Optical Waveguide Theory (I)

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Optical waveguide theory

A Photonics / integrated optics; theory, motto; phenomena, introductory examples.

B Brush up on mathematical tools.

C Maxwell equations, different formulations, interfaces, energy and power flow.

D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.

E Normal modes of dielectric optical waveguides, mode interference.

F Examples for dielectric optical waveguides.

G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.

H Bent optical waveguides; whispering gallery resonances; circular microresonators.

I Coupled mode theory, perturbation theory.

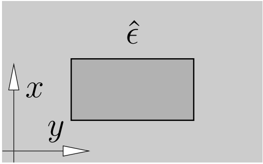
- Hybrid analytical / numerical coupled mode theory.

J A touch of photonic crystals; a touch of plasmonics.

- Oblique semi-guided waves: 2-D integrated optics.
- Summary, concluding remarks.

Perturbations of single modes

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

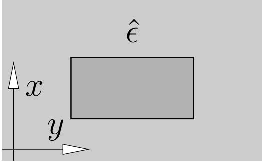


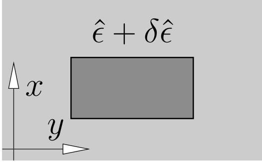
$\lambda, \ \hat{\epsilon}(x, y)$

$\beta,$
$\bar{\boldsymbol{E}}, \ \bar{\boldsymbol{H}}$

Perturbations of single modes

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)





$$\lambda,\ \ \hat{\epsilon}(x,y)$$

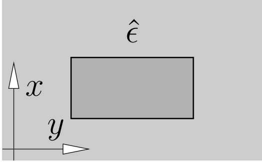
$$\beta,$$
$$\bar{\boldsymbol{E}},\ \bar{\boldsymbol{H}}$$

$$\lambda,\ \hat{\epsilon}(x,y) + \delta\hat{\epsilon}(x,y)$$

$$\beta + \delta\beta,$$
$$\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}},\ \bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}$$

Perturbations of single modes

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)



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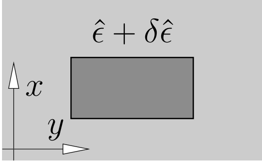


$\lambda, \ \hat{\epsilon}(x, y)$

$\beta,$
$\bar{\boldsymbol{E}}, \ \bar{\boldsymbol{H}}$

$\rightsquigarrow$

$\lambda, \ \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$

$\beta + \delta\beta,$
$\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}}, \ \bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}$

A functional for guided modes of 3-D dielectric waveguides

(→ *Exercise.*)

- $$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}, \qquad \beta \in \mathbb{R},$$
  $$\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} \to 0 \ \text{ for } \ x, y \to \pm\infty.$$

- $$(\mathsf{C} + \mathrm{i}\,\beta\mathsf{R})\bar{\boldsymbol{E}} = -\mathrm{i}\omega\mu_0\bar{\boldsymbol{H}}, \quad (\mathsf{C} + \mathrm{i}\,\beta\mathsf{R})\bar{\boldsymbol{H}} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\bar{\boldsymbol{E}},$$
  $$\mathsf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

- $$\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) := \frac{\omega\epsilon_0\langle\bar{\boldsymbol{E}}, \hat{\epsilon}\bar{\boldsymbol{E}}\rangle + \omega\mu_0\langle\bar{\boldsymbol{H}}, \bar{\boldsymbol{H}}\rangle + \mathrm{i}\,\langle\bar{\boldsymbol{E}}, \mathsf{C}\bar{\boldsymbol{H}}\rangle - \mathrm{i}\,\langle\bar{\boldsymbol{H}}, \mathsf{C}\bar{\boldsymbol{E}}\rangle}{\langle\bar{\boldsymbol{E}}, \mathsf{R}\bar{\boldsymbol{H}}\rangle - \langle\bar{\boldsymbol{H}}, \mathsf{R}\bar{\boldsymbol{E}}\rangle},$$
  $$\langle\bar{\boldsymbol{F}}, \bar{\boldsymbol{G}}\rangle = \iint \bar{\boldsymbol{F}}^* \cdot \bar{\boldsymbol{G}}\,\mathrm{d}x\,\mathrm{d}y.$$

$$\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta \ \ (*), \qquad \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}} + s\,\bar{\boldsymbol{F}}, \bar{\boldsymbol{H}} + s\,\bar{\boldsymbol{G}})\Big|_{s=0} = 0 \ \ (**)$$

at valid mode fields $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$, for arbitrary $\bar{\boldsymbol{F}}, \bar{\boldsymbol{G}}$.

