

# *Optical Waveguide Theory (I)*



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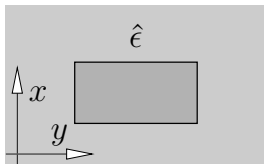
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### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I **Coupled mode theory, perturbation theory.**
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

## Perturbations of single modes

$\sim \exp(i\omega t)$  (FD)



$\lambda, \hat{\epsilon}(x, y)$

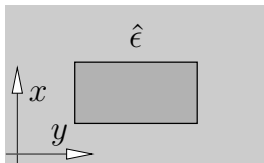


$\beta,$

$\bar{\mathbf{E}}, \bar{\mathbf{H}}$

## Perturbations of single modes

$\sim \exp(i\omega t)$  (FD)

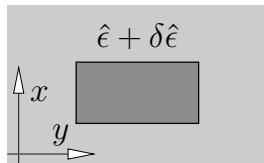


$\lambda, \hat{\epsilon}(x, y)$



$\beta,$

$\bar{\mathbf{E}}, \bar{\mathbf{H}}$



$\lambda, \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$

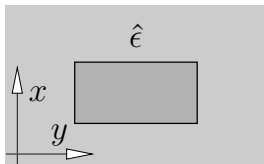


$\beta + \delta\beta,$

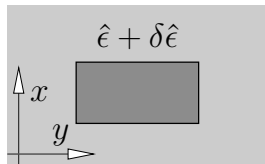
$\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}$

## Perturbations of single modes

$\sim \exp(i\omega t)$  (FD)



?



$\lambda, \hat{\epsilon}(x, y)$   
 $\beta,$   
 $\bar{\mathbf{E}}, \bar{\mathbf{H}}$



$\lambda, \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$   
 $\beta + \delta\beta,$   
 $\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}$

## A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

- $$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad \beta \in \mathbb{R},$$

$$\bar{\mathbf{E}}, \bar{\mathbf{H}} \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.$$

- $$(\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{E}} = -i\omega\mu_0\bar{\mathbf{H}}, \quad (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{H}} = i\omega\epsilon_0\hat{\epsilon}\bar{\mathbf{E}},$$

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

- $$\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) := \frac{\omega\epsilon_0\langle\bar{\mathbf{E}}, \hat{\epsilon}\bar{\mathbf{E}}\rangle + \omega\mu_0\langle\bar{\mathbf{H}}, \bar{\mathbf{H}}\rangle + i\langle\bar{\mathbf{E}}, \mathbf{C}\bar{\mathbf{H}}\rangle - i\langle\bar{\mathbf{H}}, \mathbf{C}\bar{\mathbf{E}}\rangle}{\langle\bar{\mathbf{E}}, \mathbf{R}\bar{\mathbf{H}}\rangle - \langle\bar{\mathbf{H}}, \mathbf{R}\bar{\mathbf{E}}\rangle},$$

$$\langle\bar{\mathbf{F}}, \bar{\mathbf{G}}\rangle = \iint \bar{\mathbf{F}}^* \cdot \bar{\mathbf{G}} \, dx \, dy.$$

$$\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta \quad (*), \quad \left. \frac{d}{ds} \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}} + s\bar{\mathbf{F}}, \bar{\mathbf{H}} + s\bar{\mathbf{G}}) \right|_{s=0} = 0 \quad (**)$$

at valid mode fields  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ , for arbitrary  $\bar{\mathbf{F}}, \bar{\mathbf{G}}$ .

(\*) : “arbitrary”  $\hat{\epsilon}$ .

(\*\*) : Hermitian  $\hat{\epsilon}$ .

## Perturbations of single modes

- Available: Mode  $\beta, \bar{\mathbf{E}}, \bar{\mathbf{H}}$  for parameters  $\lambda, \hat{\epsilon}$ ; ( $\hat{\epsilon} = \hat{\epsilon}^\dagger$ )  
 $\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$ ,  $\mathcal{B}_{\hat{\epsilon}}$  stationary at  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ .

