Optical Waveguide Theory (G)

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Paderborn University — Summer Semester 2018

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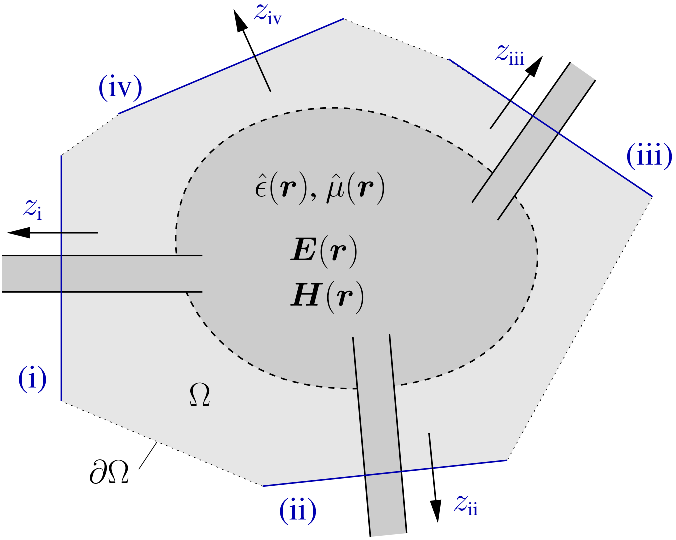
Course overview

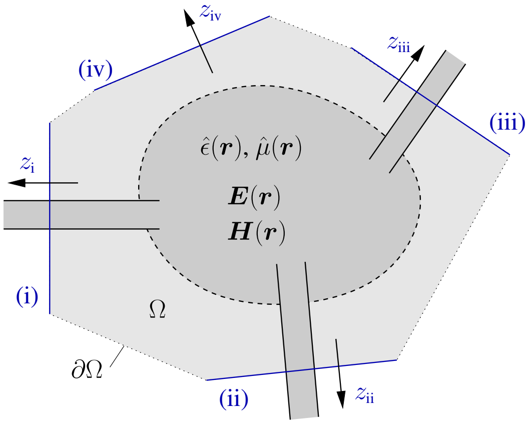
Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

PICs, OICs, scattering matrices



Scattering matrices, prerequisites



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Passive, linear circuit.

Scattering matrices, prerequisites

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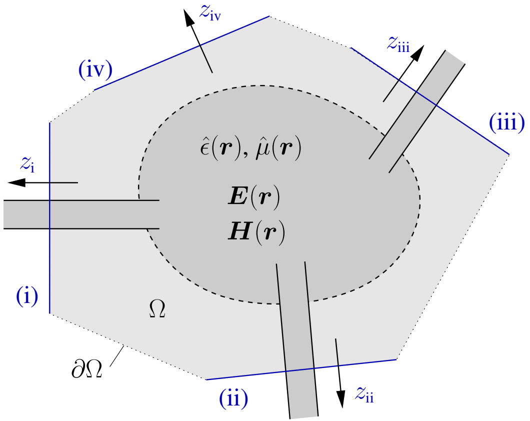
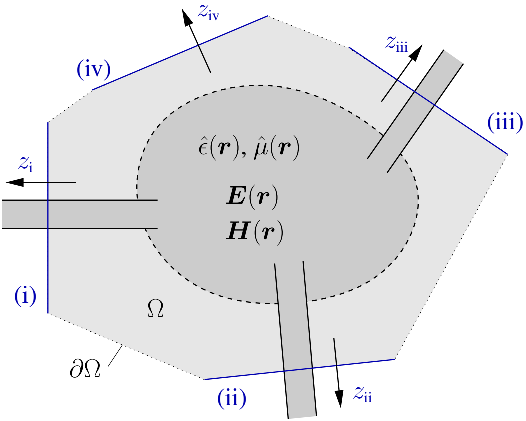


- (Computational) domain of interest Ω, its boundary $\partial\Omega$.

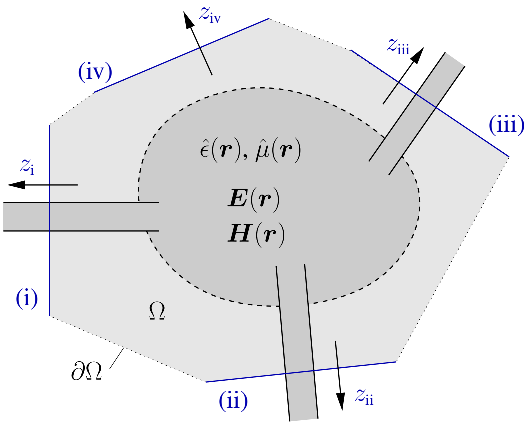
Scattering matrices, prerequisites



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Connecting channels: lossless waveguides (or “half-spaces”).

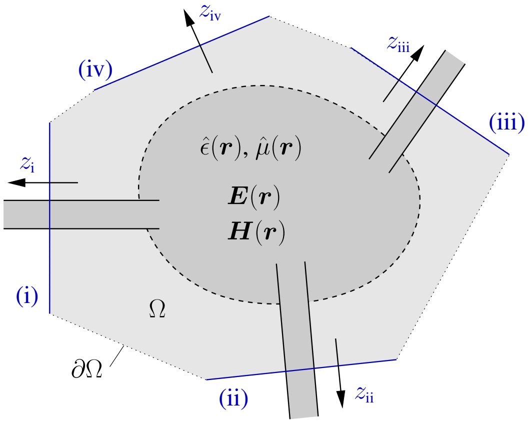
Scattering matrices, prerequisites



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Physical ports $p = \mathrm{i}, \mathrm{ii}, \ldots$: waveguide cross-section planes, local coordinates x_p, y_p, z_p; local axis z_p oriented outwards of Ω.

Scattering matrices, prerequisites

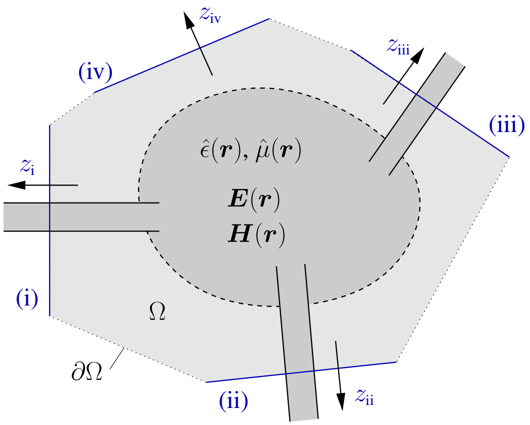


$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Establish sets $\mathcal{N}_p$ of *propagating* directional normal modes $\{\psi^d_{p,m} := (\boldsymbol{E}^d_{p,m}, \boldsymbol{H}^d_{p,m}),\ \beta_{p,m};\ d = \mathrm{f,b}\}$ on each port p.

