

# Optical Waveguide Theory (C)



Manfred Hammer\*

Theoretical Electrical Engineering  
Paderborn University, Germany

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\*Theoretical Electrical Engineering, Paderborn University Phone: +49(0)5251/60-3560  
Wärburger Straße 100, 33098 Paderborn, Germany E-mail: manfred.hammer@uni-paderborn.de

... ?

“This concerns time harmonic fields ... with angular frequency ... ,  
for vacuum wavenumber ... , speed of light ... , and wavelength ... .”

“The problem is governed by the Maxwell curl equations in the  
frequency domain for the electric field ... and magnetic field ... , for  
(lossless) uncharged dielectric, nonmagnetic linear (isotropic) media  
with (piecewise constant) relative permittivity ... :

... (.)”

[ M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016) ]

## Course overview

### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

### Maxwell equations, Fourier transform

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}$$

$$\& \quad \mathbf{F}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\mathbf{F}}(\mathbf{r}, \omega) e^{i\omega t} d\omega, \quad \tilde{\mathbf{F}}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int \mathbf{F}(\mathbf{r}, t) e^{-i\omega t} dt$$

$$\begin{aligned} & \left\{ \begin{array}{l} \mathbf{E}(\mathbf{r}, t), \mathbf{D}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t), \rho_f(\mathbf{r}, t), \mathbf{J}_f(\mathbf{r}, t) \\ \tilde{\mathbf{E}}(\mathbf{r}, \omega), \tilde{\mathbf{D}}(\mathbf{r}, \omega), \tilde{\mathbf{B}}(\mathbf{r}, \omega), \tilde{\mathbf{H}}(\mathbf{r}, \omega), \tilde{\rho}_f(\mathbf{r}, \omega), \tilde{\mathbf{J}}_f(\mathbf{r}, \omega) \end{array} \right. \end{aligned}$$

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_f, \quad \nabla \times \tilde{\mathbf{E}} = -i\omega \tilde{\mathbf{B}}, \quad \nabla \cdot \tilde{\mathbf{B}} = 0, \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}_f + i\omega \tilde{\mathbf{D}}$$

(Caution: arbitrary choice of  $\sim \exp(\pm i\omega t)$  !).

## Maxwell equations, frequency domain

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_f, \quad \nabla \times \tilde{\mathbf{E}} = -i\omega\tilde{\mathbf{B}}, \quad \nabla \cdot \tilde{\mathbf{B}} = 0, \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}_f + i\omega\tilde{\mathbf{D}}.$$

$$\mathbf{F}(\mathbf{r}, t) \in \mathbb{R} \rightsquigarrow \tilde{\mathbf{F}}(\mathbf{r}, -\omega) = (\tilde{\mathbf{F}}(\mathbf{r}, \omega))^*$$

“at frequency  $\omega_0$ ”:  $\tilde{\mathbf{F}}(\mathbf{r}, \omega) = \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}(\mathbf{r}) \delta(\omega - \omega_0) + \sqrt{\frac{\pi}{2}} \bar{\mathbf{F}}^*(\mathbf{r}) \delta(\omega + \omega_0)$

$$\curvearrowright \mathbf{F}(\mathbf{r}, t) = \frac{1}{2} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} + \bar{\mathbf{F}}^*(\mathbf{r}) e^{-i\omega_0 t} \right\},$$

$$\mathbf{F}(\mathbf{r}, t) = \text{Re} \left\{ \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} \right\},$$

$$\text{“}\mathbf{F}(\mathbf{r}, t) = \frac{1}{2} \bar{\mathbf{F}}(\mathbf{r}) e^{i\omega_0 t} + \text{c.c.}\text{”}.$$

$$\curvearrowright \bar{\mathbf{E}}(\mathbf{r}), \bar{\mathbf{D}}(\mathbf{r}), \bar{\mathbf{B}}(\mathbf{r}), \bar{\mathbf{H}}(\mathbf{r}), \tilde{\rho}_f(\mathbf{r}), \tilde{\mathbf{J}}_f(\mathbf{r}), \sim \exp(i\omega_0 t),$$

$$\nabla \cdot \bar{\mathbf{D}} = \bar{\rho}_f, \quad \nabla \times \bar{\mathbf{E}} = -i\omega_0 \bar{\mathbf{B}}, \quad \nabla \cdot \bar{\mathbf{B}} = 0, \quad \nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}}_f + i\omega_0 \bar{\mathbf{D}}.$$

Caution: Decorations  $\sim, \bar{\cdot}, \hat{\cdot}$  are usually omitted; context determines interpretation of symbols.