- Investigate parameters  $\lambda, \hat{\epsilon} + \delta\hat{\epsilon}$ , for a “small” change  $\delta\hat{\epsilon}$ :

$$\mathcal{B}_{\hat{\epsilon} + \delta\hat{\epsilon}}(\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}) = \beta + \delta\beta$$

↪ ...  $\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}) \approx \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$

↪ ...  $\delta(-)\delta(-)$

↪ 
$$\delta\beta = \frac{\omega\epsilon_0 \langle \bar{\mathbf{E}}, \delta\hat{\epsilon} \bar{\mathbf{E}} \rangle}{\langle \bar{\mathbf{E}}, \mathbf{R}\bar{\mathbf{H}} \rangle - \langle \bar{\mathbf{H}}, \mathbf{R}\bar{\mathbf{E}} \rangle}, \quad \text{or} \quad \delta\beta = \frac{\omega\epsilon_0 \iint \bar{\mathbf{E}}^* \cdot \delta\hat{\epsilon} \bar{\mathbf{E}} \, dx \, dy}{2 \operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) \, dx \, dy}.$$

(Valid for *small* perturbations: The original mode profiles are good approximations of the true fields in the modified structure.)

## Small uniform change in refractive index



- $n \rightarrow n + \delta n$  on  $\square$ ,  $n, \delta n$  constant on  $\square$

$$\hookrightarrow \beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{\omega\epsilon_0 n \iint_{\square} |\bar{\mathbf{E}}|^2 dx dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta n.$$

( $\delta\epsilon = 2n\delta n$ .)  
 (Plausible:  $\delta\beta \sim \delta n$ ,  $\delta\beta \sim |\bar{\mathbf{E}}|^2|_{\square}$ .)



## Small attenuation



- $n \rightarrow n - in''$  on  $\square$ ,  $n, n''$  constant on  $\square$ ,  $n, n'' \in \mathbb{R}$

$$\beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{-i\omega\epsilon_0 n \iint_{\square} |\bar{\mathbf{E}}|^2 dx dy}{\operatorname{Re} \iint (\bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x) dx dy} n''.$$

( $\delta\epsilon = -i2nn''$ .)

(Different attenuation for each mode.)

(Damping, power, plane wave:  $\sim \exp(-2kn''z)$ , mode:  $\not\sim \exp(-2kn''z)$ .)

## Small anisotropy



- $\hat{\epsilon} \mathbf{1} \rightarrow \hat{\epsilon} \mathbf{1} + \delta\hat{\epsilon}$  on  $\square$ ,  $\epsilon, \delta\hat{\epsilon}$  constant on  $\square$

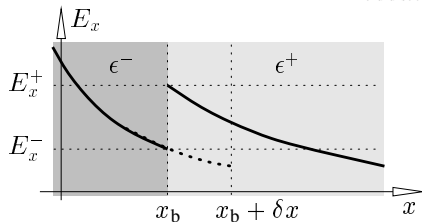
$$\beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{\omega\epsilon_0 \iint_{\square} \bar{\mathbf{E}}^* \cdot \delta\hat{\epsilon} \bar{\mathbf{E}} \, dx \, dy}{2 \operatorname{Re} \iint_{\square} (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) \, dx \, dy}.$$

(Phase shifts due to anisotropic contributions to the permittivity.)

(Polarization coupling might occur for modes with “close” propagation constants  $\leftrightarrow$  CMT.)

## Small displacements of dielectric interfaces

Interface displacement  $\leftrightarrow$  Locally *strong* thin layer perturbation.  
 Field discontinuity  $\rightsquigarrow$  Previous expressions are not directly applicable.



- $\epsilon^- \neq \epsilon^+$ ,  
 shift of interface  
 $x_b \rightarrow x_b + \delta x$ .

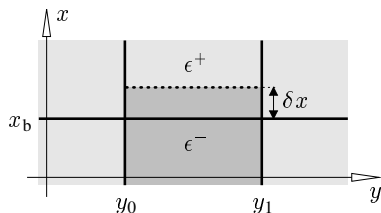
- Reposition discontinuity in field:  $E_x \rightarrow E_x + \delta E_x$ ,

$$\delta E_x(x, y) = \begin{cases} \frac{\epsilon^+ - \epsilon^-}{\epsilon^-} E_x(x, y), & \text{for } x_b < x < x_b + \delta x, \\ 0, & \text{otherwise.} \end{cases}$$

