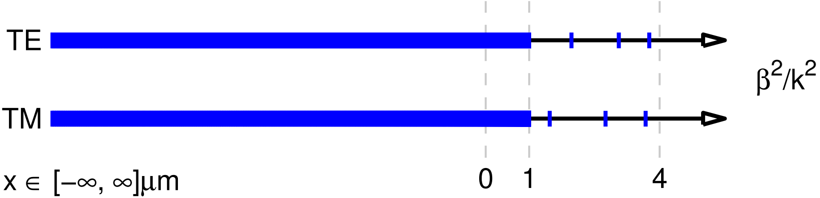
- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: guided modes.
- $0 < \beta^2/k^2 < n_s^2$: propagating radiation modes.
- $\beta^2/k^2 < 0$: evanescent radiation modes.

2-D slab waveguide, normal mode spectrum



$n_s = n_c = 1.0, \;\; n_f = 2.0,$
$d = 1.3\,\mu\text{m}, \;\; \lambda = 1.55\,\mu\text{m},$

$E_y = 0, H_y = 0$ at $x = \pm\infty$.



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: guided modes (discrete spectrum).
- $0 < \beta^2/k^2 < n_s^2$: propagating radiation modes (continuous spec.).
- $\beta^2/k^2 < 0$: evanescent radiation modes (continuous spec.).

Propagating & evanescent modes

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f,b}} \\ \bar{\boldsymbol{H}}^{\mathrm{f,b}} \end{pmatrix} (x, y)\ \mathrm{e}^{\mp \mathrm{i}\beta z}. \qquad \sim \exp(\mathrm{i}\omega t) \quad \text{(FD)}$$

- $\beta^2 > 0 \quad \leftrightsquigarrow \quad \beta = \sqrt{\beta^2},\ \beta \in \mathbb{R},\ \beta > 0,$
  $\sim\ \mathrm{e}^{\mp \mathrm{i}\beta z}$, a forward / backward propagating mode.

(Physical relevance of individual modes.)

Propagating & evanescent modes

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f,b}} \\ \bar{\boldsymbol{H}}^{\mathrm{f,b}} \end{pmatrix}(x,y)\ \mathrm{e}^{\mp \mathrm{i}\beta z}. \qquad \sim \exp(\mathrm{i}\omega t) \quad \text{(FD)}$$

- $\beta^2 > 0 \iff \beta = \sqrt{\beta^2},\ \beta \in \mathbb{R},\ \beta > 0,$
  $\sim\ \mathrm{e}^{\mp \mathrm{i}\beta z}$, a forward / backward propagating mode.

(Physical relevance of individual modes.)

- $\beta^2 < 0 \iff \beta = -\mathrm{i}\sqrt{|\beta^2|} = -\mathrm{i}\alpha,\ \alpha = \sqrt{|\beta^2|} \in \mathbb{R},\ \alpha > 0,$
  $\sim\ \mathrm{e}^{\mp \alpha z}$, a forward / backward traveling evanescent mode.

  "forward": $\sim\ \mathrm{e}^{-\alpha z}$, field decays with z,
  "backward": $\sim\ \mathrm{e}^{+\alpha z}$, field grows with z.

(Relevant for purposes of field expansions.)

Propagating & evanescent modes

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f,b}} \\ \bar{\boldsymbol{H}}^{\mathrm{f,b}} \end{pmatrix}(x,y)\ \mathrm{e}^{\mp \mathrm{i}\beta z}. \qquad \sim \exp(\mathrm{i}\omega t) \quad \text{(FD)}$$

- $\beta^2 > 0 \quad \leftrightsquigarrow \quad \beta = \sqrt{\beta^2},\ \beta \in \mathbb{R},\ \beta > 0,$
  $\sim\ \mathrm{e}^{\mp \mathrm{i}\beta z}$, a forward / backward propagating mode.

(Physical relevance of individual modes.)

- $\beta^2 < 0 \quad \leftrightsquigarrow \quad \beta = -\mathrm{i}\sqrt{|\beta^2|} = -\mathrm{i}\alpha,\ \alpha = \sqrt{|\beta^2|} \in \mathbb{R},\ \alpha > 0,$
  $\sim\ \mathrm{e}^{\mp \alpha z}$, a forward / backward traveling evanescent mode.

  "forward": $\sim\ \mathrm{e}^{-\alpha z}$, field decays with z,
  "backward": $\sim\ \mathrm{e}^{+\alpha z}$, field grows with z.

(Relevant for purposes of field expansions.)

- $\{$forward & backward, propagating & evanescent modes$\}$
  $=$ the set of normal modes.