$(*)$: "arbitrary" $\hat{\epsilon}$.
$(**)$: Hermitian $\hat{\epsilon}$.

Perturbations of single modes

- Available: Mode $\beta, \bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$ for parameters $\lambda,\ \hat{\epsilon}$; $(\hat{\epsilon} = \hat{\epsilon}^\dagger)$
  $\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta$, $\mathcal{B}_{\hat{\epsilon}}$ stationary at $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$.

- Investigate parameters $\lambda,\ \hat{\epsilon} + \delta\hat{\epsilon}$, for a "small" change $\delta\hat{\epsilon}$:

$$\mathcal{B}_{\hat{\epsilon}+\delta\hat{\epsilon}}(\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}) = \beta + \delta\beta$$

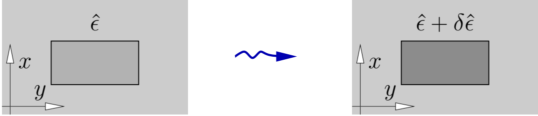
... $$\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}) \approx \mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta$$

... $$\cancel{\delta(\cdot)\,\delta(\cdot)}$$

$$\delta\beta = \frac{\omega\epsilon_0 \langle \bar{\boldsymbol{E}}, \delta\hat{\epsilon}\,\bar{\boldsymbol{E}}\rangle}{\langle \bar{\boldsymbol{E}}, \mathsf{R}\bar{\boldsymbol{H}}\rangle - \langle \bar{\boldsymbol{H}}, \mathsf{R}\bar{\boldsymbol{E}}\rangle}, \quad \text{or} \quad \delta\beta = \frac{\omega\epsilon_0 \iint \bar{\boldsymbol{E}}^* \cdot \delta\hat{\epsilon}\,\bar{\boldsymbol{E}}\,\mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re} \iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}.$$

(Valid for *small* perturbations: The original mode profiles are good approximations of the true fields in the modified structure.)

Small uniform change in refractive index



- $n \longrightarrow n + \delta n$ on $\square$, $\quad n, \delta n$ constant on $\square$

$$\beta \longrightarrow \beta + \delta\beta, \qquad \delta\beta = \frac{\omega\epsilon_0\, n \iint_{\square} |\bar{\boldsymbol{E}}|^2 \,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x\right) \,\mathrm{d}x\,\mathrm{d}y}\, \delta n\,.$$

($\delta\epsilon = 2n\delta n$.)
(Plausible: $\delta\beta \sim \delta n$, $\delta\beta \sim |\bar{\boldsymbol{E}}|^2|_{\square}$.)

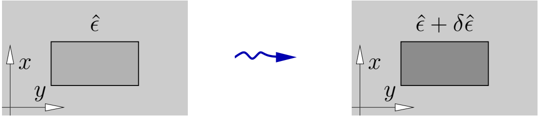
Small attenuation

- $n \longrightarrow n - \mathrm{i}n''$ on $\square$, n, n'' constant on $\square$, $n, n'' \in \mathbb{R}$

$$\beta \longrightarrow \beta + \delta\beta, \quad \delta\beta = \frac{-\mathrm{i}\omega\epsilon_0\, n \iint_{\square} |\bar{\boldsymbol{E}}|^2 \,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x\right) \,\mathrm{d}x\,\mathrm{d}y}\, n''\,.$$

($\delta\epsilon = -\mathrm{i}\,2nn''$.)
(Different attenuation for each mode.)
(Damping, power, plane wave: $\sim \exp(-2kn''z)$, mode: $\not\sim \exp(-2kn''z)$.)

Small anisotropy



- $\epsilon\hat{1} \longrightarrow \epsilon\hat{1} + \delta\hat{\epsilon}$ on $\Box$, $\epsilon, \delta\hat{\epsilon}$ constant on $\Box$

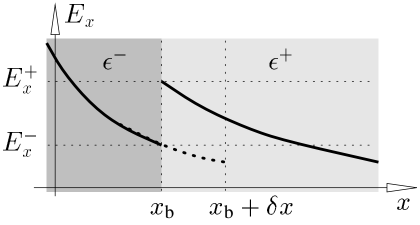
$$\beta \longrightarrow \beta + \delta\beta, \qquad \delta\beta = \frac{\omega\epsilon_0 \iint_{\Box} \bar{\boldsymbol{E}}^* \cdot \delta\hat{\epsilon}\, \bar{\boldsymbol{E}}\, \mathrm{d}x\, \mathrm{d}y}{2\,\mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x\right)\, \mathrm{d}x\, \mathrm{d}y}.$$

(Phase shifts due to anisotropic contributions to the permittivity.)