- Use functional with locally modified field

$\curvearrowright$  ... (omitted) ...  $\rightsquigarrow$

## Small displacements of dielectric interfaces

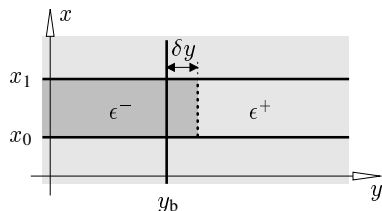


- Displacement of the interface at  $x_b$  between  $y_0$  and  $y_1$  by  $\delta x$ :

$$\hookrightarrow \beta \longrightarrow \beta + \delta\beta,$$

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{y_0}^{y_1} \left( \frac{1}{\epsilon^- \epsilon^+} |\epsilon \bar{E}_x|^2 + |\bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x_b, y) dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta x.$$

## Small displacements of dielectric interfaces

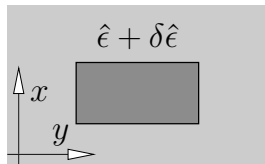
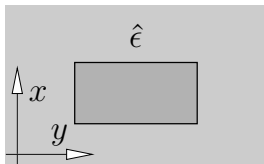


- Displacement of the interface at  $y_b$  between  $x_0$  and  $x_1$  by  $\delta y$ :

$$\curvearrowright \beta \longrightarrow \beta + \delta\beta,$$

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} \left( |\bar{E}_x|^2 + \frac{1}{\epsilon^- \epsilon^+} |\epsilon^- \bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x, y_b) dx}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta y.$$

## Perturbations of single modes



↪  $\lambda, \hat{\epsilon}(x, y)$   
 $\beta,$   
 $\bar{\mathbf{E}}, \bar{\mathbf{H}}$

↪  $\lambda, \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$   
 $\beta + \delta\beta,$   
 $\approx \bar{\mathbf{E}}, \approx \bar{\mathbf{H}}$

- View  $\frac{\delta\beta}{\delta p}$  as  $\frac{\partial\beta}{\partial p}$ : slope of the dispersion curves  $\beta$  vs.  $p$ .
- Depending on the parametrization, change of a parameter value might require several perturbations.
- First order theory: In case of multiple perturbations, add the effects of the individual expressions.
- Estimation of fabrication tolerances: The phase shifts  $\delta\beta$  enter into respective scattering matrix models.
- Wavelength shifts . . . ?

## Small shift of frequency or vacuum wavelength

(\*): Explicit frequency dependence of  $\mathcal{B}$  & dependence through  $\hat{\epsilon}$ .

(\*\*): Frequency dependence of  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ .

$$\beta(\omega) = \mathcal{B}_{\hat{\epsilon}}(\omega; \bar{\mathbf{E}}(\omega), \bar{\mathbf{H}}(\omega))$$

$$\begin{aligned} \hookrightarrow \frac{\partial \beta}{\partial \omega} &= \frac{\partial \mathcal{B}_{\hat{\epsilon}}}{\partial \omega} \Big|_{s=0} (*) + \frac{\partial}{\partial s} \mathcal{B}_{\hat{\epsilon}} \left( \omega; \bar{\mathbf{E}} + s \frac{\partial \bar{\mathbf{E}}}{\partial \omega}, \bar{\mathbf{H}} \right) \Big|_{s=0} (**) \\ &\quad + \frac{\partial}{\partial s} \mathcal{B}_{\hat{\epsilon}} \left( \omega; \bar{\mathbf{E}}, \bar{\mathbf{H}} + s \frac{\partial \bar{\mathbf{H}}}{\partial \omega} \right) \Big|_{s=0} (**) \\ &= \frac{\partial \mathcal{B}_{\hat{\epsilon}}}{\partial \omega}, \end{aligned} \quad \text{(Stationarity of } \mathcal{B} \text{ at } \bar{\mathbf{E}}, \bar{\mathbf{H}}.)$$

$$\hookrightarrow \frac{\partial \beta}{\partial \omega} = \frac{\iint \left( \epsilon_0 \bar{\mathbf{E}}^* \cdot \frac{\partial(\omega \hat{\epsilon})}{\partial \omega} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2 \right) dx dy}{2 \operatorname{Re} \iint \left( \bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x \right) dx dy}.$$

