Optical Waveguide Theory (F)

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Course overview

Optical waveguide theory

A  Photonics / integrated optics; theory, motto; phenomena, introductory examples.
B  Brush up on mathematical tools.
C  Maxwell equations, different formulations, interfaces, energy and power flow.
D  Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
E  Normal modes of dielectric optical waveguides, mode interference.
F  Examples for dielectric optical waveguides.
G  Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
H  Bent optical waveguides; whispering gallery resonances; circular microresonators.
   I  Coupled mode theory, perturbation theory.
      ● Hybrid analytical / numerical coupled mode theory.
J  A touch of photonic crystals; a touch of plasmonics.
      ● Oblique semi-guided waves: 2-D integrated optics.
      ● Summary, concluding remarks.
2-D waveguide configurations

- 2-D waveguide, 1-D cross section.
- Permittivity $\epsilon = n^2$, refractive index $n(x)$. (1-D waveguide)
- $\partial_y \epsilon = 0 \quad \Rightarrow \quad \partial_y E = 0, \, \partial_y H = 0, \, 2$-D TE/TM setting.
- $\partial_z \epsilon = 0 \quad \Rightarrow \quad$ Modal solutions that vary harmonically with $z$:

\[
\begin{pmatrix}
E \\
H
\end{pmatrix}(x, z) = \begin{pmatrix}
\bar{E} \\
\bar{H}
\end{pmatrix}(x) \, e^{-i \beta z},
\]

- mode profile $\bar{E}, \bar{H}$,
- propagation constant $\beta$,
- effective index $n_{\text{eff}} = \beta/k$. 

$\epsilon \in \mathbb{R}, \, \mu = 1, \sim \exp(i \omega t) \, (\text{FD})$
2-D waveguide configurations

\( \epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i\omega t) \) (FD)

- 2-D waveguide, 1-D cross section.
- Permittivity \( \epsilon = n^2 \), refractive index \( n(x) \).

(1-D waveguide)

- \( \partial_y \epsilon = 0 \) \( \Leftrightarrow \) \( \partial_y \mathbf{E} = 0, \partial_y \mathbf{H} = 0 \), 2-D TE / TM setting.
- \( \partial_z \epsilon = 0 \) \( \Leftrightarrow \) Modal solutions that vary harmonically with \( z \):

\[
\begin{pmatrix}
E \\
H
\end{pmatrix}(x, z) = \begin{pmatrix}
\bar{E} \\
\bar{H}
\end{pmatrix}(x) e^{-i\beta z}, \quad \text{mode profile } \bar{E}, \bar{H}, \quad \text{propagation constant } \beta, \quad \text{effective index } n_{\text{eff}} = \beta/k.
\]

(TE): principal component \( \bar{E}_y \),

\[
\partial_x^2 \bar{E}_y + (k^2 \epsilon - \beta^2) \bar{E}_y = 0,
\]

\[
\begin{align*}
\bar{E}_x &= 0, \quad \bar{E}_z = 0, \\
\bar{H}_x &= \frac{-\beta}{\omega \mu_0} \bar{E}_y, \\
\bar{H}_y &= 0, \\
\bar{H}_z &= \frac{i}{\omega \mu_0} \partial_x \bar{E}_y,
\end{align*}
\]

\( \bar{E}_y \) \& \( \partial_x \bar{E}_y \) continuous at dielectric interfaces.
2-D waveguide configurations

\( \epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i \omega t) \) (FD)

- 2-D waveguide, 1-D cross section.
- Permittivity \( \epsilon = n^2 \), refractive index \( n(x) \).