(Restriction to propagating fields: a condition on port positioning / a model assumption.)

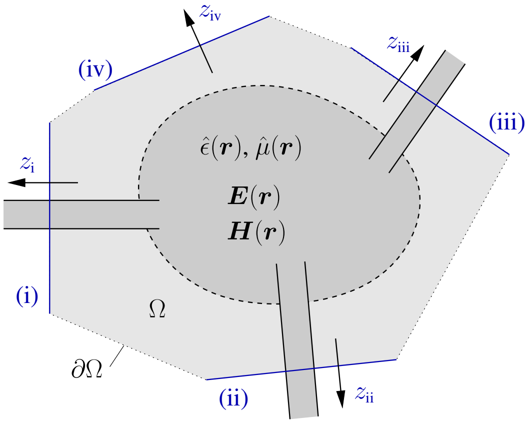
Scattering matrices, prerequisites



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Ports & modes are such that all mode fields vanish
  - on all "other" port planes, and
  - on $\partial\Omega$ outside the ports.

Scattering matrices, prerequisites



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

Field on port plane p and "outside":

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x_p, y_p, z_p) = \sum_{m\in\mathcal{N}_p} F_{p,m}\boldsymbol{\psi}^{\mathrm{f}}_{p,m}(x_p, y_p)\ \mathrm{e}^{-\mathrm{i}\,\beta_{p,m}z_p} + B_{p,m}\boldsymbol{\psi}^{\mathrm{b}}_{p,m}(x_p, y_p)\ \mathrm{e}^{\mathrm{i}\,\beta_{p,m}z_p}.$$

Scattering matrices, prerequisites

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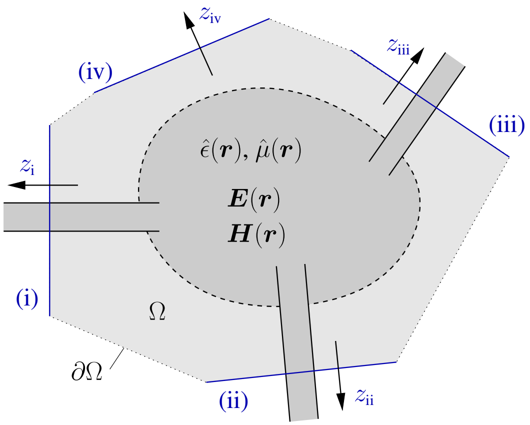
Scattering matrices

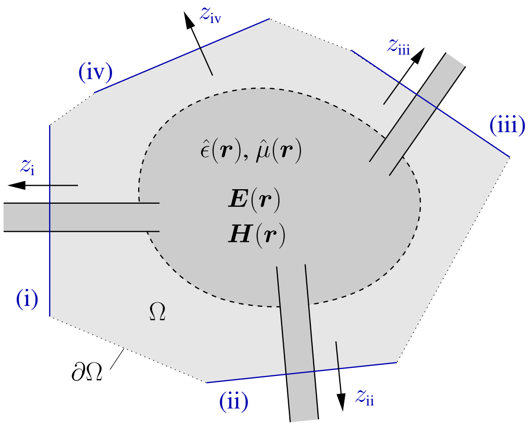


$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Merge all mode indices $\{m\}$ and port IDs $\{p\}$ into one set of mode identifiers $\{\nu\}$, $\quad \mathcal{N} = \cup_p \mathcal{N}_p$.

Scattering matrices

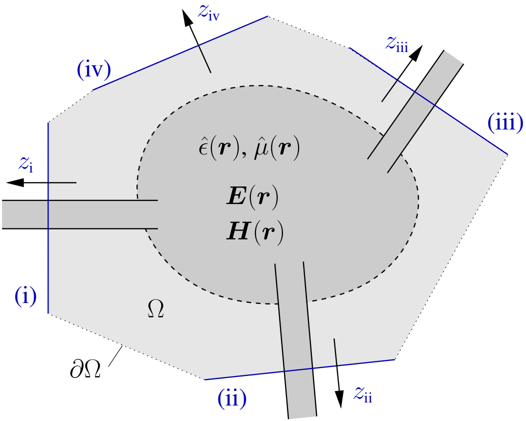
$\sim \exp(\mathrm{i}\,\omega t)$ (FD)



- Assert that $\psi^{\cdot}_{p,\cdot}(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \in \partial\Omega$, $\boldsymbol{r} \notin$ port p.

Scattering matrices

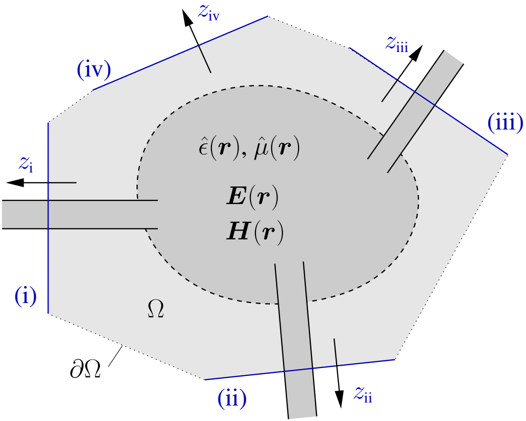
$\sim \exp(\mathrm{i}\,\omega t)$ (FD)



- Field on $\partial\Omega$:
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_{\nu \in \mathcal{N}} \left\{ F_\nu \boldsymbol{\psi}_\nu^{\mathrm{f}} + B_\nu \boldsymbol{\psi}_\nu^{\mathrm{b}} \right\}.$$

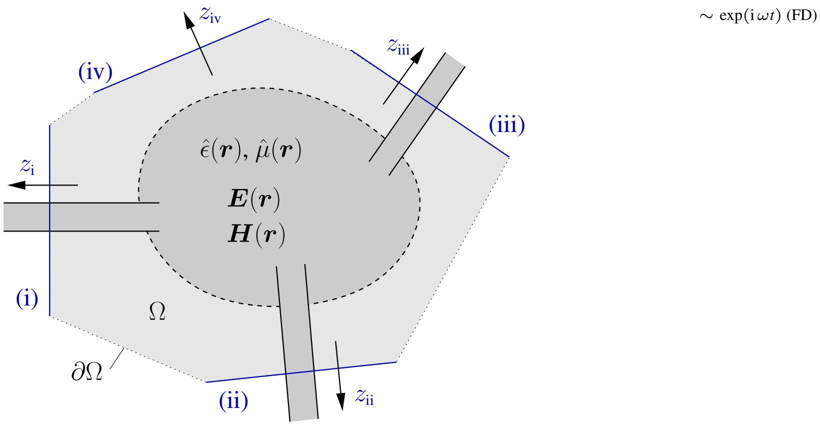
(Position arguments omitted.)