## Small shift of frequency or vacuum wavelength

If dispersion can be neglected,  $\partial_\omega \hat{\epsilon} = 0$ :

$$\hookrightarrow \frac{\partial \beta}{\partial \omega} = \frac{\iint (\epsilon_0 \bar{\mathbf{E}}^* \cdot \hat{\epsilon} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2) dx dy}{2 \operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy},$$

$$\hookrightarrow \frac{\partial \beta}{\partial \lambda} = -\frac{\pi c}{\lambda^2} \frac{\iint (\epsilon_0 \bar{\mathbf{E}}^* \cdot \hat{\epsilon} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2) dx dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy}.$$

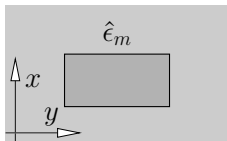
$(\omega = 2\pi c / \lambda \leftrightarrow \partial_\lambda \omega = -2\pi c / \lambda^2)$   
(Compare with expression based on homogeneity, H, 12.)



## Coupled mode theory (CMT)

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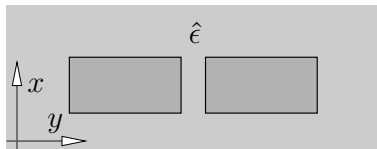
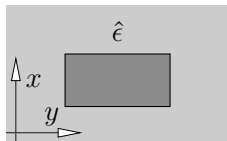
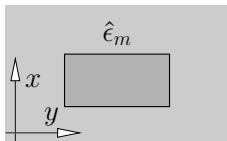
$\sim \exp(i\omega t)$  (FD)



$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$

## Coupled mode theory (CMT)

$\sim \exp(i\omega t)$  (FD)

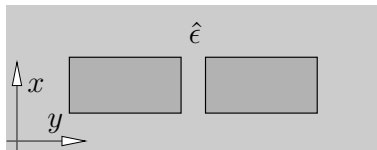
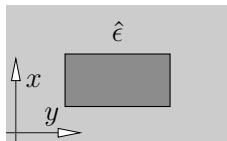
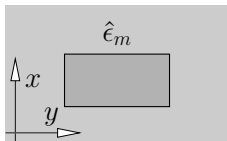


$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z)$$

# Coupled mode theory (CMT)

$\sim \exp(i\omega t)$  (FD)



?

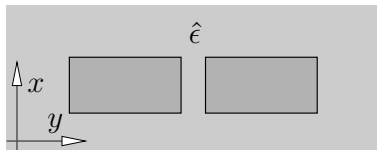
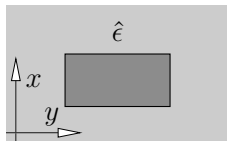
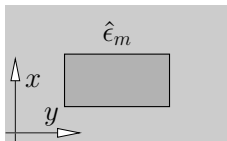
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$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z)$$

# Coupled mode theory (CMT)

$\sim \exp(i\omega t)$  (FD)



?

$$\left\{ \hat{\epsilon}_m; \beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) \right\}$$



$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z)$$

(Next: One of many variants of approaches to CMT.)

(Propagation & interaction of basis fields along a common propagation coordinate.)

[D.G. Hall, B.J. Thompson, *Selected papers on Coupled-Mode Theory in Guided-Wave Optics*, SPIE Milestone series MS 84 (1993)]

(Codirectional coupling (here), versus contradirectional coupling, coupling to radiation modes, nonlinear coupling.)

(Hybrid variant (HCMT): separate lecture.)

## Coupled mode theory (CMT)

- Investigate a permittivity  $\hat{\epsilon}$ , look for fields  $\mathbf{E}, \mathbf{H}$  with

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}.$$

( $\hat{\epsilon}(x, y, z)$ , in general.)

- Available: A set of fields  $\{\mathbf{E}_m, \mathbf{H}_m\}$  for permittivities  $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$ ;

$$\nabla \times \mathbf{E}_m = -i\omega\mu_0\mathbf{H}_m, \quad \nabla \times \mathbf{H}_m = i\omega\epsilon_0\hat{\epsilon}_m\mathbf{E}_m.$$

(Not necessarily “modes”.)

## Coupled mode theory (CMT)

- Investigate a permittivity  $\hat{\epsilon}$ , look for fields  $\mathbf{E}, \mathbf{H}$  with

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$$\nabla \times \mathbf{E}_m = -i\omega\mu_0\mathbf{H}_m, \quad \nabla \times \mathbf{H}_m = i\omega\epsilon_0\hat{\epsilon}_m\mathbf{E}_m.$$

(Not necessarily “modes”.)