(1-D waveguide)

\[ \partial_y \epsilon = 0 \quad \partial_y E = 0, \quad \partial_y H = 0, \quad \text{2-D TE/TM setting.} \]

\[ \partial_z \epsilon = 0 \quad \text{Modal solutions that vary harmonically with } z : \]

\[
\begin{pmatrix}
  E \\
  H
\end{pmatrix}(x, z) = \left( \begin{pmatrix}
  \bar{E} \\
  \bar{H}
\end{pmatrix}(x) \right) e^{-i \beta z},
\]

mode profile \( \bar{E}, \bar{H} \), propagation constant \( \beta \), effective index \( n_{\text{eff}} = \beta / k \).

(TM): principal component \( \bar{H}_y \),

\[
\epsilon \frac{1}{\epsilon} \frac{\partial_x}{\partial z} \bar{H}_y + (k^2 \epsilon - \beta^2) \bar{H}_y = 0,
\]

\[
\bar{E}_x = \frac{\beta}{\omega \epsilon_0 \epsilon} \bar{H}_y, \quad \bar{E}_y = 0, \quad \bar{E}_z = \frac{-i}{\omega \epsilon_0 \epsilon} \partial_x \bar{H}_y, \quad \bar{H}_x = 0, \quad \bar{H}_z = 0,
\]

\( \bar{H}_y \) & \( \epsilon^{-1} \partial_x \bar{H}_y \) continuous at dielectric interfaces.
Guided 2-D TE/TM modes, orthogonality properties

- A set (index \( m \)) of guided modes of a 2-D waveguide (\( \epsilon \)),
  \[ \psi_m^p = (\bar{E}_m, \bar{H}_m), \quad p=\text{TE,TM} \quad \& \quad \beta_m, \quad \beta_m \neq \beta_l, \text{ if } l \neq m. \]

- \( (E_1, H_1; E_2, H_2) := \frac{1}{4} \int (E_{1x}^*H_{2y} - E_{1y}^*H_{2x} + H_{1y}^*E_{2x} - H_{1x}^*E_{2y}) \, dx. \)

- Power \( P_m \) per lateral (\( y \)) unit length carried by mode \( \psi_m^p, \beta_m \):
  \[ P_m := \int S_z \, dx = (\psi_m^p; \psi_m^p) = \begin{cases} \frac{\beta_m}{2\omega \mu_0} \int |E_{m,y}|^2 \, dx, & \text{if } p = \text{TE}, \\ \frac{\beta_m}{2\omega \epsilon_0} \int \frac{1}{\epsilon} |H_{m,y}|^2 \, dx, & \text{if } p = \text{TM}. \end{cases} \]

\[ (\psi_l^{\text{TE}}; \psi_m^{\text{TM}}) = 0, \quad (\psi_l^{\text{TE}}; \psi_m^{\text{TE}}) = \frac{\beta_m}{2\omega \mu_0} \int E_{l,y}^* E_{m,y} \, dx = \delta_{lm} P_m, \]
\[ (\psi_l^{\text{TM}}; \psi_m^{\text{TE}}) = 0, \quad (\psi_l^{\text{TM}}; \psi_m^{\text{TM}}) = \frac{\beta_m}{2\omega \epsilon_0} \int \frac{1}{\epsilon} H_{l,y}^* H_{m,y} \, dx = \delta_{lm} P_m. \]
Dielectric multilayer slab waveguide

$\epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i\omega t) \ (2\text{-D, FD})$

- $N$ interior layers, piecewise constant $\epsilon = n^2$:
  
  $n(x) = \begin{cases} 
  n_{N+1} & \text{if } h_N < x, \\
  n_l & \text{if } h_{l-1} < x < h_l, \\
  n_0 & \text{if } x < h_0.
  \end{cases}$

- Principal component $\phi(x)$
  (TE: $\phi = \bar{E}_y$, TM: $\phi = \bar{H}_y$).

- $\partial_x^2 \phi + (k^2 n_l^2 - \beta^2) \phi = 0, \quad x \in \text{layer } l, \quad l = 0, \ldots, N + 1$
  (Half-infinte substrate ($l = 0$) and cover ($l = N + 1$) layers.)