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)



- B_ν : $\sim$ incident modes, traveling towards the interior of Ω.
  F_ν : $\sim$ outgoing modes, traveling towards the exterior of Ω.
  Combine into amplitude vectors $\boldsymbol{B}$, $\boldsymbol{F}$.

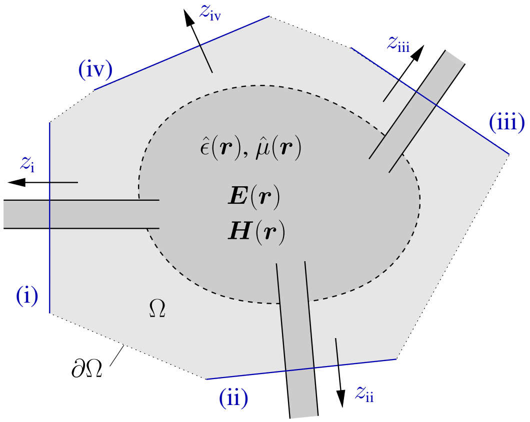
Scattering matrices



Linear circuit ⟷ linear dependence of $\boldsymbol{F}$ on $\boldsymbol{B}$,

Scattering matrix S of the circuit: $\boldsymbol{F} = \mathsf{S}\,\boldsymbol{B}$, $\mathsf{S} = (S_{\nu\mu})$.

Scattering matrices



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- $S_{\nu\nu}: \ \sim (\nu, \mathrm{b}) \to (\nu, \mathrm{f})$, reflection coefficient for mode ν.
- $S_{\nu\mu}: \ \sim (\mu, \mathrm{b}) \to (\nu, \mathrm{f})$, transmission coefficient for modes μ, ν.

Scattering matrices

- Merge all mode indices $\{m\}$ and port IDs $\{p\}$ $\quad\sim \exp(\mathrm{i}\,\omega t)$ (FD)
  into one set of mode identifiers $\{\nu\}$, $\quad \mathcal{N} = \cup_p \mathcal{N}_p$.
- Assert that $\psi^{\cdot}_{p,\cdot}(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \in \partial\Omega$, $\boldsymbol{r} \notin$ port p.
- Field on $\partial\Omega$:
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \sum_{\nu\in\mathcal{N}} \{F_\nu \psi_\nu^{\mathrm{f}} + B_\nu \psi_\nu^{\mathrm{b}}\}.$$
(Position arguments omitted.)
- B_ν : $\sim$ incident modes, traveling towards the interior of Ω.
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  Combine into amplitude vectors $\boldsymbol{B}$, $\boldsymbol{F}$.

Linear circuit $\longleftrightarrow$ linear dependence of $\boldsymbol{F}$ on $\boldsymbol{B}$,
Scattering matrix S of the circuit: $\quad \boldsymbol{F} = \mathsf{S}\,\boldsymbol{B}$, $\qquad \mathsf{S} = (S_{\nu\mu})$.

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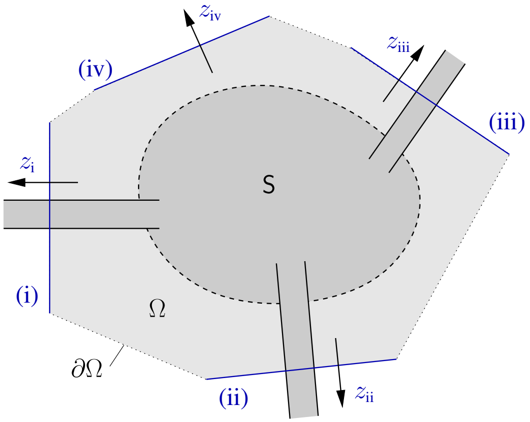
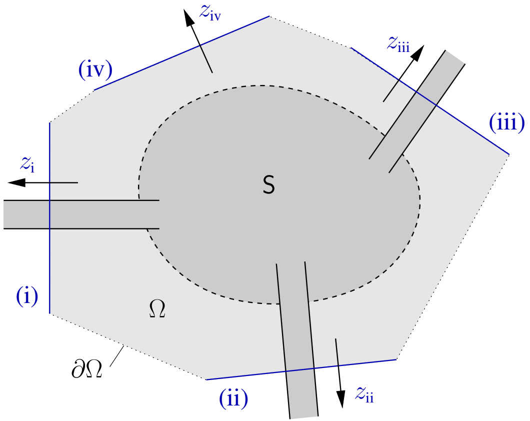


$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Scenario: Full matrix S, including guided and radiation modes, large $\dim \mathsf{S} \leftrightarrow$ theoretical results.

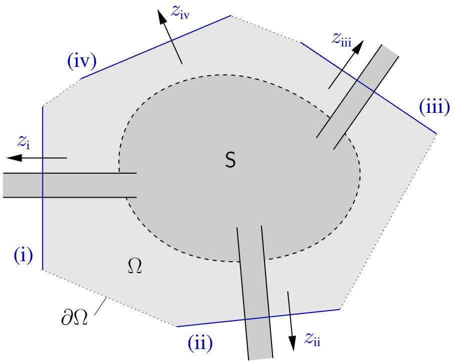


$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Scenario: Restrict to a specific set of (guided) modes, or: Only a small set of guided modes are relevant: small $\dim \mathsf{S} = N \times N \;\leftrightarrow\;$ an N-port circuit, a 2-N-pole.

(N: the total number of relevant modes, not the number of ports.)

Scattering matrices, port plane positions



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Shift port plane of mode ν by Δz_ν: $\quad F_\nu \;\rightarrow\; F'_\nu = F_\nu\,\mathrm{e}^{-\mathrm{i}\,\beta_\nu \Delta z_\nu}$,

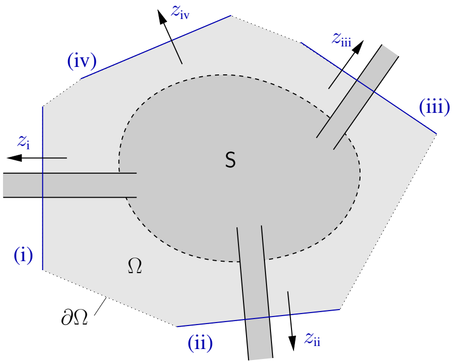
  Shift port plane of mode μ by Δz_μ: $\quad B_\mu \;\rightarrow\; B'_\mu = B_\mu\,\mathrm{e}^{\mathrm{i}\,\beta_\mu \Delta z_\mu}$,

  $$F'_\nu = S'_{\nu\mu}\, B'_\mu, \quad S'_{\nu\mu} = S_{\nu\mu}\,\mathrm{e}^{-\mathrm{i}\,(\beta_\nu \Delta z_\nu + \beta_\mu \Delta z_\mu)}.$$

(Moving port planes $\leftrightarrow$ Phase change in reflection / transmission coefficients.)

(Moving port planes $\leftrightarrow$ No effect on reflectances / transmittances.)