(Polarization coupling might occur for modes with “close” propagation constants CMT.)

Small displacements of dielectric interfaces

Interface displacement ⟷ Locally *strong* thin layer perturbation.
Field discontinuity ⇝ Previous expressions are not directly applicable.



- $\epsilon^- \neq \epsilon^+$, shift of interface $x_b \to x_b + \delta x$.

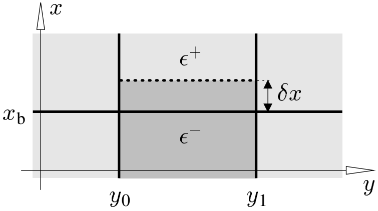
- Reposition discontinuity in field: $E_x \to E_x + \delta E_x$,

$$\delta E_x(x,y) = \begin{cases} \dfrac{\epsilon^+ - \epsilon^-}{\epsilon^-} E_x(x,y), & \text{for } x_b < x < x_b + \delta x, \\ 0, & \text{otherwise.} \end{cases}$$

- Use functional with locally modified field

↪ . . . (omitted) . . . ⇝

Small displacements of dielectric interfaces

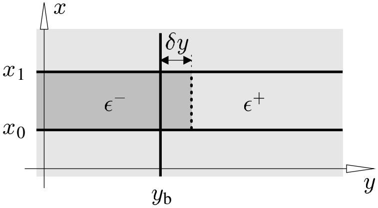


- Displacement of the interface at x_b between y_0 and y_1 by δx:

$$\beta \longrightarrow \beta + \delta\beta,$$

$$\delta\beta = \frac{\omega\epsilon_0}{2}\,\frac{(\epsilon^- - \epsilon^+)\displaystyle\int_{y_0}^{y_1}\left(\frac{1}{\epsilon^-\epsilon^+}|\epsilon\bar{E}_x|^2 + |\bar{E}_y|^2 + |\bar{E}_z|^2\right)(x_\mathrm{b}, y)\,\mathrm{d}y}{\mathrm{Re}\displaystyle\iint\left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}\,\delta x\,.$$

Small displacements of dielectric interfaces

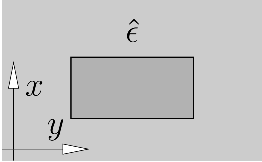


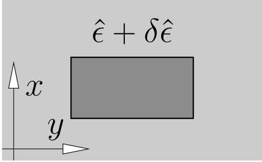
- Displacement of the interface at y_b between x_0 and x_1 by δy:

$$\beta \longrightarrow \beta + \delta\beta,$$

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} \left(|\bar{E}_x|^2 + \frac{1}{\epsilon^-\epsilon^+} |\epsilon \bar{E}_y|^2 + |\bar{E}_z|^2 \right)(x, y_\mathrm{b})\,\mathrm{d}x}{\mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x \right)\,\mathrm{d}x\,\mathrm{d}y}\,\delta y\,.$$

Perturbations of single modes





$$\lambda,\ \hat{\epsilon}(x,y) \quad \rightsquigarrow \quad \lambda,\ \hat{\epsilon}(x,y) + \delta\hat{\epsilon}(x,y)$$

$$\beta, \qquad\qquad \beta + \delta\beta,$$

$$\bar{\boldsymbol{E}},\ \bar{\boldsymbol{H}} \qquad\qquad \approx \bar{\boldsymbol{E}},\ \approx \bar{\boldsymbol{H}}$$

- View $\frac{\delta\beta}{\delta p}$ as $\frac{\partial\beta}{\partial p}$: slope of the dispersion curves β vs. p.
- Depending on the parametrization, change of a parameter value might require several perturbations.
- First order theory: In case of multiple pertubations, add the effects of the individual expressions.
- Estimation of fabrication tolerances: The phase shifts $\delta\beta$ enter into respective scattering matrix models.
- Wavelength shifts . . . ?