## Magnetization

$\tilde{\mathbf{M}}$ : density of magnetic dipole moments (bound currents).

$$\tilde{\mathbf{H}} = \frac{1}{\mu_0} \tilde{\mathbf{B}} - \tilde{\mathbf{M}}, \quad [\tilde{\mathbf{H}}] = [\mathbf{M}] = \frac{\text{A m}^2}{\text{m}^3}, \quad [\tilde{\mathbf{B}}] = \text{T} = \frac{\text{Vs}}{\text{m}^2},$$

$$\blacktriangleright \text{vacuum permeability } \mu_0 = 4\pi \cdot 10^{-7} \left[ \frac{\text{N}}{\text{A}^2} = \frac{\text{Vs}}{\text{Am}} \right].$$

• Local dipoles induced by  $\tilde{\mathbf{H}} \rightsquigarrow \tilde{\mathbf{M}}(\tilde{\mathbf{H}})$ .

• Linear magnetic media:

$$\tilde{\mathbf{M}} = \hat{\chi}_m \tilde{\mathbf{H}}, \quad \hat{\chi}_m: \text{magnetic susceptibility}, \quad [\hat{\chi}_m] = \hat{1}.$$

$$\curvearrowright \tilde{\mathbf{B}} = \mu_0 (\hat{1} + \hat{\chi}_m) \tilde{\mathbf{H}} = \mu_0 \hat{\mu} \tilde{\mathbf{H}}, \quad \hat{\mu}: \text{relative permeability}, \quad [\hat{\mu}] = \hat{1}.$$

•  $\hat{\chi}_m(\mathbf{r}, \omega), \hat{\mu}(\mathbf{r}, \omega)$  are determined in the frequency domain.

• Complications: manifold.

• Traditional integrated optics (frequencies, media):  $\hat{\mu}(\boldsymbol{\neq}) = \hat{1}$ .

## Polarization

$\tilde{\mathbf{P}}$ : density of electric dipole moment (bound charges).

$$\tilde{\mathbf{D}} = \epsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}, \quad [\tilde{\mathbf{D}}] = [\tilde{\mathbf{P}}] = \frac{\text{As m}}{\text{m}^3}, \quad [\tilde{\mathbf{E}}] = \frac{\text{V}}{\text{m}},$$

$$\blacktriangleright \text{vacuum permittivity } \epsilon_0 = 8.854187817 \dots \cdot 10^{-12} \left[ \frac{\text{F}}{\text{m}} = \frac{\text{As}}{\text{Vm}} \right].$$

• Local dipoles induced by  $\tilde{\mathbf{E}} \rightsquigarrow \tilde{\mathbf{P}}(\tilde{\mathbf{E}})$ .

• Linear dielectrics:

$$\tilde{\mathbf{P}} = \epsilon_0 \hat{\chi}_e \tilde{\mathbf{E}}, \quad \hat{\chi}_e: \text{dielectric susceptibility}, \quad [\hat{\chi}_e] = \hat{1}.$$

$$\curvearrowright \tilde{\mathbf{D}} = \epsilon_0 (\hat{1} + \hat{\chi}_e) \tilde{\mathbf{E}} = \epsilon_0 \hat{\epsilon} \tilde{\mathbf{E}}, \quad \hat{\epsilon}: \text{relative permittivity}, \quad [\hat{\epsilon}] = \hat{1}.$$

•  $\hat{\chi}_e(\mathbf{r}, \omega), \hat{\epsilon}(\mathbf{r}, \omega)$  are determined in the frequency domain.