- $\phi$ & $\eta \partial_x \phi$ continuous at $x = h_l, \quad $ (TE: $\eta = 1$, TM: $\eta = n^{-2}$).
Dielectric multilayer slab waveguide

- Interior layer $l$, $h_{l-1} < x < h_l$, local refractive index $n_l$,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_l^2) \phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_l^2$ $\leadsto$ $\partial_x^2 \phi = -\kappa_l^2 \phi$, $\kappa_l := \sqrt{k^2 n_l^2 - \beta^2}$,
  $\phi(x) = A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x)$.

- $\beta^2 > k^2 n_l^2$ $\leadsto$ $\partial_x^2 \phi = \kappa_l^2 \phi$, $\kappa_l := \sqrt{\beta^2 - k^2 n_l^2}$,
  $\phi(x) = A_l e^{\kappa_l x} + B_l e^{-\kappa_l x}$.

- Unknowns $A_l, B_l \in \mathbb{C}$. (Local coordinate offsets required to cope with the exponentials.)
Dielectric multilayer slab waveguide, guided modes

![Diagram of a layered waveguide with labels for layers and refractive indices]

- **Substrate region,**
  \[ x < h_0, \]
  local refractive index \( n_0 \),
  \[ \partial_x^2 \phi = (\beta^2 - k^2 n_0^2)\phi. \]

- Consider a trial value \( \beta^2 \in \mathbb{R} \).

  \[ \beta^2 < k^2 n_0^2 \quad \implies \quad \partial_x^2 \phi = -\kappa_0^2 \phi, \quad \kappa_0 := \sqrt{k^2 n_0^2 - \beta^2}, \]
  \[ \phi(x) = A_0 \sin(\kappa_0 x) + B_0 \cos(-\kappa_0 x). \]

  \[ \beta^2 > k^2 n_0^2 \quad \implies \quad \partial_x^2 \phi = \kappa_0^2 \phi, \quad \kappa_0 := \sqrt{\beta^2 - k^2 n_0^2}, \]
  \[ \phi(x) = A_0 e^{\kappa_0 x} + B_0 e^{-\kappa_0 x}. \]

- Unknown \( A_0 \in \mathbb{C} \).

  Guided modes: \( n_{\text{eff}} = \beta/k > n_0 \).
**Dielectric multilayer slab waveguide, guided modes**

- **Cover region,**
  \( h_N < x, \) local refractive index \( n_{N+1}, \)

- \( \partial_x^2 \phi = (\beta^2 - k^2 n_{N+1}^2) \phi. \)

- Consider a trial value \( \beta^2 \in \mathbb{R}. \)

- \( \beta^2 < k^2 n_{N+1}^2 \quad \Rightarrow \quad \partial_x^2 \phi = -\kappa_{N+1}^2 \phi, \quad \kappa_{N+1} := \sqrt{k^2 n_{N+1}^2 - \beta^2}, \)
  \[ \phi(x) = A_{N+1} \sin(\kappa_{N+1} x) + B_{N+1} \cos(-\kappa_{N+1} x). \]

- \( \beta^2 > k^2 n_{N+1}^2 \quad \Rightarrow \quad \partial_x^2 \phi = \kappa_{N+1}^2 \phi, \quad \kappa_{N+1} := \sqrt{\beta^2 - k^2 n_{N+1}^2}, \)
  \[ \phi(x) = A_{N+1} e^{\kappa_{N+1} x} + B_{N+1} e^{-\kappa_{N+1} x}. \]

- Unknown \( B_{N+1} \in \mathbb{C}. \) Guided modes: \( n_{\text{eff}} = \beta / k > n_{N+1}. \)
Dielectric multilayer slab waveguide

\[ \phi(x) = \begin{cases} B_{N+1} e^{-\kappa_{N+1} x}, & \text{for } h_N < x, \\ A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x), & \text{if } \beta^2 < k^2 n_l^2, \\ A_l e^{\kappa_l x} + B_l e^{-\kappa_l x}, & \text{if } \beta^2 > k^2 n_l^2, \\ A_0 e^{\kappa_0 x}, & \text{for } x < h_0. \end{cases} \]