Small shift of frequency or vacuum wavelength

$(*)$: Explicit frequency dependence of $\mathcal{B}$ & dependence through $\hat{\epsilon}$.
$(**)$: Frequency dependence of $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$.

$$\beta(\omega) = \mathcal{B}_{\hat{\epsilon}}(\omega;\ \bar{\boldsymbol{E}}(\omega),\ \bar{\boldsymbol{H}}(\omega))$$

$$\begin{aligned}\frac{\partial\beta}{\partial\omega} &= \frac{\partial\mathcal{B}_{\hat{\epsilon}}}{\partial\omega}_{\ (*)} + \frac{\partial}{\partial s}\mathcal{B}_{\hat{\epsilon}}\left(\omega;\ \bar{\boldsymbol{E}} + s\frac{\partial\bar{\boldsymbol{E}}}{\partial\omega},\ \bar{\boldsymbol{H}}\right)\bigg|_{s=0}\ (**) \\ &\qquad + \frac{\partial}{\partial s}\mathcal{B}_{\hat{\epsilon}}\left(\omega;\ \bar{\boldsymbol{E}},\ \bar{\boldsymbol{H}} + s\frac{\partial\bar{\boldsymbol{H}}}{\partial\omega}\right)\bigg|_{s=0}\ (**) \\ &= \frac{\partial\mathcal{B}_{\hat{\epsilon}}}{\partial\omega},\end{aligned}$$

(Stationarity of $\mathcal{B}$ at $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$.)

$$\frac{\partial\beta}{\partial\omega} = \frac{\iint\left(\epsilon_0\,\bar{\boldsymbol{E}}^* \cdot \frac{\partial(\omega\hat{\epsilon})}{\partial\omega}\,\bar{\boldsymbol{E}} + \mu_0\,|\bar{\boldsymbol{H}}|^2\right)\mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re}\iint\left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}.$$

Small shift of frequency or vacuum wavelength

If dispersion can be neglected, $\partial_\omega \hat{\epsilon} = 0$:

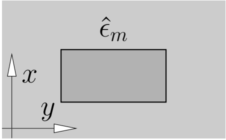
$$\frac{\partial \beta}{\partial \omega} = \frac{\iint \left(\epsilon_0 \, \bar{\boldsymbol{E}}^* \cdot \hat{\epsilon} \, \bar{\boldsymbol{E}} + \mu_0 \, |\bar{\boldsymbol{H}}|^2 \right) \mathrm{d}x \, \mathrm{d}y}{2 \, \mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x \right) \, \mathrm{d}x \, \mathrm{d}y},$$

$$\frac{\partial \beta}{\partial \lambda} = -\frac{\pi \mathrm{c}}{\lambda^2} \frac{\iint \left(\epsilon_0 \, \bar{\boldsymbol{E}}^* \cdot \hat{\epsilon} \, \bar{\boldsymbol{E}} + \mu_0 \, |\bar{\boldsymbol{H}}|^2 \right) \mathrm{d}x \, \mathrm{d}y}{\mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x \right) \, \mathrm{d}x \, \mathrm{d}y}.$$

($\omega = 2\pi c/\lambda \longleftrightarrow \partial_\lambda \omega = -2\pi c/\lambda^2$)
(Compare with expression based on homogeneity, H, 12.)

Coupled mode theory (CMT)

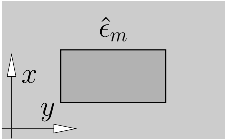
$\sim \exp(\mathrm{i}\,\omega t)$ (FD)

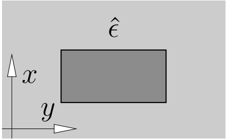


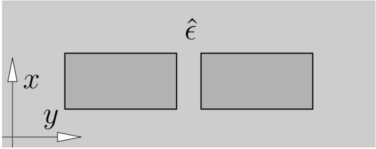
$$\left\{ \hat{\epsilon}_m;\ \beta_m, (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) \right\}$$

Coupled mode theory (CMT)

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)





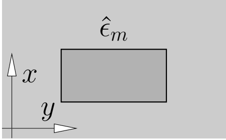


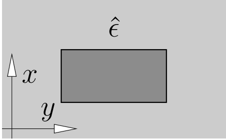
$$\left\{\hat{\epsilon}_m;\; \beta_m, (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m)\right\}$$

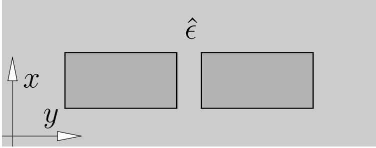
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} (x, y, z)$$

Coupled mode theory (CMT)

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)







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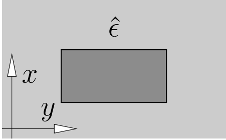
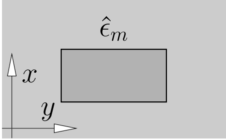
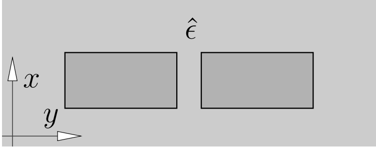
$$\left\{\hat{\epsilon}_m;\ \beta_m, (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m)\right\} \quad \rightsquigarrow \quad \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} (x, y, z)$$

Coupled mode theory (CMT)

$\sim \exp(\mathrm{i}\,\omega t)$ (FD)







?

$$\left\{ \hat{\epsilon}_m;\ \beta_m, (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) \right\} \quad \rightsquigarrow \quad \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} (x, y, z)$$

(Next: One of many variants of approaches to CMT.)