- Assume that  $(\mathbf{E}, \mathbf{H})$  can be well approximated by

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) \approx \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z),$$

$C_m$ : unknown amplitudes, common propagation coordinate  $z$ .

(Choose  $\hat{\epsilon}_m$  as close as possible to  $\hat{\epsilon}$ .)

## Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”.)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for  $\mathbf{E}, \mathbf{H}$ .)

 ...

( $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)

 ...

(Apply identity  $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}$ .)

 ...

( $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)

 ...

(Manipulate, arrange terms, tidy up.)

## Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”.)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

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↪ ...

( $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)

↪ ...

(Manipulate, arrange terms, tidy up.)

$$\sum_m o_{lm} \partial_z C_m = -i \sum_m k_{lm} C_m \quad \forall l, \quad \text{coupled mode equations.}$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$



## Coupled mode theory (CMT)

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(Starting point: a “reciprocity identity”.)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for  $\mathbf{E}, \mathbf{H}$ .)

↪ ...

( $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)

↪ ...

(Apply identity  $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}_m$ .)

↪ ...

( $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)

↪ ...

(Manipulate, arrange terms, tidy up.)

---

$$\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K}\mathbf{C},$$

coupled mode equations.

$$\mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

## Coupled mode theory (CMT)

(Variational derivation of CMT equations.)

$$\mathcal{F}(\mathbf{E}, \mathbf{H}) = \iiint \left\{ \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E}^* \cdot (\nabla \times \mathbf{H}) + i\omega\mu_0 \mathbf{H}^* \cdot \mathbf{H} + i\omega\epsilon_0 \mathbf{E}^* \cdot \hat{\epsilon} \mathbf{E} \right\} dx dy dz,$$

$$\delta \mathcal{F} = 0 \quad \forall \delta \mathbf{E}, \delta \mathbf{H} \quad \iff \quad \nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0 \hat{\epsilon} \mathbf{E}.$$

(Restrict  $\mathcal{F}$  to the CMT ansatz for  $\mathbf{E}, \mathbf{H} \rightsquigarrow \mathcal{F}_c(\mathbf{C})$ , require  $\delta \mathcal{F}_c = 0 \quad \forall \delta \mathbf{C}$ )



$$\dots \quad (\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}_m, \iint dx dy, \mathbf{E}_m, \mathbf{H}_m \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.)$$

...

(Manipulate, arrange terms, tidy up.)

$$\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K} \mathbf{C},$$

coupled mode equations.

$$\mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

## Coupled mode equations

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...

$$\curvearrowright \quad \mathbf{O} \partial_z \mathbf{C} = -i \mathbf{K} \mathbf{C}, \quad \mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z \, dx \, dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega \epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m \, dx \, dy.$$

- A set of coupled *ordinary* linear differential equations, of first order. (Here.)
  - $o_{lm}$ : **power coupling coefficients** (field overlaps). (No reason to assume  $o_{lm} = \delta_{lm}$ , in general.)
  - $k_{lm}$ : **coupling coefficients**.
  - $z$ -dependence of  $\hat{\epsilon}$ ,  $\hat{\epsilon}_m$ ,  $\mathbf{E}_m$ ,  $\mathbf{H}_m$   $\rightsquigarrow$   $o_{lm}(z)$ ,  $k_{lm}(z)$ ,  $\mathbf{O}(z)$ ,  $\mathbf{K}(z)$ . (Compare the bend-straight couplers, Lecture H.)
- 

... to be solved by numerical procedures.

(In general.)

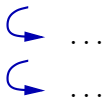
## CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

basis: guided modes  $\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z},$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \sum_m C_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y).$$

$$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$$



$$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$$

(Symmetrize coefficients.)

## CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

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$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y).$$

$$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$$



...

$$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$$



...