Scattering matrices, port mode orthogonality



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

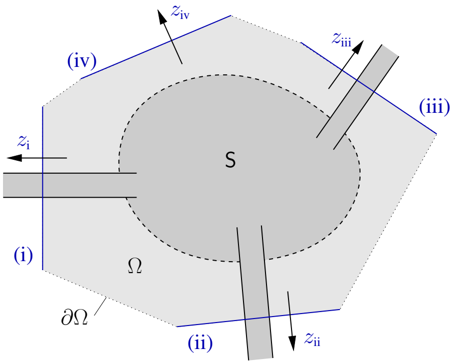
- Orthogonality relations on port plane p:

$$(\boldsymbol{E}_a, \boldsymbol{H}_a; \boldsymbol{E}_b, \boldsymbol{H}_b) = \frac{1}{4} \iint_p \left(E^*_{ax} H_{by} - E^*_{ay} H_{bx} + H^*_{ay} E_{bx} - H^*_{ax} E_{by} \right) \mathrm{d}x_p\, \mathrm{d}y_p$$

$$(\psi^d_{p,l}; \psi^r_{p,m}) = \pm \delta_{dr} \delta_{lm} P_{p,m}.$$

(Things restricted to propagating modes.)

Scattering matrices, port mode orthogonality



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

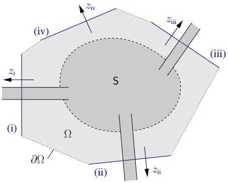
- Extend to the full boundary $\partial\Omega$:

$$(\boldsymbol{E}_a, \boldsymbol{H}_a; \boldsymbol{E}_b, \boldsymbol{H}_b) := \frac{1}{4} \int_{\partial\Omega} (\boldsymbol{E}_a^* \times \boldsymbol{H}_b + \boldsymbol{E}_b \times \boldsymbol{H}_a^*) \cdot \mathrm{d}\boldsymbol{a}$$

$$(\psi^d_{p,l}; \psi^r_{q,m}) = \pm\delta_{dr}\delta_{pq}\delta_{lm}P_{p,m} \quad \text{or} \quad (\psi^d_\nu; \psi^r_\mu) = \pm\delta_{dr}\delta_{\nu\mu}P_\nu .$$

(Modes belonging to different ports are mutually orthogonal.)

Scattering matrices, power balance

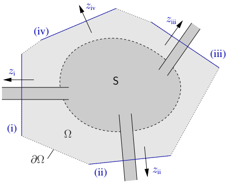


$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H}) \; = \sum_p \sum_{m \in \mathcal{N}_p} \left(|F_{p,m}|^2 - |B_{p,m}|^2 \right) P_{p,m}$$

Scattering matrices, power balance



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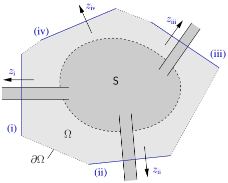
$$= \sum_{\nu\in\mathcal{N}} \left(|F_\nu|^2 - |B_\nu|^2\right) P_\nu \,,$$

Scattering matrices, power balance



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$$= \sum_{\nu \in \mathcal{N}} \left(|F_\nu|^2 - |B_\nu|^2 \right) P_\nu \,,$$

$|B_\mu|^2 P_\mu$: incident power carried by mode μ,
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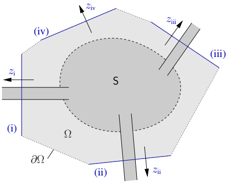
Scattering matrices, power balance



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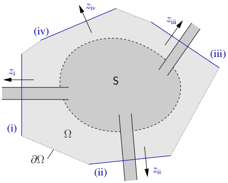
$$|S_{\nu\mu}|^2 \frac{P_\nu}{P_\mu} = \frac{|F_\nu|^2 P_\nu}{|B_\mu|^2 P_\mu}\,, \qquad \begin{array}{ll} \mu\neq\nu: & \text{power transmittance } \mu\to\nu, \\ \mu=\nu: & \text{power reflectance for mode } \nu. \end{array}$$

(Uniform normalized modes, $P_\nu = P_\mu$: transmittances are directly given by elements of the scattering matrix).

Scattering matrices, power balance



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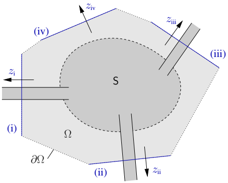
- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H}) = P_0 \left(\boldsymbol{B}^* \cdot (\mathsf{S}^\dagger \mathsf{S} - 1)\boldsymbol{B} \right),$$

uniform normalization, $P_\nu = P_0$ for all ν.

Scattering matrices, power balance



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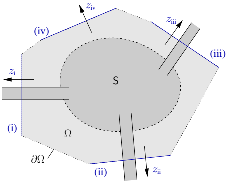
uniform normalization, $P_\nu = P_0$ for all ν.

- Lossless circuit $\longleftrightarrow$ $\int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = 0$ $\longleftrightarrow$ $\mathsf{S}^\dagger \mathsf{S} = 1$,
the scattering matrix of a lossless circuit is unitary.

Scattering matrices, power balance



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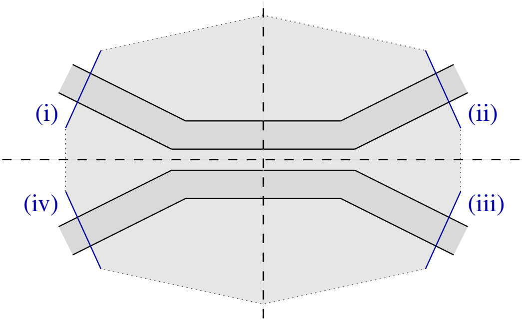
- Lossless circuit $\longleftrightarrow \int_{\partial\Omega} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} = 0 \longleftrightarrow \mathsf{S}^\dagger\mathsf{S} = \mathsf{1}$,

  the scattering matrix of a lossless circuit is unitary.

- Lossy circuit $\longleftrightarrow \int_{\partial\Omega} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} \le 0 \rightsquigarrow \boldsymbol{B}^*\cdot\mathsf{S}^\dagger\mathsf{S}\,\boldsymbol{B} \le \boldsymbol{B}^*\boldsymbol{B}$,

  $\sum_\nu |S_{\nu\mu}|^2 \le 1$ for all μ. (The sum of transmittances mode μ to all other modes ν is less than one.)
  (Interior lossy media, or radiative losses: outgoing propagating modes not taken into account.)