Evanescent modes

$$\epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Evanescent modes

$\beta = -\mathrm{i}\alpha,\ \alpha \in \mathbb{R}$ $\epsilon \in \mathbb{R}$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Evanescent modes

$\beta = -\mathrm{i}\alpha, \ \alpha \in \mathbb{R}$ $\qquad\qquad$ $\epsilon \in \mathbb{R}$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Plane mode profiles: real PDE for E_x, E_y, E_z, $\mathrm{i}H_x, \mathrm{i}H_y, \mathrm{i}H_z$; common phase with real E_x, E_y, E_z, imaginary H_x, H_y, H_z.

Evanescent modes

$\beta = -\mathrm{i}\alpha, \ \alpha \in \mathbb{R}$ $\qquad\qquad$ $\epsilon \in \mathbb{R}$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Plane mode profiles: real PDE for E_x, E_y, E_z, $\mathrm{i}H_x, \mathrm{i}H_y, \mathrm{i}H_z$;
  common phase with real E_x, E_y, E_z, imaginary H_x, H_y, H_z.
- Directional evanescent modes:
  $\{E_x, E_y, E_z, \ H_x, H_y, H_z; \ \alpha\}^{\mathrm{f}} \rightsquigarrow \{E_x, E_y, -E_z, \ -H_x, -H_y, H_z; \ -\alpha\}^{\mathrm{b}}$.

Evanescent modes

$\beta = -\mathrm{i}\alpha,\ \alpha \in \mathbb{R}$ $\qquad\qquad \epsilon \in \mathbb{R}$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Plane mode profiles: real PDE for $E_x, E_y, E_z,\ \mathrm{i}H_x, \mathrm{i}H_y, \mathrm{i}H_z$;
  common phase with real E_x, E_y, E_z, imaginary H_x, H_y, H_z.

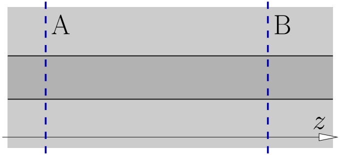
- Directional evanescent modes:
  $\{E_x, E_y, E_z,\ H_x, H_y, H_z;\ \alpha\}^{\mathrm{f}} \rightsquigarrow \{E_x, E_y, -E_z,\ -H_x, -H_y, H_z;\ -\alpha\}^{\mathrm{b}}$.

- Modal power:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\alpha z}, \qquad \begin{aligned} \bar{\boldsymbol{E}} &= a(E'_x, E'_y, E'_z), \\ \bar{\boldsymbol{H}} &= \mathrm{i}a(H'_x, H'_y, H'_z), \\ E'_x, \ldots, H'_z &\in \mathbb{R},\quad a \in \mathbb{C} \end{aligned}$$

$$\hookrightarrow \quad S_z = \frac{1}{2}\mathrm{Re}\,\left(E_x^* H_y - E_y^* H_x\right) = 0\,, \qquad \iint S_z\,\mathrm{d}x\,\mathrm{d}y = 0\,.$$

Completeness of normal modes



$\epsilon \in \mathbb{R}$, $\sim \exp(\mathrm{i}\omega t)$ (FD)

A lossless, z-homogeneous waveguide configuration; general solution of the Maxwell equations between cross sectional planes A and B:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \sum_{m\in\mathcal{N}} F_m \begin{pmatrix} \bar{\boldsymbol{E}}_m^{\mathrm{f}} \\ \bar{\boldsymbol{H}}_m^{\mathrm{f}} \end{pmatrix}(x,y)\, \mathrm{e}^{-\mathrm{i}\beta_m z} + \sum_{m\in\mathcal{N}} B_m \begin{pmatrix} \bar{\boldsymbol{E}}_m^{\mathrm{b}} \\ \bar{\boldsymbol{H}}_m^{\mathrm{b}} \end{pmatrix}(x,y)\, \mathrm{e}^{+\mathrm{i}\beta_m z},$$

$\sum \rightarrow \sum\!\!\!\!\!\!\int$

$\mathcal{N}$: the set of forward normal modes supported by the waveguide.

("Solution": obvious; "general": without proof.)

Completeness of normal modes

Stronger statement:
"any" transverse 2-component field on a cross sectional plane can be expanded into alternatively

- the transverse electric components of forward normal modes,
- the transverse magnetic components of forward normal modes,
- the transverse electric components of backward normal modes,
- the transverse magnetic components of backward normal modes.