(Symmetrize coefficients.)

$$\sum_m \sigma_{lm} \partial_z c_m = -i \sum_m (b_{lm} + \kappa_{lm}) c_m \quad \forall l,$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z dx dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m dx dy,$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

$$b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

## CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

basis: guided modes  $\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) e^{-i\beta_m z},$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix}(x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y).$$

$$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$$



$$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$$



(Symmetrize coefficients.)

$$\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q}) \mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}),$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z dx dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m dx dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2},$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

## Longitudinally constant structures, coupled mode equations

$$\dots \quad (\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$$

$$\curvearrowright \mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z \, dx \, dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega \epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \bar{\mathbf{E}}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$
$$\delta \hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- A set of coupled *ordinary* linear differential equations, of first order (Here.)
- $\sigma_{lm}$ : **power coupling coefficients** (field overlaps). (No reason to assume  $\sigma_{lm} = \delta_{lm}$ , in general.)
- $\kappa_{lm}$ : **coupling coefficients.**
- $\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0 \rightsquigarrow \partial_z \sigma_{lm} = \partial_z b_{lm} = \partial_z \kappa_{lm} = 0.$

(ODEs with **constant coefficients**.)

... quasi-analytical solutions.

## Longitudinally constant structures, coupled mode equations

...

$$(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$$

$$\hookrightarrow \mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z \, dx \, dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega \epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \bar{\mathbf{E}}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta \hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- $\sigma_{ml}^* = \sigma_{lm}, \quad b_{ml}^* = b_{lm}; \quad \kappa_{ml}^* = \kappa_{lm},$  if  $\hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m,$   
 $\mathbf{S}^\dagger = \mathbf{S}, \quad \mathbf{B}^\dagger = \mathbf{B}; \quad \mathbf{Q}^\dagger = \mathbf{Q},$  if  $\hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m.$

- Power:  $P = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_{l,m} c_l^* (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) c_m = \mathbf{c}^* \cdot \mathbf{S} \mathbf{c}$

$$\hookrightarrow \partial_z P = i \mathbf{c}^* \cdot ((\mathbf{B} + \mathbf{Q})^\dagger - (\mathbf{B} + \mathbf{Q})) \mathbf{c}, \quad \partial_z P = 0 \text{ for } \mathbf{B}^\dagger = \mathbf{B}, \quad \mathbf{Q}^\dagger = \mathbf{Q}.$$

(For lossless waveguides the scheme is power conservative.)




## Longitudinally constant structures, formal solution

$$\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c},$$

$$\partial_z \mathbf{S} = \partial_z \mathbf{B} = \partial_z \mathbf{Q} = 0.$$

Ansatz:  $\mathbf{c}(z) = \mathbf{a} e^{-ibz}$ ,

$\mathbf{a}, b$  constants.


  $(\mathbf{B} + \mathbf{Q})\mathbf{a} = b \mathbf{S}\mathbf{a},$

a generalized eigenvalue problem.

(Dimension: number of basis modes included.)

---

Solutions:  $\{\mathbf{a}, b\}$ ,

 “supermodes”  $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \left( \sum_m a_m \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix}(x, y) \right) e^{-ibz}.$

(Superpositions of the original mode profiles with constant coefficients.)

(As many supermodes as there are basis modes.)

(Formalism can be continued: power/orthogonality of supermodes . . .)

## Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes  $(\mathbf{E}_1, \mathbf{H}_1)$ ,  $(\mathbf{E}_2, \mathbf{H}_2)$ :

(Example: two modes supported by the same isotropic waveguide ( $\hat{\epsilon}_1 = \hat{\epsilon}_2$ ); interaction due to small anisotropy ( $\hat{\epsilon}$ .)

(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0. \quad (\text{Orthogonal modes, uniform normalization } P_m = P_0.)$$

(Or: apply inverse of  $\mathbf{S}$  to CM equations, continue with redefined expressions for  $\beta_m, \kappa_{lm}$ .)

$$\begin{aligned} \curvearrowright \begin{pmatrix} \partial_z c_1 \\ \partial_z c_2 \end{pmatrix} &= -i \begin{pmatrix} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, & \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ & & \kappa &= \kappa_{12}/P_0. \end{aligned}$$

## Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes  $(\mathbf{E}_1, \mathbf{H}_1)$ ,  $(\mathbf{E}_2, \mathbf{H}_2)$ :

(Example: two modes supported by the same isotropic waveguide ( $\hat{\epsilon}_1 = \hat{\epsilon}_2$ ); interaction due to small anisotropy ( $\hat{\epsilon}$ .)

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$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0.$$

(Orthogonal modes, uniform normalization  $P_m = P_0$ .)