Scattering matrices, symmetry

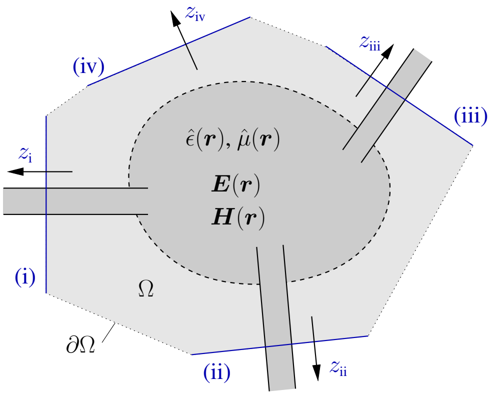


Circuit with specific spatial symmetry
& symmetrical setting of the port planes

↪ respective symmetry in related coefficients of S, symmetric power transmission properties.

Scattering matrices, reciprocity

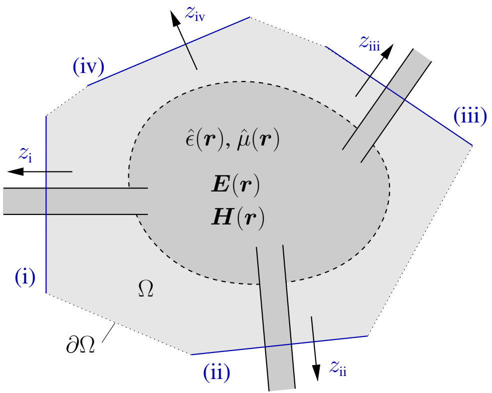
$\sim \exp(\mathrm{i}\,\omega t)$ (FD)



Circuit properties
for reversed
wave propagation ?

$S_{\nu\mu} \longleftrightarrow S_{\mu\nu}$?

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)



Circuit properties
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$S_{\nu\mu} \leftrightarrow S_{\mu\nu}$?

- $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$ solve $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H},\ \boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}.$

$\hookrightarrow$ $\boldsymbol{\nabla} \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{H}_1 \times \boldsymbol{E}_2) = 0,$ if $\hat{\epsilon}$ and $\hat{\mu}$ are symmetric.

(i.e. if $\hat{\epsilon}^{\mathsf{T}} = \epsilon,\ \hat{\mu}^{\mathsf{T}} = \mu$.)

(Note: order of factors, no complex conjugates.)

Scattering matrices, reciprocity

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Scattering matrices, reciprocity

- $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$ solve $\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H}$, $\nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}$

$$\hookrightarrow \quad \nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{H}_1 \times \boldsymbol{E}_2) = 0, \quad \text{if } \hat{\epsilon} \text{ and } \hat{\mu} \text{ are symmetric,}$$

$$\hookrightarrow \quad 0 = \int_\Omega \nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{H}_1 \times \boldsymbol{E}_2)\, \mathrm{d}^3 r = \int_{\partial\Omega} (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{H}_1 \times \boldsymbol{E}_2) \cdot \mathrm{d}\boldsymbol{a}.$$

Scattering matrices, reciprocity

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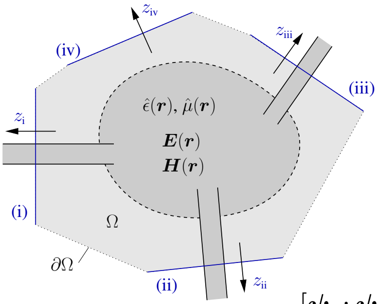
$$0 = \int_\Omega \nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{H}_1 \times \boldsymbol{E}_2)\,\mathrm{d}^3 r = \int_{\partial\Omega} (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{H}_1 \times \boldsymbol{E}_2) \cdot \mathrm{d}\boldsymbol{a}.$$

- Fields on $\partial\Omega$: $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}_j = \sum_{\nu \in \mathcal{N}} \{F_{j,\nu}\psi_\nu^{\mathrm{f}} + B_{j,\nu}\psi_\nu^{\mathrm{b}}\}, \qquad j = 1, 2,$

$$[\psi_a; \psi_b] := \int_{\partial\Omega} (\boldsymbol{E}_a \times \boldsymbol{H}_b + \boldsymbol{H}_a \times \boldsymbol{E}_b) \cdot \mathrm{d}\boldsymbol{a},$$

$$0 = \sum_\nu \sum_\mu \Big(F_{1,\nu}F_{2,\mu}[\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{f}}] + F_{1,\nu}B_{2,\mu}[\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{b}}] + B_{1,\nu}F_{2,\mu}[\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{f}}] + B_{1,\nu}B_{2,\mu}[\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{b}}] \Big).$$

Scattering matrices, reciprocity



$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

$$[\psi_a;\psi_b] := \int_{\partial\Omega} (\boldsymbol{E}_a \times \boldsymbol{H}_b + \boldsymbol{H}_a \times \boldsymbol{E}_b) \cdot \mathrm{d}\boldsymbol{a}.$$

- $[\psi^{\cdot}_{\nu};\psi^{\cdot}_{\mu}] = 0,$ if ν and μ relate to different ports.
- If ν and μ relate to the same port plane p:
  $$[\psi^r_\nu;\psi^d_\mu] = \iint_p \left(E^r_{\nu x}H^d_{\mu y} - E^r_{\nu y}H^d_{\mu x} - H^r_{\nu y}E^d_{\mu x} + H^r_{\nu x}E^d_{\mu y}\right) \mathrm{d}x_p\, \mathrm{d}y_p.$$

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p:

$$[\psi_\nu^r; \psi_\mu^d] = \iint_p \left(E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d\right) \mathrm{d}x_p \, \mathrm{d}y_p.$$

Scattering matrices, reciprocity

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$$[\psi_\nu^r;\psi_\mu^d] = \iint_p \left(E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d\right) \mathrm{d}x_p \, \mathrm{d}y_p.$$

- Compare with the modal orthogonality relations on port plane p,

$$(\psi_\nu^r;\psi_\mu^d) = \frac{1}{4}\iint_p \left((E_{\nu x}^r)^* H_{\mu y}^d - (E_{\nu y}^r)^* H_{\mu x}^d + (H_{\nu y}^r)^* E_{\mu x}^d - (H_{\nu x}^r)^* E_{\mu y}^d\right) \mathrm{d}x_p \, \mathrm{d}y_p.$$

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p:
$$[\psi_\nu^r; \psi_\mu^d] = \iint_p \left(E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d\right) \mathrm{d}x_p \, \mathrm{d}y_p.$$

- Compare with the modal orthogonality relations on port plane p, for propagating modes with real transverse components:
$$(\psi_\nu^r; \psi_\mu^d) = \frac{1}{4} \iint_p \left(E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d + H_{\nu y}^r E_{\mu x}^d - H_{\nu x}^r E_{\mu y}^d\right) \mathrm{d}x_p \, \mathrm{d}y_p,$$
$$(\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{f}}) = \delta_{\nu\mu} P_\nu, \quad (\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{b}}) = -\delta_{\nu\mu} P_\nu, \quad (\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{b}}) = (\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{f}}) = 0.$$

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p:

$$[\psi_\nu^r;\psi_\mu^d] = \iint_p \left(E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d\right) \mathrm{d}x_p \, \mathrm{d}y_p.$$