(Propagation & interaction of basis fields along a common propagation coordinate.)

[D.G. Hall, B.J. Thompson, *Selected papers on Coupled-Mode Theory in Guided-Wave Optics*, SPIE Milestone series MS 84 (1993)]

(Codirectional coupling (here), versus contradirectional coupling, coupling to radiation modes, nonlinear coupling.)

(Hybrid variant (HCMT): separate lecture.)

Coupled mode theory (CMT)

- Investigate a permittivity $\hat{\epsilon}$, look for fields $\boldsymbol{E}, \boldsymbol{H}$ with
  $\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}.$

  ($\hat{\epsilon}(x, y, z)$, in general.)

- Available: A set of fields $\{\boldsymbol{E}_m, \boldsymbol{H}_m\}$ for permittivities $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$;
  $\nabla \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0\boldsymbol{H}_m, \quad \nabla \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0\hat{\epsilon}_m\boldsymbol{E}_m.$

  (Not necessarily "modes".)

Coupled mode theory (CMT)

- Investigate a permittivity $\hat{\epsilon}$, look for fields $\boldsymbol{E}, \boldsymbol{H}$ with
  $\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0 \hat{\epsilon} \boldsymbol{E}.$

  ($\hat{\epsilon}(x, y, z)$, in general.)

- Available: A set of fields $\{\boldsymbol{E}_m, \boldsymbol{H}_m\}$ for permittivities $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$;
  $\nabla \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m, \quad \nabla \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0 \hat{\epsilon}_m \boldsymbol{E}_m.$

  (Not necessarily "modes".)

- Assume that $(\boldsymbol{E}, \boldsymbol{H})$ can be well approximated by

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) \approx \sum_m C_m(z) \begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x, y, z),$$

  C_m : unknown amplitudes, common propagation coordinate z.

  (Choose $\hat{\epsilon}_m$ as close as possible to $\hat{\epsilon}$.)

Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{H} \times \boldsymbol{E}_l^* - \boldsymbol{E} \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l)\boldsymbol{E}.$$

(Insert CMT ansatz for $\boldsymbol{E}$, $\boldsymbol{H}$.)

$$\ldots$$

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$$\ldots$$

(Apply identity $\boldsymbol{\nabla} \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}.$)

$$\ldots$$

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$$\ldots$$

(Manipulate, arrange terms, tidy up.)

Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\nabla \cdot (\boldsymbol{H} \times \boldsymbol{E}_l^* - \boldsymbol{E} \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l)\boldsymbol{E}.$$

(Insert CMT ansatz for $\boldsymbol{E}$, $\boldsymbol{H}$.)

$\ldots$

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$\ldots$

(Apply identity $\nabla \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}.$)

$\ldots$

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$\ldots$

(Manipulate, arrange terms, tidy up.)

$$\sum_m o_{lm}\,\partial_z C_m = -\mathrm{i} \sum_m k_{lm}\, C_m \quad \forall\, l, \qquad \text{coupled mode equations.}$$

$$o_{lm} = \frac{1}{4} \iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z\, \mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \boldsymbol{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)\boldsymbol{E}_m\, \mathrm{d}x\,\mathrm{d}y.$$

Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\nabla \cdot (\boldsymbol{H} \times \boldsymbol{E}_l^* - \boldsymbol{E} \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l)\boldsymbol{E}.$$

(Insert CMT ansatz for $\boldsymbol{E}$, $\boldsymbol{H}$.)

$\ldots$

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$\ldots$

(Apply identity $\nabla \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}.$)

$\ldots$

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$\ldots$

(Manipulate, arrange terms, tidy up.)

$$\mathsf{O}\,\partial_z \boldsymbol{C} = -\mathrm{i}\mathsf{K}\boldsymbol{C},$$

coupled mode equations.

$$\boldsymbol{C} = (C_m),\ \mathsf{O} = (o_{lm}),\ \mathsf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4}\iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4}\iint \boldsymbol{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)\boldsymbol{E}_m \,\mathrm{d}x\,\mathrm{d}y.$$

Coupled mode theory (CMT)

(Variational derivation of CMT equations.)