- Trial value \( \beta^2 \in \mathbb{R}, \)
  \( \beta/k > n_0, n_{N+1}, \)
- \( \kappa_l, \ l = 0, \ldots, N + 1. \)

- 2N + 2 unknowns \( A_0, A_1, B_1, \ldots, A_N, B_N, B_{N+1}. \)
- Continuity of \( \phi, \eta \partial_x \phi \) at \( N + 1 \) interfaces \( \sim \) 2N + 2 equations.
Dielectric multilayer slab waveguide

Trial value $\beta^2 \in \mathbb{R}$, $\beta/k > n_0, n_{N+1}$.

- $2N + 2$ unknowns $A_0, A_1, B_1, \ldots, A_N, B_N, B_{N+1}$.
- Continuity of $\phi, \eta \partial_x \phi$ at $N + 1$ interfaces $\sim \sim 2N + 2$ equations.
- Arrange as linear system of equations $M(\beta^2)(A_0, \ldots, B_{N+1})^T = 0$.
- Identify propagation constants where $M(\beta^2)$ becomes singular. (Equations relate to the series of interfaces $\leftrightarrow$ A transfer-matrix technique can be applied.)
- Choose e.g. $A_0 = 1$, fill $A_1, \ldots, B_{N+1}$, normalize. ($\ldots, \ldots$)

Guided modes $\{\beta_m, (\vec{E}_m, \vec{H}_m)\}$. 
A nonsymmetric 3-layer slab waveguide

\[ n_s = 1.45, \quad n_f = 1.99, \quad n_c = 1.0, \]
\[ d = 1.5 \, \mu m, \quad \lambda = 1.55 \, \mu m. \]

\[ \text{TE}_0: \quad n_{\text{eff}} = 1.944, \quad \text{TM}_0: \quad n_{\text{eff}} = 1.933, \]
\[ \text{TE}_1: \quad n_{\text{eff}} = 1.804, \quad \text{TM}_1: \quad n_{\text{eff}} = 1.759, \]
\[ \text{TE}_2: \quad n_{\text{eff}} = 1.562, \quad \text{TM}_2: \quad n_{\text{eff}} = 1.490. \]
**Dielectric multilayer slab waveguide, nodal properties**

(Fixed polarization, TE/TM.)

\[
\partial_x (\partial_x \phi) = -(k^2 n^2 - \beta^2) \phi.
\]

\(k^2 n^2 - \beta^2\) determines the rate of change of the slope of \(\phi\).

Imagine a numerical ODE algorithm of “shooting-type”.

- Guided modes with a growing number of nodes \((x\text{ with }\phi(x) = 0)\) with decreasing effective indices
  
  \(\text{mode indices} = \text{number of nodes in }\phi\).

  “Quantum numbers”. 
Dielectric multilayer slab waveguide, nodal properties

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  “Quantum numbers”.

• A fundamental mode with zero nodes and highest effective index.
Dielectric multilayer slab waveguide, nodal properties

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\[ \partial_x (\partial_x \phi) = -(k^2 n^2 - \beta^2) \phi. \]

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Imagine a numerical ODE algorithm of “shooting-type”.

• Guided modes with a growing number of nodes (\( x \) with \( \phi(x) = 0 \))
  with decreasing effective indices
  \( \text{mode indices} = \text{number of nodes in} \ \phi \).

  “Quantum numbers”.

• A fundamental mode with zero nodes and highest effective index.

• Modes of the same polarization are non-degenerate.
Dielectric multilayer slab waveguide, nodal properties

(Fixed polarization, TE/TM.)

\[
\partial_x (\partial_x \phi) = -(k^2 n^2 - \beta^2) \phi.
\]

\(k^2 n^2 - \beta^2\) determines the rate of change of the slope of \(\phi\).