• Complications:  $\text{Im } \epsilon, \hat{\epsilon}(T), \hat{\epsilon}(\mathbf{F}), \chi_{jkl}^{(2)} E_k E_l, \chi_{jklm}^{(3)} E_k E_l E_m, \dots$

• Simpler cases:  $\hat{\epsilon}(\boldsymbol{\neq}), \hat{\epsilon} = \epsilon \hat{1}$ .

## Maxwell equations, dispersion

(Material) dispersion:  $\hat{\epsilon}(\mathbf{r}, \omega), \hat{\mu}(\mathbf{r}, \omega)$  are frequency dependent.

$$\tilde{\mathbf{D}}(\mathbf{r}, \omega) = \epsilon_0 \hat{\epsilon}(\mathbf{r}, \omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega), \quad \tilde{\mathbf{B}}(\mathbf{r}, \omega) = \mu_0 \hat{\mu}(\mathbf{r}, \omega) \tilde{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\curvearrowright \mathbf{D}(\mathbf{r}, t) = \epsilon_0 \int \hat{\epsilon}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{E}(\mathbf{r}, t') dt',$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \int \hat{\mu}_{\text{TD}}(\mathbf{r}, t - t') \mathbf{H}(\mathbf{r}, t') dt'.$$

## Helmholtz equations

Linear dielectric media without free charges or currents,  
time dependence  $\sim \exp(i\omega t)$ , fields  $\mathbf{E}(\mathbf{r})$ ,  $\mathbf{D}(\mathbf{r})$ ,  $\mathbf{B}(\mathbf{r})$ ,  $\mathbf{H}(\mathbf{r})$ ,  
material properties  $\hat{\epsilon}(\mathbf{r})$ ,  $\hat{\mu}(\mathbf{r})$ :

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -i\omega \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = i\omega \mathbf{D},$$

$$\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}, \quad \mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}.$$

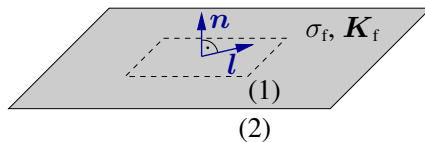
$$\nabla \times \mathbf{E} = -i\omega \mu_0 \hat{\mu} \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega \epsilon_0 \hat{\epsilon} \mathbf{E}, \quad \nabla \cdot \hat{\epsilon} \mathbf{E} = 0, \quad \nabla \cdot \hat{\mu} \mathbf{H} = 0.$$

$$\nabla \times (\hat{\mu}^{-1} \nabla \times \mathbf{E}) = \omega^2 \epsilon_0 \mu_0 \hat{\epsilon} \mathbf{E} \quad \text{or} \quad \nabla \times (\hat{\epsilon}^{-1} \nabla \times \mathbf{H}) = \omega^2 \epsilon_0 \mu_0 \hat{\mu} \mathbf{H}.$$

Where  $\hat{\epsilon} = \epsilon \hat{1}$ ,  $\nabla \epsilon = 0$ ,  $\hat{\mu} = \mu \hat{1}$ ,  $\nabla \mu = 0$ : (!)

$$\Delta \mathbf{E} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{E} = 0 \quad \text{or} \quad \Delta \mathbf{H} + \frac{\omega^2}{c^2} \epsilon \mu \mathbf{H} = 0, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

## Interface conditions



Surface between media (1) and (2), surface normal  $\mathbf{n}$ , tangents  $\mathbf{l}$ ,  
surface charge density  $\sigma_f$ , surface current density  $\mathbf{K}_f$ :

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{l} \cdot (\mathbf{K}_f \times \mathbf{n}).$$

## Plane harmonic waves

Where  $\hat{\epsilon} = \epsilon \hat{1}$ ,  $\nabla \epsilon = 0$ ,  $\hat{\mu} = \mu \hat{1}$ ,  $\nabla \mu = 0$ : (!)  
Components of  $\mathbf{E}$ ,  $\mathbf{H}$  satisfy

$$\psi(\mathbf{r}, t) = \psi_0 e^{-i(\mathbf{k}_m \cdot \mathbf{r} - \omega t)}, \quad \Delta \psi + \frac{\omega^2}{c^2} \epsilon \mu \psi = 0,$$