$$\mathcal{F}(\boldsymbol{E},\boldsymbol{H}) = \iiint \Big\{ \boldsymbol{H}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{E}^* \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) + \mathrm{i}\omega\mu_0 \boldsymbol{H}^* \cdot \boldsymbol{H} + \mathrm{i}\omega\epsilon_0 \boldsymbol{E}^* \cdot \hat{\epsilon}\boldsymbol{E} \Big\}\, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\delta\mathcal{F} = 0 \;\; \forall\, \delta\boldsymbol{E}, \delta\boldsymbol{H} \quad \leftrightsquigarrow \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\boldsymbol{H}, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}.$$

(Restrict $\mathcal{F}$ to the CMT ansatz for $\boldsymbol{E}, \boldsymbol{H}$ $\rightsquigarrow$ $\mathcal{F}_\mathrm{c}(\boldsymbol{C})$, require $\delta\mathcal{F}_\mathrm{c} = 0 \;\; \forall \delta\boldsymbol{C}$.)

$\hookrightarrow$ ...

($\boldsymbol{\nabla} \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0\boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}$, $\iint \mathrm{d}x\,\mathrm{d}y$, $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

$\hookrightarrow$ ...

(Manipulate, arrange terms, tidy up.)

$$\mathsf{O}\,\partial_z\boldsymbol{C} = -\mathrm{i}\mathsf{K}\boldsymbol{C}, \qquad \text{coupled mode equations.}$$

$$\boldsymbol{C} = (C_m),\;\; \mathsf{O} = (o_{lm}),\;\; \mathsf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4}\iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z\,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4}\iint \boldsymbol{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)\boldsymbol{E}_m\,\mathrm{d}x\,\mathrm{d}y.$$

$\ldots$

$$\mathsf{O}\,\partial_z \boldsymbol{C} = -\mathrm{i}\mathsf{K}\boldsymbol{C}, \qquad \boldsymbol{C} = (C_m),\ \mathsf{O} = (o_{lm}),\ \mathsf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4}\iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4}\iint \boldsymbol{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)\boldsymbol{E}_m \,\mathrm{d}x\,\mathrm{d}y.$$

- A set of coupled *ordinary* linear differential equations, of first order. (Here.)
- o_{lm} : power coupling coefficients (field overlaps). (No reason to assume $o_{lm} = \delta_{lm}$, in general.)
- k_{lm} : coupling coefficients.
- z-dependence of $\hat{\epsilon}, \hat{\epsilon}_m, \boldsymbol{E}_m, \boldsymbol{H}_m$ $\rightsquigarrow$ $o_{lm}(z),\ k_{lm}(z),\ \mathsf{O}(z),\ \mathsf{K}(z)$. (Compare the bend-straight couplers, Lecture H.)

$\ldots$ to be solved by numerical procedures. (In general.)

CMT for longitudinally homogeneous structures

$\partial_z \hat{\epsilon} = 0,\ \ \partial_z \hat{\epsilon}_m = 0,$

basis: guided modes $\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\,\beta_m z},$

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \sum_m C_m(z)\ \begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \sum_m c_m(z)\ \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y).$$

($c_m(z) = C_m(z)\exp(-\mathrm{i}\,\beta_m z)$, rewrite CMT equations for $c_m(z)$.)

$\ldots$

($\boldsymbol{\nabla}\cdot(\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}$, integrate, rewrite for $\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m$.)

$\ldots$

(Symmetrize coefficients.)

CMT for longitudinally homogeneous structures

$\partial_z \hat{\epsilon} = 0, \;\; \partial_z \hat{\epsilon}_m = 0,$

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$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \sum_m C_m(z) \begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \sum_m c_m(z) \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y).$$

($c_m(z) = C_m(z)\exp(-\mathrm{i}\,\beta_m z)$, rewrite CMT equations for $c_m(z)$.)

$\hookrightarrow$ …

($\boldsymbol{\nabla} \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}$, integrate, rewrite for $\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m$.)

$\hookrightarrow$ …

(Symmetrize coefficients.)

$$\sum_m \sigma_{lm}\, \partial_z c_m = -\mathrm{i} \sum_m (b_{lm} + \kappa_{lm})\; c_m \quad \forall\, l,$$

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y\,, \qquad b_{lm} = \sigma_{lm}\,\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

CMT for longitudinally homogeneous structures

$\partial_z \hat{\epsilon} = 0$, $\partial_z \hat{\epsilon}_m = 0$,

basis: guided modes $\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}$,

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \sum_m C_m(z) \begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,y,z) = \sum_m c_m(z) \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x,y).$$

($c_m(z) = C_m(z)\exp(-\mathrm{i}\,\beta_m z)$, rewrite CMT equations for $c_m(z)$.)