Imagine a numerical ODE algorithm of “shooting-type”.

- A sign change of \(\partial_x \phi\) is required to form a guided mode.
  - There must be some region (layer) with \(k^2 n^2 - \beta^2 > 0\).

Interval for effective indices \(n_{\text{eff}}\) of guided modes:

\[
\max\{n_0, n_{N+1}\} < n_{\text{eff}} < \max_l\{n_l\}.
\]
3-layer slab waveguide, dispersion curves

Symmetric waveguide, moderate refractive index contrast,
\( n_s = 1.45, \quad n_f = 1.99, \quad n_c = 1.45. \)

(Caution: \( \partial_\lambda \epsilon = 0 \) assumed!)
3-layer slab waveguide, dispersion curves

Nonsymmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.0$.

(Caution: $\partial \lambda \epsilon = 0$ assumed!)
Symmetric waveguide, high refractive index contrast, 

\( n_s = 1.45, \ n_f = 3.45, \ n_c = 1.45. \)
3-layer slab waveguide, dispersion curves

Nonsymmetric waveguide, high refractive index contrast, 
\( n_s = 1.45, \ n_f = 3.45, \ n_c = 1.0. \)
3-layer slab waveguide, dispersion curves

Remarks / observations:

• At large core thicknesses, or short wavelengths, for all modes: $n_{\text{eff}}$ approaches the level $n_f$ of bulk waves in the core material.

• Modes of higher order at the same $n_{\text{eff}}$ supported by waveguides with thickness increased by specific distances.

Guided mode, layer $l$ with $\kappa_l^2 = (k^2 n^2 - \beta^2) > 0$, field $\phi(x) \sim \cos(\kappa_l x + \chi)$ for $x \in \text{layer } l$;
increase layer thickness by $\Delta x = \pi / \kappa_l$, such that $\kappa_l (x + \Delta x) = \kappa_l x + \pi$
$\rightarrow$ the thicker waveguide supports a mode of order $+1$ with the same propagation constant.

• Cutoff thicknesses at fixed wavelength.

Nonsymmetric 3-layer waveguide $n_s \neq n_c$: There exist cutoff thicknesses for all modes.
Symmetric 3-layer waveguide $n_s = n_c$: Cutoff thicknesses exist for all modes of order $\geq 1$,
no cutoff thickness for the fundamental TE/TM modes.

• $\lambda$ is the “length-defining” quantity; wavelength scaling, factor $a$:

$n_{\text{eff}}(\lambda, d) = n_{\text{eff}}(a\lambda, ad), \quad \beta(\lambda, d) = a^{-1} \beta(a\lambda, ad)$.

• Cutoff wavelengths for waveguides with fixed thickness.

For all modes; exception: no cutoff wavelength for the fundamental TE/TM modes in a symmetric 3-layer waveguide.
3-layer slab waveguide, mode confinement

Symmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \, \mu m$, $d = 1.50 \, \mu m$, TE$_0$: $n_{\text{eff}} = 1.946$. 
3-layer slab waveguide, mode confinement

Symmetric waveguide, moderate refractive index contrast, 

\( n_s = 1.45, \; n_f = 1.99, \; n_c = 1.45, \; \lambda = 1.55 \, \mu m, \)

\( d = 1.25 \, \mu m, \; TE_0: \; n_{eff} = 1.932. \)
3-layer slab waveguide, mode confinement

Symmetric waveguide, moderate refractive index contrast,
\( n_s = 1.45, \quad n_f = 1.99, \quad n_c = 1.45, \quad \lambda = 1.55 \text{ µm}, \)
\( d = 1.00 \text{ µm}, \quad \text{TE}_0: \quad n_{\text{eff}} = 1.908. \)
3-layer slab waveguide, mode confinement

Symmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \, \mu m$, $d = 0.75 \, \mu m$, $TE_0$: $n_{\text{eff}} = 1.868$. 