(Or: apply inverse of  $\mathbf{S}$  to CM equations, continue with redefined expressions for  $\beta_m, \kappa_{lm}$ .)

$$\begin{aligned} \left( \begin{array}{c} \partial_z c_1 \\ \partial_z c_2 \end{array} \right) &= -i \left( \begin{array}{cc} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right), & \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ & & \kappa &= \kappa_{12}/P_0. \end{aligned}$$

...

$$\left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) (z) = e^{-i \frac{(\beta'_1 + \beta'_2)}{2} z} \left( \begin{array}{cc} \cos \rho z - i \frac{\Delta \beta'}{2\rho} \sin \rho z & -i \frac{\kappa}{\rho} \sin \rho z \\ -i \frac{\kappa^*}{\rho} \sin \rho z & \cos \rho z + i \frac{\Delta \beta'}{2\rho} \sin \rho z \end{array} \right) \left( \begin{array}{c} c_{10} \\ c_{20} \end{array} \right),$$

$$\Delta \beta' = \beta'_1 - \beta'_2, \quad \rho = \sqrt{\left( \frac{\Delta \beta'}{2} \right)^2 + |\kappa|^2}.$$

## Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes  $(\mathbf{E}_1, \mathbf{H}_1)$ ,  $(\mathbf{E}_2, \mathbf{H}_2)$ :

(Example: two modes supported by the same isotropic waveguide ( $\hat{\epsilon}_1 = \hat{\epsilon}_2$ ); interaction due to small anisotropy ( $\hat{\epsilon}$ ).  
(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0. \quad (\text{Orthogonal modes, uniform normalization } P_m = P_0.)$$

(Or: apply inverse of  $\mathbf{S}$  to CM equations, continue with redefined expressions for  $\beta_m, \kappa_{lm}$ .)

$$\begin{aligned} \left( \begin{array}{c} \partial_z c_1 \\ \partial_z c_2 \end{array} \right) &= -i \left( \begin{array}{cc} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right), & \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ & & \kappa &= \kappa_{12}/P_0. \end{aligned}$$

- $c_{20} = 0 \rightsquigarrow \left| \frac{c_2(z)}{c_1(0)} \right|^2 = \eta_{\max} \sin^2(\rho z), \quad \eta_{\max} = \frac{|\kappa|^2}{|\kappa|^2 + (\Delta\beta'/2)^2}.$

- Maximum conversion**  $\eta_{\max}$  at  $z = L_c$  with  $\rho L_c = \pi/2$ ,

**coupling length**  $L_c = \frac{\pi}{\sqrt{(\Delta\beta')^2 + 4|\kappa|^2}},$  (Conversion length, half-beat length.)

- In case of **phase matching**  $\Delta\beta' = \beta'_1 - \beta'_2 = 0$ :  $\eta_{\max} = 1$ ,  $L_c = \frac{\pi}{2|\kappa|}.$

(Here the *phase-shifted* propagation constants are relevant.)  
(Small interaction (small maximum conversion) for out-of-phase modes, i.e. for  $|\Delta\beta'|^2 \gg |\kappa|^2$ .)

## Longitudinally constant structures, one “coupled” mode

CMT with one basis mode:  $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = c_1(z) \begin{pmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{H}}_1 \end{pmatrix}(x, y)$

↪  $\partial_z c_1 = -i \frac{b_{11} + \kappa_{11}}{\sigma_{11}} c_1,$

$$\frac{b_{11}}{\sigma_{11}} = \beta_1, \quad \frac{\kappa_{11}}{\sigma_{11}} = \frac{\omega \epsilon_0 \iint \bar{\mathbf{E}}_1^* \cdot (\hat{\epsilon} - \hat{\epsilon}_1) \bar{\mathbf{E}}_1 \, dx \, dy}{2 \operatorname{Re} \iint (\bar{E}_{1x}^* \bar{H}_{1y} - \bar{E}_{1y}^* \bar{H}_{1x}) \, dx \, dy} =: \delta\beta_1,$$

↪  $\partial_z c_1 = -i(\beta_1 + \delta\beta_1)c_1,$

↪  $c_1(z) = c_1(0) e^{-i(\beta_1 + \delta\beta_1)z}.$

↔ Theory of single mode perturbations.

## Upcoming

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Next lectures:

- Hybrid analytical / numerical coupled mode theory.
- A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.