$\hookrightarrow$ …

($\boldsymbol{\nabla}\cdot(\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\,\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}$, integrate, rewrite for $\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m$.)

$\hookrightarrow$ …

(Symmetrize coefficients.)

$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}\,(\mathsf{B}+\mathsf{Q})\,\boldsymbol{c}$, $\boldsymbol{c} = (c_m)$, $\mathsf{S} = (\sigma_{lm})$, $\mathsf{B} = (b_{lm})$, $\mathsf{Q} = (\kappa_{lm})$,

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y, \qquad b_{lm} = \sigma_{lm}\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

Longitudinally constant structures, coupled mode equations

. . . $(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$

$$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}(\mathsf{B}+\mathsf{Q})\boldsymbol{c}, \quad \boldsymbol{c} = (c_m), \ \mathsf{S} = (\sigma_{lm}), \ \mathsf{B} = (b_{lm}), \ \mathsf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y\,, \qquad b_{lm} = \sigma_{lm}\,\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- A set of coupled *ordinary* linear differential equations, of first order (Here.)
- σ_{lm} : power coupling coefficients (field overlaps). (No reason to assume $\sigma_{lm} = \delta_{lm}$, in general.)
- κ_{lm} : coupling coefficients.
- $\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0 \rightsquigarrow \partial_z \sigma_{lm} = \partial_z b_{lm} = \partial_z \kappa_{lm} = 0.$

(ODEs with constant coefficents.)

. . . quasi-analytical solutions.

Longitudinally constant structures, coupled mode equations

$\ldots$ $(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$

$$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}(\mathsf{B}+\mathsf{Q})\boldsymbol{c}, \quad \boldsymbol{c} = (c_m), \; \mathsf{S} = (\sigma_{lm}), \; \mathsf{B} = (b_{lm}), \; \mathsf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y\,, \qquad b_{lm} = \sigma_{lm}\,\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- $\sigma_{ml}^* = \sigma_{lm}, \quad b_{ml}^* = b_{lm}; \quad \kappa_{ml}^* = \kappa_{lm}, \text{ if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \; \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m,$
  $\mathsf{S}^\dagger = \mathsf{S}, \quad \mathsf{B}^\dagger = \mathsf{B}; \quad \mathsf{Q}^\dagger = \mathsf{Q}, \text{ if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \; \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m.$
- Power: $P = (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H}) = \sum_{l,m} c_l^* (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) c_m = \boldsymbol{c}^* \cdot \mathsf{S}\boldsymbol{c}$

$$\partial_z P = \mathrm{i}\boldsymbol{c}^* \cdot \big((\mathsf{B}+\mathsf{Q})^\dagger - (\mathsf{B}+\mathsf{Q})\big)\boldsymbol{c}, \quad \partial_z P = 0 \text{ for } \mathsf{B}^\dagger = \mathsf{B}, \; \mathsf{Q}^\dagger = \mathsf{Q}.$$

(For lossless waveguides the scheme is power conservative.)

Longitudinally constant structures, formal solution

$$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}\,(\mathsf{B}+\mathsf{Q})\boldsymbol{c}, \qquad \partial_z\mathsf{S} = \partial_z\mathsf{B} = \partial_z\mathsf{Q} = 0.$$

Ansatz: $\boldsymbol{c}(z) = \boldsymbol{a}\;\mathrm{e}^{-\mathrm{i}bz}$, $\boldsymbol{a}, b$ constants.

$$(\mathsf{B}+\mathsf{Q})\boldsymbol{a} = b\,\mathsf{S}\boldsymbol{a},$$ a generalized eigenvalue problem.

(Dimension: number of basis modes included.)

Solutions: $\{\boldsymbol{a}, b\}$,

"supermodes" $$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \left(\sum_m a_m \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\right)\;\mathrm{e}^{-\mathrm{i}bz}.$$

(Superpositions of the original mode profiles with constant coefficients.)

(As many supermodes as there are basis modes.)

(Formalism can be continued: power / orthogonality of supermodes . . .)

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)
(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) = \delta_{lm} P_0.$$

(Orthogonal modes, uniform normalization $P_m = P_0$.)
(Or: apply inverse of S to CM equations, continue with redefined expressions for β_m, κ_{lm}.)

$$\begin{pmatrix} \partial_z c_1 \\ \partial_z c_2 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \beta_1' & \kappa \\ \kappa^* & \beta_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \begin{aligned} \beta_l' &= \beta_l + \kappa_{ll}/P_0, \\ \kappa &= \kappa_{12}/P_0. \end{aligned}$$

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)

(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) = \delta_{lm} P_0.$$

(Orthogonal modes, uniform normalization $P_m = P_0$.)