![Graph showing mode confinement with labels for $x$, $n_c$, $n_f$, $d$, $n_s$ and $E_y$ vs. $x$ in $\mu m$]
Symmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \, \mu\text{m}$, $d = 0.50 \, \mu\text{m}$, TE$_0$: $n_{\text{eff}} = 1.791$. 
3-layer slab waveguide, mode confinement

Symmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55$ $\mu$m, $d = 0.25$ $\mu$m, $TE_0$: $n_{eff} = 1.630$. 
3-layer slab waveguide, mode confinement

Symmetric waveguide,  
moderate refractive index contrast, 
\( n_s = 1.45, \ n_f = 1.99, \ n_c = 1.45, \ \lambda = 1.55 \ \mu m, \ 
\ d = 0.10 \ \mu m, \ \ TE_0: \ \ n_{eff} = 1.494. \)
3-layer slab waveguide, mode confinement

Symmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \, \mu m$, $d = 0.05 \, \mu m$, TE$_0$: $n_{eff} = 1.462$. 

![Graph showing mode profile](image-url)
Symmetric waveguide, moderate refractive index contrast, $n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55 \, \mu m$, $d = 0.02 \, \mu m$, $TE_0$: $n_{eff} = 1.452$. 

![Diagram of a 3-layer slab waveguide with mode confinement. The diagram shows a cross-sectional view of the waveguide with layers labeled $n_s$, $n_f$, and $n_c$, separated by a distance $d$. The field $E_y$ is plotted against $x$ [\mu m] with a linear scale.]
Symmetric waveguide, moderate refractive index contrast, 

\( n_s = 1.45, \; n_f = 1.99, \; n_c = 1.45, \; \lambda = 1.55 \, \mu\text{m}, \)

\( d = 0.01 \, \mu\text{m}, \; \text{TE}_0: \; n_{\text{eff}} = 1.450. \)
Field in the core:

\[ \sim a_u e^{-i(\kappa x + \beta z)} + a_d e^{-i(-\kappa x + \beta z)}, \quad k^2 n_f^2 = \beta^2 + \kappa^2 \]

\[\text{propagation angle } \theta \quad \text{with} \quad \beta = kn_f \cos \theta, \quad \kappa = kn_f \sin \theta.\]

Guided mode formation:
- Repeated total internal reflection of waves in the core at upper and lower interfaces
- Calculate optical phase gain, including phase jumps for reflection at interfaces (polarization dependent).
- Phase gain of \(2\pi\) for one “round trip”, “transverse resonance condition” \(\longleftrightarrow\) constructive interference of waves.

(A frequently encountered intuitive model . . . of very limited applicability.)
3-layer slab waveguide, ray model

Field in the core:
\[ \sim a_u e^{-i(\kappa x + \beta z)} + a_d e^{-i(-\kappa x + \beta z)}, \quad k^2 n_f^2 = \beta^2 + \kappa^2 \]

propagation angle \( \theta \) with \( \beta = k n_f \cos \theta, \) \( \kappa = k n_f \sin \theta. \)

Guided mode formation:
- Repeated total internal reflection of waves in the core at upper and lower interfaces
- Calculate optical phase gain, including phase jumps for reflection at interfaces (polarization dependent).
- Phase gain of \( 2\pi \) for one “round trip”, “transverse resonance condition” \( \longleftrightarrow \) constructive interference of waves.