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$$(\psi_\nu^{\mathrm{f}};\psi_\mu^{\mathrm{f}}) = \delta_{\nu\mu}P_\nu, \quad (\psi_\nu^{\mathrm{b}};\psi_\mu^{\mathrm{b}}) = -\delta_{\nu\mu}P_\nu, \quad (\psi_\nu^{\mathrm{f}};\psi_\mu^{\mathrm{b}}) = (\psi_\nu^{\mathrm{b}};\psi_\mu^{\mathrm{f}}) = 0.$$

- $$\begin{aligned} \psi^{\mathrm{f}} &= (E_x, E_y, \quad \mathrm{i}E_z, \quad H_x, \quad H_y, \mathrm{i}H_z)^{\mathsf{T}} \\ \longleftrightarrow \quad \psi^{\mathrm{b}} &= (E_x, E_y, -\mathrm{i}E_z, \; -H_x, -H_y, \mathrm{i}H_z)^{\mathsf{T}}. \end{aligned}$$

(Real components).

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p:

$$[\psi_\nu^r; \psi_\mu^d] = \iint_p \left(E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d\right) \mathrm{d}x_p\, \mathrm{d}y_p.$$

- Compare with the modal orthogonality relations on port plane p, for propagating modes with real transverse components:

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$$(\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{f}}) = \delta_{\nu\mu} P_\nu, \quad (\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{b}}) = -\delta_{\nu\mu} P_\nu, \quad (\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{b}}) = (\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{f}}) = 0.$$

- $$\begin{aligned} \psi^{\mathrm{f}} &= (E_x, E_y, \ \mathrm{i}E_z, \ \ H_x, \ \ H_y, \mathrm{i}H_z)^{\mathsf{T}} \\ \leftrightarrow \quad \psi^{\mathrm{b}} &= (E_x, E_y, -\mathrm{i}E_z, -H_x, -H_y, \mathrm{i}H_z)^{\mathsf{T}}. \end{aligned}$$

(Real components).

$$[\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{f}}] = [\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{b}}] = 0, \quad [\psi_\nu^{\mathrm{f}}; \psi_\mu^{\mathrm{b}}] = -\delta_{\nu\mu} 4P_\nu, \quad [\psi_\nu^{\mathrm{b}}; \psi_\mu^{\mathrm{f}}] = \delta_{\nu\mu} 4P_\nu.$$

Scattering matrices, reciprocity

$$0 = \sum_\nu 4P_\nu \big(B_{1,\nu}F_{2,\nu} - F_{1,\nu}B_{2,\nu}\big),$$

uniform normalization $P_\nu = P_0$,

$$0 = \sum_\nu \big(B_{1,\nu}F_{2,\nu} - F_{1,\nu}B_{2,\nu}\big),$$

$$0 = \boldsymbol{B}_1 \cdot \boldsymbol{F}_2 - \boldsymbol{F}_1 \cdot \boldsymbol{B}_2,$$

$$\boldsymbol{F}_j = \mathsf{S}\,\boldsymbol{B}_j,$$

$$0 = \boldsymbol{B}_1 \cdot \mathsf{S}\,\boldsymbol{B}_2 - (\mathsf{S}\,\boldsymbol{B}_1) \cdot \boldsymbol{B}_2,$$

$$0 = \boldsymbol{B}_1 \cdot \mathsf{S}\,\boldsymbol{B}_2 - \boldsymbol{B}_1 \cdot \mathsf{S}^\mathsf{T}\,\boldsymbol{B}_2,$$

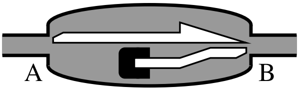
$$0 = \boldsymbol{B}_1 \cdot \big(\mathsf{S} - \mathsf{S}^\mathsf{T}\big)\boldsymbol{B}_2 \quad \text{for all } \boldsymbol{B}_1, \boldsymbol{B}_2.$$

$$\mathsf{S} = \mathsf{S}^\mathsf{T}, \qquad S_{\nu\mu} = S_{\mu\nu} \text{ for all } \nu, \mu.$$

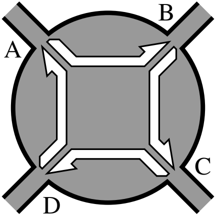
The scattering matrix of a reciprocal circuit is symmetric.

Reciprocal circuit: made of reciprocal media, with $\hat{\epsilon} = \hat{\epsilon}^\mathsf{T}$, $\hat{\mu} = \hat{\mu}^\mathsf{T}$.

Nonreciprocal devices



Isolator:
unidirectional transmission,
$S_{\text{BA}} = 1, \quad S_{\text{AB}} = 0.$



Circulator:
transmission cycle,
$S_{\text{BA}} = 1, \quad S_{\text{CB}} = 1, \quad S_{\text{DC}} = 1, \quad S_{\text{AD}} = 1,$
$S_{..} = 0$ otherwise.

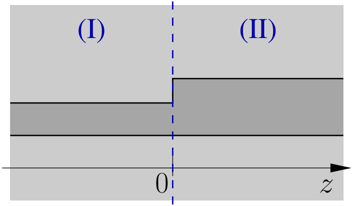
Required: nonreciprocal media with $\hat{\epsilon} \neq \hat{\epsilon}^{\mathsf{T}}$,
$\leftrightsquigarrow$ magnetooptic media, Faraday effect.

Nonreciprocal devices

What about, for example,

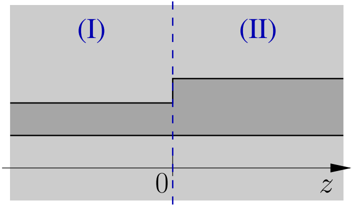
- a long, “adiabatic” Y-junction ?
- a junction between a single mode core and a wider multimode waveguide ?

Waveguide discontinuities



Half-infinite waveguides (I), (II), discontinuity at $z = 0$.

Waveguide discontinuities



Half-infinite waveguides (I), (II), discontinuity at $z = 0$.

- Expand into local normal modes $\{\psi^d_{s,m}, \beta_{s,m}\}$, $m \in \mathcal{N}_s$, $s = \mathrm{I}, \mathrm{II}$:

Transverse boundary conditions $\leftrightsquigarrow$ discrete sets.

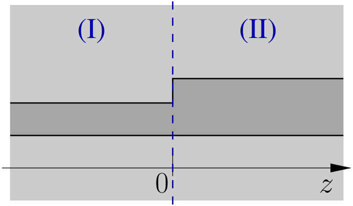
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}_s (x,y,z) = \sum_{m \in \mathcal{N}_s} \left\{ f_{s,m} \psi^{\mathrm{f}}_{s,m}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{s,m} z} + b_{s,m} \psi^{\mathrm{b}}_{s,m}(x,y)\ \mathrm{e}^{+\mathrm{i}\beta_{s,m} z} \right\},$$

$z < 0$: $s = \mathrm{I}$, $f_{\mathrm{I},m}$ given influx, $b_{\mathrm{I},m}$ unknown,

$z > 0$: $s = \mathrm{II}$, $f_{\mathrm{II},m}$ unknown, $b_{\mathrm{II},m}$ given influx.