(Or: apply inverse of S to CM equations, continue with redefined expressions for β_m, κ_{lm}.)

$$\begin{pmatrix} \partial_z c_1 \\ \partial_z c_2 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \begin{aligned} \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ \kappa &= \kappa_{12}/P_0. \end{aligned}$$

$\ldots$

$\ldots$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(z) = \mathrm{e}^{-\mathrm{i}\frac{(\beta'_1+\beta'_2)}{2}z} \begin{pmatrix} \cos\ \rho z - \mathrm{i}\dfrac{\Delta\beta'}{2\rho}\sin\rho z & -\mathrm{i}\dfrac{\kappa}{\rho}\sin\rho z \\ -\mathrm{i}\dfrac{\kappa^*}{\rho}\sin\rho z & \cos\ \rho z + \mathrm{i}\dfrac{\Delta\beta'}{2\rho}\sin\rho z \end{pmatrix} \begin{pmatrix} c_{10} \\ c_{20} \end{pmatrix},$$

$$\Delta\beta' = \beta'_1 - \beta'_2, \quad \rho = \sqrt{\left(\frac{\Delta\beta'}{2}\right)^2 + |\kappa|^2}.$$

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)
(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) = \delta_{lm} P_0.$$

(Orthogonal modes, uniform normalization $P_m = P_0$.)
(Or: apply inverse of S to CM equations, continue with redefined expressions for β_m, κ_{lm}.)

$$\begin{pmatrix} \partial_z c_1 \\ \partial_z c_2 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \beta_1' & \kappa \\ \kappa^* & \beta_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \begin{aligned} \beta_l' &= \beta_l + \kappa_{ll}/P_0, \\ \kappa &= \kappa_{12}/P_0. \end{aligned}$$

- $c_{20} = 0 \rightsquigarrow \left| \dfrac{c_2(z)}{c_1(0)} \right|^2 = \eta_{\max} \sin^2(\rho z), \quad \eta_{\max} = \dfrac{|\kappa|^2}{|\kappa|^2 + (\Delta\beta'/2)^2}.$

- Maximum conversion $\eta_{\max}$ at $z = L_\mathrm{c}$ with $\rho L_\mathrm{c} = \pi/2$,

  coupling length $L_\mathrm{c} = \dfrac{\pi}{\sqrt{(\Delta\beta')^2 + 4|\kappa|^2}}$,

  (Conversion length, half-beat length.)

- In case of phase matching $\Delta\beta' = \beta_1' - \beta_2' = 0$: $\eta_{\max} = 1$, $L_\mathrm{c} = \dfrac{\pi}{2|\kappa|}$.

(Here the *phase-shifted* propagation constants are relevant.)
(Small interaction (small maximum conversion) for out-of-phase modes, i.e. for $|\Delta\beta'|^2 \gg |\kappa|^2$).)

Longitudinally constant structures, one "coupled" mode

CMT with one basis mode: $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) = c_1(z) \begin{pmatrix} \bar{\boldsymbol{E}}_1 \\ \bar{\boldsymbol{H}}_1 \end{pmatrix}(x, y)$

$$\partial_z c_1 = -\mathrm{i}\,\frac{b_{11} + \kappa_{11}}{\sigma_{11}} c_1,$$

$$\frac{b_{11}}{\sigma_{11}} = \beta_1, \qquad \frac{\kappa_{11}}{\sigma_{11}} = \frac{\omega\epsilon_0 \displaystyle\iint \bar{\boldsymbol{E}}_1^* \cdot (\hat{\epsilon} - \hat{\epsilon}_1)\bar{\boldsymbol{E}}_1 \,\mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re} \displaystyle\iint \left(\bar{E}_{1x}^* \bar{H}_{1y} - \bar{E}_{1y}^* \bar{H}_{1x}\right)\,\mathrm{d}x\,\mathrm{d}y} =: \delta\beta_1,$$

$$\partial_z c_1 = -\mathrm{i}(\beta_1 + \delta\beta_1) c_1,$$

$$c_1(z) = c_1(0)\,\mathrm{e}^{-\mathrm{i}(\beta_1 + \delta\beta_1) z}.$$

$\leftrightsquigarrow$ Theory of single mode perturbations.

Upcoming

Next lectures:

- Hybrid analytical / numerical coupled mode theory.
- A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.