(A frequently encountered intuitive model . . . of very limited applicability.)
Cross sections (2-D) of typical integrated-optical waveguides.
3-D rectangular waveguides

No analytical solutions:
- numerical mode solvers.
- approximations.
Effective index method

Divide into slices $\rho = I, II, III$:

- $n(x, y) = n_\rho(x)$ if $y \in$ slice $\rho$.
- Compute polarized modes $X_\rho(x)$, $\beta_\rho$, $X_\rho'' + (k^2 n_\rho^2 - \beta_\rho^2) X_\rho = 0$, $N_\rho = \beta_\rho / k$.
- Consider a scalar mode equation for the principal component $\Psi$ of the 3-D waveguide $\partial_x^2 \Psi + \partial_y^2 \Psi + (k^2 n^2 - \beta^2) \Psi = 0$, $\Psi = E_y (TE)$, $\Psi = H_y (TM)$.
- Ansatz: $\Psi(x, y) = X_\rho(x) Y(y)$, if $y \in$ slice $\rho$; require continuity of $Y$ and $Y'$.
- Effective index profile: $N(y) := N_\rho$, if $y \in$ slice $\rho$.

$Y'' + (k^2 N^2 - \beta^2) Y = 0$, a 1-D mode equation for $Y$, $\beta$ with the effective index profile $N$ in place of the refractive indices.
**Effective index method**

Outline:

- Divide into slices $\rho = I, II, III$: $n(x, y) = n_\rho(x)$, if $y \in$ slice $\rho$.
- Compute polarized modes $X_\rho(x), \beta_\rho$, $X'_\rho + (k^2 n^2_\rho - \beta^2_\rho)X_\rho = 0$, $N_\rho = \beta_\rho/k$.
- Consider a scalar mode equation for the principal component $\Psi$ of the 3-D waveguide
  \[ \partial_x^2 \Psi + \partial_y^2 \Psi + (k^2 n^2 - \beta^2)\Psi = 0, \quad \Psi = E_y \text{ (TE)}, \quad \Psi = H_y \text{ (TM)}. \]
- Ansatz: $\Psi(x, y) = X_\rho(x) Y(y)$, if $y \in$ slice $\rho$; require continuity of $Y$ and $Y'$.
- Effective index profile: $N(y) := N_\rho$, if $y \in$ slice $\rho$.

\[ Y'' + (k^2 N^2 - \beta^2)Y = 0, \]

a 1-D mode equation for $Y, \beta$ with the effective index profile $N$ in place of the refractive indices.
**Effective index method, schematically**

\[ n(x, y) \]

\[ N(y) \rightarrow \beta \]

\[ N_I \quad N_{II} \quad N_{III} \]

Remarks / issues:
- A popular, quite intuitive method.
- Frequently an (often informal) basis for discussion of waveguide properties.
- Relevance of the slab waveguide model.
- Manifold variants / ways of improvements exist.
- What if a slice does not support a guided slab mode?
- What about higher order modes?
- How to evaluate modal fields? What about other than principal components?
- ...
Variational effective index method

Outline:

• Identify a reference slice, refractive index profile $n_r(x)$.

• Compute polarized guided slab modes $(\bar{E}, \bar{H})_r$ for the reference slice.

• For each reference slab mode:
  ...

• Choose an ansatz: (VEIM) $E_x, E_y, E_z, H_x, H_y, H_z(x, y, z) = Y_{E_x}(y), Y_{H_x}(y)$.
Variational effective index method

Outline:

• Identify a reference slice, refractive index profile \( n_r(x) \).
• Compute polarized guided slab modes \( (\vec{E}, \vec{H})_r, \beta_r \) for the reference slice.
• For each each reference slab mode: \( \ldots \)
• Choose an ansatz: \( \text{(VEIM)} \)

\[
\begin{pmatrix}
E_x, E_y, E_z \\
H_x, H_y, H_z
\end{pmatrix}
(x, y, z) =
\begin{pmatrix}
0, & \bar{E}_{r,y}(x)Y^E_y(y), & \bar{E}_{r,y}(x)Y^E_z(y) \\
\bar{H}_{r,x}(x)Y^H_x(y), & \bar{H}_{r,z}(x)Y^H_y(y), & \bar{H}_{r,z}(x)Y^H_z(y)
\end{pmatrix}
\] \( \text{(TE)} \)