$\hookrightarrow$ $(\boldsymbol{E}, \boldsymbol{H})_{\mathrm{I,II}}$ are solutions for $z < 0$ and $z > 0$.

Waveguide discontinuities



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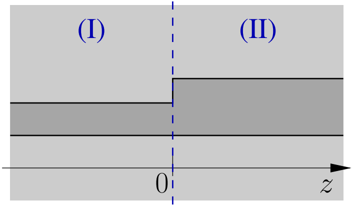
$\hookrightarrow$ $(\boldsymbol{E}, \boldsymbol{H})_{\mathrm{I,II}}$ are solutions for $z < 0$ and $z > 0$.

- Continuity of the tangential components of $\boldsymbol{E}, \boldsymbol{H}$ at the interface

$\leftrightsquigarrow$ formally equate expressions for $(\boldsymbol{E}, \boldsymbol{H})_{\mathrm{I,II}}$ at $z = 0$.

(Only equality of E_x, E_y, H_x, H_y will be relevant.)



Half-infinite waveguides (I), (II), discontinuity at $z = 0$.

- Expand into local normal modes $\{\psi^d_{s,m}, \beta_{s,m}\}$, $m \in \mathcal{N}_s$, $s = \mathrm{I}, \mathrm{II}$:

  Transverse boundary conditions $\leftrightsquigarrow$ discrete sets.

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}_s (x,y,z) = \sum_{m\in\mathcal{N}_s} \left\{ f_{s,m}\psi^{\mathrm{f}}_{s,m}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{s,m}z} + b_{s,m}\psi^{\mathrm{b}}_{s,m}(x,y)\ \mathrm{e}^{+\mathrm{i}\beta_{s,m}z} \right\},$$

$z < 0$: $s = \mathrm{I}$, $f_{\mathrm{I},m}$ given influx, $b_{\mathrm{I},m}$ unknown,
$z > 0$: $s = \mathrm{II}$, $f_{\mathrm{II},m}$ unknown, $b_{\mathrm{II},m}$ given influx.

$\hookrightarrow$ $(\boldsymbol{E},\boldsymbol{H})_{\mathrm{I,II}}$ are solutions for $z < 0$ and $z > 0$.

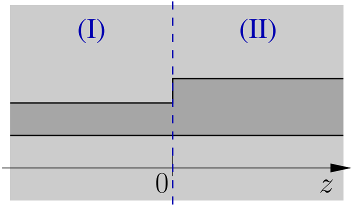
- Continuity of the tangential components of $\boldsymbol{E},\boldsymbol{H}$ at the interface

  $\leftrightsquigarrow$ formally equate expressions for $(\boldsymbol{E},\boldsymbol{H})_{\mathrm{I,II}}$ at $z = 0$.

  (Only equality of E_x, E_y, H_x, H_y will be relevant.)

- Project on $\psi^d_{s,l}$ to extract coefficients ...

Waveguide discontinuities, scattering matrix



(Global coordinate $z \neq$ former local coordinate on port I.)
(One variant of a projection procedure.)

- $(\psi^{\mathrm{b}}_{\mathrm{I},l};\ \cdot = \cdot\ ),\quad l \in \mathcal{N}_{\mathrm{I}}:$

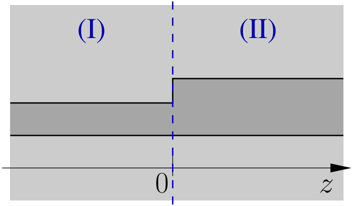
$$\sum_{m\in\mathcal{N}_{\mathrm{I}}} \Big[f_{\mathrm{I},m}(\psi^{\mathrm{b}}_{\mathrm{I},l}; \psi^{\mathrm{f}}_{\mathrm{I},m}) + b_{\mathrm{I},m}(\psi^{\mathrm{b}}_{\mathrm{I},l}; \psi^{\mathrm{b}}_{\mathrm{I},m}) \Big] = \sum_{m\in\mathcal{N}_{\mathrm{II}}} \Big[f_{\mathrm{II},m}(\psi^{\mathrm{b}}_{\mathrm{I},l}; \psi^{\mathrm{f}}_{\mathrm{II},m}) + b_{\mathrm{II},m}(\psi^{\mathrm{b}}_{\mathrm{I},l}; \psi^{\mathrm{b}}_{\mathrm{II},m}) \Big]\,,$$

- $(\psi^{\mathrm{f}}_{\mathrm{II},l};\ \cdot = \cdot\ ),\quad l \in \mathcal{N}_{\mathrm{II}}:$

$$\sum_{m\in\mathcal{N}_{\mathrm{I}}} \Big[f_{\mathrm{I},m}(\psi^{\mathrm{f}}_{\mathrm{II},l}; \psi^{\mathrm{f}}_{\mathrm{I},m}) + b_{\mathrm{I},m}(\psi^{\mathrm{f}}_{\mathrm{II},l}; \psi^{\mathrm{b}}_{\mathrm{I},m}) \Big] = \sum_{m\in\mathcal{N}_{\mathrm{II}}} \Big[f_{\mathrm{II},m}(\psi^{\mathrm{f}}_{\mathrm{II},l}; \psi^{\mathrm{f}}_{\mathrm{II},m}) + b_{\mathrm{II},m}(\psi^{\mathrm{f}}_{\mathrm{II},l}; \psi^{\mathrm{b}}_{\mathrm{II},m}) \Big]\,,$$

$$\hookrightarrow \quad \dots \quad \rightsquigarrow \qquad \begin{pmatrix} \boldsymbol{b}_{\mathrm{I}} \\ \boldsymbol{f}_{\mathrm{II}} \end{pmatrix} = \mathsf{S} \begin{pmatrix} \boldsymbol{f}_{\mathrm{I}} \\ \boldsymbol{b}_{\mathrm{II}} \end{pmatrix} = \begin{pmatrix} \mathsf{S}_{\mathrm{I,I}} & \mathsf{S}_{\mathrm{I,II}} \\ \mathsf{S}_{\mathrm{II,I}} & \mathsf{S}_{\mathrm{II,II}} \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_{\mathrm{I}} \\ \boldsymbol{b}_{\mathrm{II}} \end{pmatrix}.$$

Waveguide discontinuities, overlap model



Most simplified variant:
Unidirectional overlap model.