Orthogonality of normal modes

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}. \qquad \sim \exp(\mathrm{i}\omega t) \quad \text{(FD)}$$

	$\bar{\boldsymbol{E}}$	$\bar{\boldsymbol{H}}$	β
[prop., f]	$(E'_x, E'_y, \mathrm{i}E'_z)$	$(H'_x, H'_y, \mathrm{i}H'_z)$	$\beta > 0$
[prop., b]	$(E'_x, E'_y, -\mathrm{i}E'_z)$	$(-H'_x, -H'_y, \mathrm{i}H'_z)$	$\beta < 0$
[evan., f]	(E'_x, E'_y, E'_z)	$(\mathrm{i}H'_x, \mathrm{i}H'_y, \mathrm{i}H'_z)$	$\beta = -\mathrm{i}\alpha,\ \alpha > 0$
[evan., b]	$(E'_x, E'_y, -E'_z)$	$(-\mathrm{i}H'_x, -\mathrm{i}H'_y, \mathrm{i}H'_z)$	$\beta = \mathrm{i}\alpha,\ \alpha > 0$

individual $E'_x, \dots H'_z \in \mathbb{R}$.

$$(\boldsymbol{E}_a, \boldsymbol{H}_a; \boldsymbol{E}_b, \boldsymbol{H}_b) := \frac{1}{4} \iint \left(E^*_{ax}H_{by} - E^*_{ay}H_{bx} + H^*_{ay}E_{bx} - H^*_{ax}E_{by}\right) \mathrm{d}x\,\mathrm{d}y$$

$$\begin{pmatrix} \boldsymbol{E}_{1,2} \\ \boldsymbol{H}_{1,2} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_{1,2} \\ \bar{\boldsymbol{H}}_{1,2} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{1,2} z}, \qquad \begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E}_{1,2} &= -\mathrm{i}\omega\mu_0 \boldsymbol{H}_{1,2}, \\ \boldsymbol{\nabla} \times \boldsymbol{H}_{1,2} &= \mathrm{i}\omega\epsilon_0\epsilon \boldsymbol{E}_{1,2}, \end{aligned}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{E}^*_1 \times \boldsymbol{H}_2 + \boldsymbol{E}_2 \times \boldsymbol{H}^*_1) = 0 \rightsquigarrow 0 = (\beta^*_1 - \beta_2)\,(\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2).$$

↪ …

Orthogonality of normal modes

Nondegenerate directional normal modes of the same waveguide (ϵ):

$$\begin{pmatrix} \boldsymbol{E}_m^{\mathrm{f,b}} \\ \boldsymbol{H}_m^{\mathrm{f,b}} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m^{\mathrm{f,b}} \\ \bar{\boldsymbol{H}}_m^{\mathrm{f,b}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m^{\mathrm{f,b}} z}, \qquad \begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{E}_m &= -\mathrm{i}\omega\mu_0\boldsymbol{H}_m, \\ \boldsymbol{\nabla}\times\boldsymbol{H}_m &= \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_m, \end{aligned}$$

$$\beta_l \neq \beta_m, \ \text{if } l \neq m.$$

- A propagating mode m :

$$(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}) =: P_m, \ \ (\bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}; \bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}) = -P_m, \qquad P_m \in \mathbb{R},$$
$$(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}) = (\bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}; \bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}) = 0,$$
$$(\bar{\boldsymbol{E}}_m^{\mathrm{d}}, \bar{\boldsymbol{H}}_m^{\mathrm{d}}; \bar{\boldsymbol{E}}_l^{\mathrm{r}}, \bar{\boldsymbol{H}}_l^{\mathrm{r}}) = (\bar{\boldsymbol{E}}_l^{\mathrm{r}}, \bar{\boldsymbol{H}}_l^{\mathrm{r}}; \bar{\boldsymbol{E}}_m^{\mathrm{d}}, \bar{\boldsymbol{H}}_m^{\mathrm{d}}) = 0 \ \ \text{for all } \ l \neq m, \ \mathrm{d,r} = \mathrm{f,b}.$$

- An evanescent mode m :

$$(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}) = (\bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}; \bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}) = 0,$$
$$(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}) =: P_m, \ \ (\bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}; \bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}) = -P_m, \qquad P_m \notin \mathbb{R},$$
$$(\bar{\boldsymbol{E}}_m^{\mathrm{d}}, \bar{\boldsymbol{H}}_m^{\mathrm{d}}; \bar{\boldsymbol{E}}_l^{\mathrm{r}}, \bar{\boldsymbol{H}}_l^{\mathrm{r}}) = (\bar{\boldsymbol{E}}_l^{\mathrm{r}}, \bar{\boldsymbol{H}}_l^{\mathrm{r}}; \bar{\boldsymbol{E}}_m^{\mathrm{d}}, \bar{\boldsymbol{H}}_m^{\mathrm{d}}) = 0 \ \ \text{for all } \ l \neq m, \ \mathrm{d,r} = \mathrm{f,b}.$$

(This implies orthogonality of propagating and evanescent modes.)
($1/\sqrt{|P_m|}$ is frequently used for mode normalization.)