\[
\begin{pmatrix}
E_x, E_y, E_z \\
H_x, H_y, H_z
\end{pmatrix}
(x, y, z) =
\begin{pmatrix}
\bar{E}_{r,x}(x)Y^E_x(y), & \bar{E}_{r,z}(x)Y^E_y(y), & \bar{E}_{r,z}(x)Y^E_z(y) \\
0, & \bar{H}_{r,y}(x)Y^H_y(y), & \bar{H}_{r,y}(x)Y^H_z(y)
\end{pmatrix}
\] \( \text{(TM)} \)

\( Y^E(y) = ? \)
A functional for guided modes of 3-D dielectric waveguides

• \((\vec{E}, \vec{H}) (x, y, z) = (\vec{E}(x, y) e^{-i\beta z}, \vec{H}(x, y))\),  
  \(\beta \in \mathbb{R}\),  
  \(\vec{E}, \vec{H} \to 0\) for \(x, y \to \pm \infty\).

• \((C + i\beta R)\vec{E} = -i\omega \mu_0 \vec{H}, \quad (C + i\beta R)\vec{H} = i\omega \epsilon_0 \epsilon \vec{E}\),
  \(R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\),  
  \(C = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}\).

• \(B(\vec{E}, \vec{H}) := \frac{\omega \epsilon_0 \langle \vec{E}, \epsilon \vec{E} \rangle + \omega \mu_0 \langle \vec{H}, \vec{H} \rangle + i \langle \vec{E}, CH \rangle - i \langle \vec{H}, CE \rangle}{\langle \vec{E}, R\vec{H} \rangle - \langle \vec{H}, R\vec{E} \rangle}\),
  \(\langle F, G \rangle = \int \int F^* \cdot G \, dx \, dy\).

\[ B(\vec{E}, \vec{H}) = \beta, \quad \frac{d}{ds}B(\vec{E} + s \delta \vec{E}, \vec{H} + s \delta \vec{H}) \bigg|_{s=0} = 0 \]

at valid mode fields \(\vec{E}, \vec{H}\), for arbitrary \(\delta \vec{E}, \delta \vec{H}\).
Variational effective index method

Outline, continued:

- Restrict $B$ to the VEIM ansatz, require stationarity with respect to the $\{Y\cdot\}$.

$1$-D mode ("-like") equations for principal unknowns $Y^{Hx}$ (TE) and $Y^{Ex}$ (TM) with effective quantities in place of refractive indices, all other $Y\cdot$ can be computed.
Optical fibers

[ Optical Communication A-D ]
Circular step index optical fibers

Circular symmetry

\[ \varepsilon = n^2, \quad n(r) = \begin{cases} n_{\text{core}}, & r \leq a, \\ n_{\text{cladding}}, & r > a. \end{cases} \]

Circular and axial symmetry:

\[
\left( \frac{E}{H} \right)(r, \theta, z) = \left( \frac{\bar{E}}{\bar{H}} \right)(r) e^{-i l \theta - i \beta z}, \quad l \in \mathbb{Z}, \beta \in \mathbb{R}.
\]

Where \( \partial \varepsilon = 0 \):

\[
\Delta \psi + k^2 n^2 \psi = 0, \quad \psi \in \{E_r, \ldots H_z\}.
\]

\[
\partial_r^2 \phi + \frac{1}{r} \partial_r \phi + \left( k^2 n^2 - \beta^2 - \frac{l^2}{r^2} \right) \phi = 0, \quad \phi \in \{\bar{E}_r, \ldots \bar{H}_z\}
\]

(An ODE of Bessel type.)

\& vectorial interface conditions at \( r = a \).

(Alternatively: Scalar theory, LP modes.)
Next lectures:

- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- Bent optical waveguides; whispering gallery resonances; circular microresonators.
- Coupled mode theory, perturbation theory.