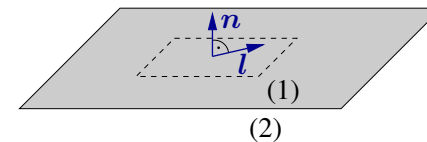
$$-\mathbf{k}_m^2 + \frac{\omega^2}{c^2} \epsilon \mu = 0.$$

(Mixture of TD and FD expressions:  $\nabla \cdot \nabla = \nabla^2$ , Re, 1/2, c.c. omitted; sloppy, but common.)

- Medium: refractive index:  $n = \sqrt{\epsilon \mu}$
  - Periodicity in time: angular frequency:  $\omega$ ,  
frequency:  $f = \omega / (2\pi)$ ,  
period:  $T = 1/f = 2\pi / \omega$ ,
  - Spatial periodicity: wave vector:  $\mathbf{k}_m$ ,  $k_m = |\mathbf{k}_m|$ ,  
wavenumber:  $k_m = \omega / c_m = (\omega / c) n = k n$ ,  
vacuum wavenumber:  $k = \omega / c$ ,  
vacuum wavelength:  $\lambda = 2\pi / k = 2\pi c / \omega$ ,  
wavelength in the medium:  $\lambda_m = 2\pi / k_m = 2\pi / (k n) = \lambda / n$ .
  - Phase velocity: speed of light in vacuum:  $c = 1 / \sqrt{\epsilon_0 \mu_0} = \lambda f$ ,  
in the medium:  $c_m = c / n = \lambda_m f$ .
- (Use of symbols depends highly on context.)

Electromagnetic spectrum ▶ ▶

## Interface conditions

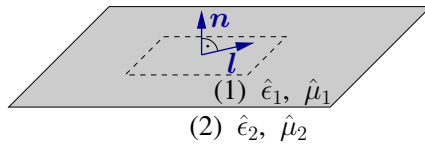


Surface between media (1) and (2), surface normal  $\mathbf{n}$ , tangents  $\mathbf{l}$ ,  
surface without free charges or currents:

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = 0,$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = 0.$$

## Interface conditions

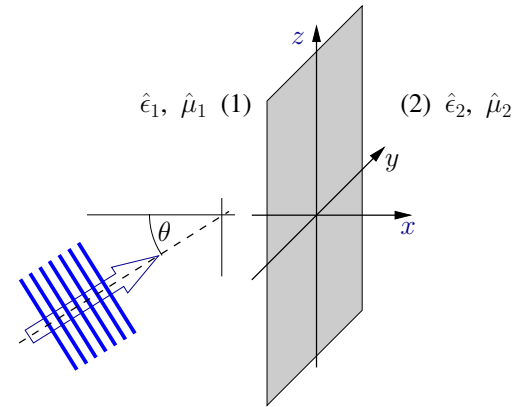


Surface between media (1) and (2), surface normal  $\mathbf{n}$ , tangents  $\mathbf{l}$ , linear media with permittivities  $\hat{\epsilon}_1, \hat{\epsilon}_2$ , and permeabilities  $\hat{\mu}_1, \hat{\mu}_2$ :

$$\begin{aligned} \mathbf{n} \cdot (\hat{\epsilon}_1 \mathbf{E}_1 - \hat{\epsilon}_2 \mathbf{E}_2) &= 0, & \mathbf{l} \cdot (\mathbf{E}_1 - \mathbf{E}_2) &= 0, \\ \mathbf{n} \cdot (\hat{\mu}_1 \mathbf{H}_1 - \hat{\mu}_2 \mathbf{H}_2) &= 0, & \mathbf{l} \cdot (\mathbf{H}_1 - \mathbf{H}_2) &= 0. \end{aligned}$$

## Reflection and transmission of plane waves at dielectric interfaces

(FD)



- $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$ ,  
 $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$
- $\hat{\epsilon}(\mathbf{r})$  and  $\hat{\mu}(\mathbf{r})$  are constant along  $y, z$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$$