Power flow associated with a normal mode expansion

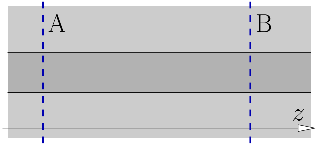
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \sum_{m\in\mathcal{N}} \left\{ F_m \begin{pmatrix} \bar{\boldsymbol{E}}_m^{\mathrm{f}} \\ \bar{\boldsymbol{H}}_m^{\mathrm{f}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z} + B_m \begin{pmatrix} \bar{\boldsymbol{E}}_m^{\mathrm{b}} \\ \bar{\boldsymbol{H}}_m^{\mathrm{b}} \end{pmatrix}(x,y)\ \mathrm{e}^{+\mathrm{i}\beta_m z} \right\}$$

Power carried along z :

$$P = \iint S_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E},\boldsymbol{H};\boldsymbol{E},\boldsymbol{H})$$

$$= \sum_{m \text{ propag.}} \left(|F_m|^2 - |B_m|^2\right) P_m \; + \sum_{m \text{ evanesc.}} \left(F_m^* B_m - B_m^* F_m\right) P_m \,.$$

- P is indepedent of z.
- Individual contributions from forward and backward propagating modes.
- Contributions from evanescent modes require forward and backward fields to be present.
- Unidirectional field (forward: $B_m = 0$ for all m): Only propagating modes carry power.



$\boldsymbol{E}, \boldsymbol{H}$: a solution of the Maxwell equations for the z-homogeneous waveguide between two cross sectional planes A and B.

↪ Extract local mode amplitudes by projection onto normal modes:

- A propagating mode m, $\beta_m > 0$:
  $(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \boldsymbol{E}, \boldsymbol{H}) = F_m P_m \, \mathrm{e}^{-\mathrm{i}\beta z}$,
  $(\bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}; \boldsymbol{E}, \boldsymbol{H}) = -B_m P_m \, \mathrm{e}^{\mathrm{i}\beta z}$.

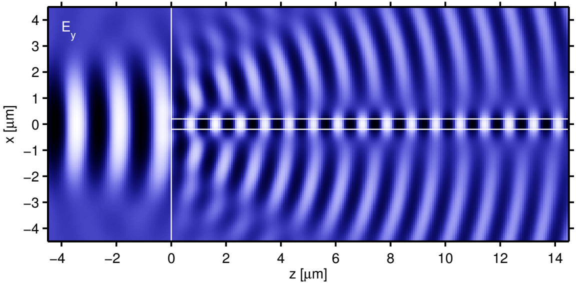
  $$F_m \, \mathrm{e}^{-\mathrm{i}\beta z} = \frac{(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \boldsymbol{E}, \boldsymbol{H})}{(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}})}$$

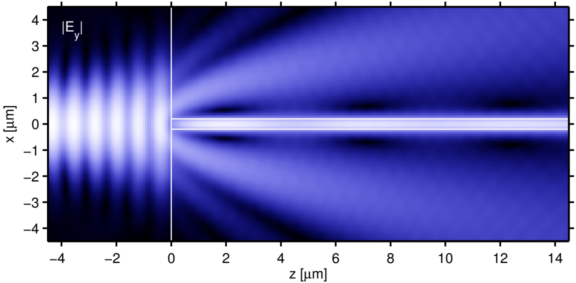
- An evanescent mode m, $\beta_m = -\mathrm{i}\alpha_m$, $\alpha_m > 0$:
  $(\bar{\boldsymbol{E}}_m^{\mathrm{f}}, \bar{\boldsymbol{H}}_m^{\mathrm{f}}; \boldsymbol{E}, \boldsymbol{H}) = B_m P_m \, \mathrm{e}^{\alpha z}$, $(\bar{\boldsymbol{E}}_m^{\mathrm{b}}, \bar{\boldsymbol{H}}_m^{\mathrm{b}}; \boldsymbol{E}, \boldsymbol{H}) = -F_m P_m \, \mathrm{e}^{-\alpha z}$.

↭ Ports of a photonic integrated circuit.

Waveguide facet: Port definition

(2-D, TE)





Upcoming

Next lectures:

- Examples for dielectric optical waveguides.
- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- Bent optical waveguides; whispering gallery resonances; circular microresonators.