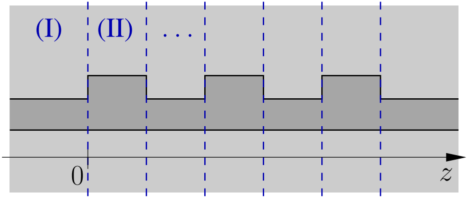
- (I) : Incoming guided mode ψ_{I}, reflections & radiation neglected.
  (II) : Outgoing guided modes $\psi_{\mathrm{II},m}$, radiation neglected.

- $$f_{\mathrm{I}}\,\psi_{\mathrm{I}} \approx \sum_m f_{\mathrm{II},m}\,\psi_{\mathrm{II},m} \quad \text{at } z = 0.$$

$$f_{\mathrm{II},m} = \frac{(\psi_{\mathrm{II},m};\psi_{\mathrm{I}})}{(\psi_{\mathrm{II},m};\psi_{\mathrm{II},m})}\,f_{\mathrm{I}}, \quad \text{or} \quad f_{\mathrm{II},m} = \frac{1}{P_{\mathrm{II},m}}\,(\psi_{\mathrm{II},m};\psi_{\mathrm{I}})\,f_{\mathrm{I}}.$$

(Transmission is given directly by the “overlaps” ↔ Relevance of the mode products $(\,\cdot\,;\,\cdot\,)$.)
(Cf. explicit expressions for overlaps of 2-D modes, involving only principal mode profile components.)

A sequence of waveguide discontinuities



- Divide into segments.
- Establish local normal mode expansions.
- Project on local modes.

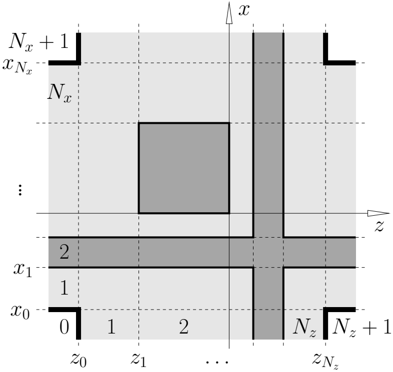
↪ Linear system of equations for all local mode amplitudes.

↪ Solve (. . .) $\rightsquigarrow$ $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z)$.

Bidirectional eigenmode propagation (BEP),
Eigenmode expansion method (EME),
. . . .

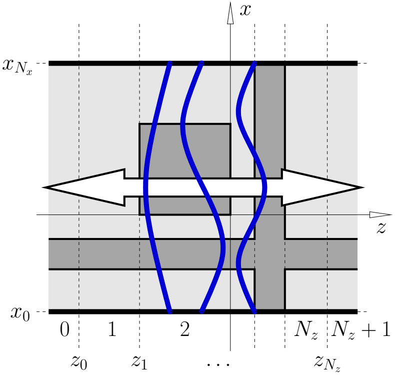
(Radiated outgoing fields: Open boundary conditions required (PMLs) $\leftrightsquigarrow$ Complex eigenmodes.)
(2-D: ok. 3-D: ?)

Rectangular 2-D circuits



- Divide into slices & layers.

Rectangular 2-D circuits

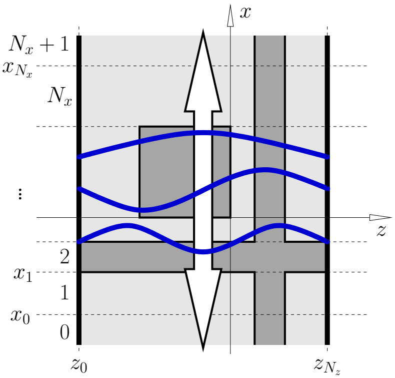


- Divide into slices & layers.
- Establish local modes:

  Propagation along $\pm z$,

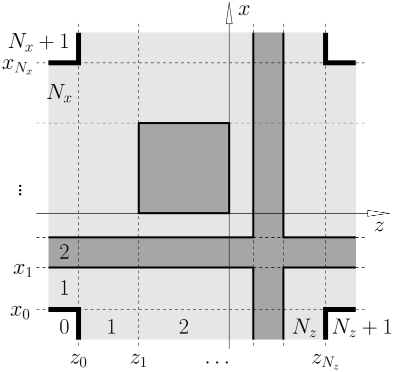
  boundary conditions $\phi = 0$.

Rectangular 2-D circuits



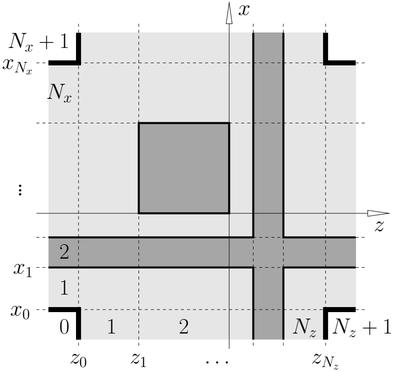
- Divide into slices & layers.
- Establish local modes:
  Propagation along $\pm z$,
  & Propagation along $\pm x$,
  boundary conditions $\phi = 0$.

Rectangular 2-D circuits



- Divide into slices & layers.
- Establish local modes:
  Propagation along $\pm z$,
  & Propagation along $\pm x$,
  boundary conditions $\phi = 0$.
- Project at horizontal
  & vertical interfaces.

Rectangular 2-D circuits



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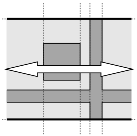
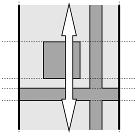
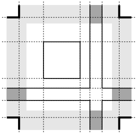
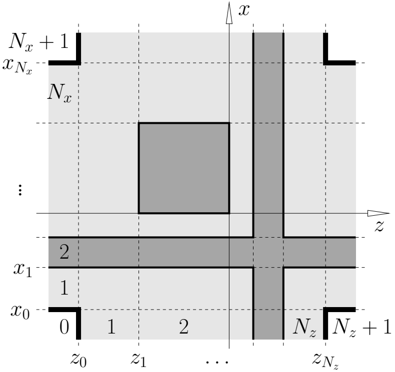






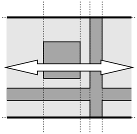
horizontal BEP, vertical BEP, continuity at x_0, x_N, z_0, z_N.

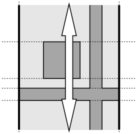
Rectangular 2-D circuits

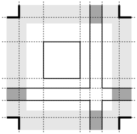


Quadridirectional Eigenmode Propagation (QUEP)

- Divide into slices & layers.
- Establish local modes:
  Propagation along $\pm z$,
  & Propagation along $\pm x$,
  boundary conditions $\phi = 0$.
- Project at horizontal
  & vertical interfaces.







horizontal BEP, vertical BEP, continuity at x_0, x_N, z_0, z_N.

Upcoming

Next lectures:

- Bent optical waveguides; whispering gallery resonances; circular microresonators.
- Coupled mode theory, perturbation theory.
- A touch of photonic crystals; a touch of plasmonics.