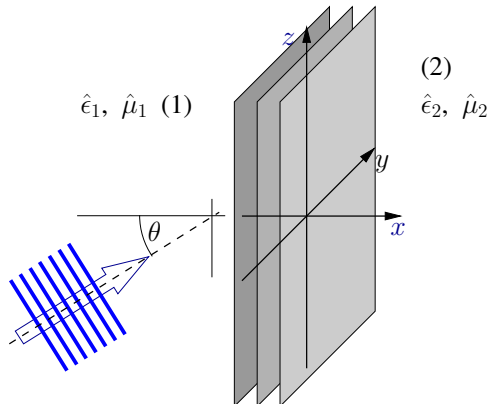
1-D problem for  $\mathbf{E}'$ ,  $\mathbf{H}'$ .

(incoming plane wave at angle  $\theta$ )  
(orient coordinates ( $k_y = 0$ ), plane of incidence, distinguish polarizations)  
(write ansatz functions for incoming, reflected, and transmitted waves)  
(interface conditions determine the amplitudes)

Fresnel equations.

## Dielectric multilayer structures

(FD)



- $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$ ,  
 $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$
- $\hat{\epsilon}(\mathbf{r})$  and  $\hat{\mu}(\mathbf{r})$  are constant along  $y, z$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(x) e^{-i(k_y y + k_z z)}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}'(x) e^{-i(k_y y + k_z z)}$$

1-D problem for  $\mathbf{E}'$ ,  $\mathbf{H}'$ .

Reflectance and transmittance properties.

## Energy of electromagnetic fields

(TD)

- Force on a particle with charge  $q$ , velocity  $\mathbf{v}$ , in a field  $\mathbf{E}, \mathbf{B}$ :  
 $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ ,
- work for shifting the particle by  $d\mathbf{r} = \mathbf{v} dt$ :  
 $dW = \mathbf{F} \cdot d\mathbf{r} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt = q\mathbf{E} \cdot \mathbf{v} dt$ ,
- respective power:  $\frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{v}$ .

For a charge density  $\rho_f(\mathbf{r}, t)$ :

- force density  $\mathbf{f} = \rho_f(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ ,
- power density  $\mathbf{f} \cdot \mathbf{v} = \rho_f \mathbf{E} \cdot \mathbf{v} = \mathbf{J}_f \cdot \mathbf{E}$ ,

total work per time unit done in  $\mathcal{V}$ :

$$\frac{dW_{\mathcal{V}}}{dt} = \int_{\mathcal{V}} \mathbf{J}_f \cdot \mathbf{E} dV.$$

## Power & energy density, Poynting theorem

(TD)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$

$$\hookrightarrow \frac{d}{dt} W_V^{\text{mech}} = \int_V \mathbf{J}_f \cdot \mathbf{E} dV = - \int_V (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}}) dV - \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV,$$

- Poynting vector:  $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ , (energy flux density, power density)
- energy density:  $w = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$ ,  $W_V^{\text{field}} = \int_V w dV$ ,
- $\hat{\epsilon}^\dagger = \hat{\epsilon}$ ,  $\hat{\epsilon}(\omega)$ ,  $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}$ ,  $\hat{\mu}^\dagger = \hat{\mu}$ ,  $\hat{\mu}(\omega)$ ,  $\mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$  (!)  
 $\rightsquigarrow \dot{w} = (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}})$

$$\hookrightarrow \overset{V \text{ arbitrary}}{\dot{w} + \nabla \cdot \mathbf{S} = -\mathbf{J}_f \cdot \mathbf{E}}, \quad \frac{d}{dt} (W_V^{\text{mech}} + W_V^{\text{field}}) = - \oint_{\partial V} \mathbf{S} \cdot d\mathbf{a}.$$

## Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:

Ohm's law  $\mathbf{J}_f = \sigma \mathbf{E}$ ,  $\sigma$ : conductivity of the material.

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \dot{\mathbf{D}}.$$

$$\hookrightarrow \dot{\rho}_f = -\frac{\sigma}{\epsilon_0 \epsilon} \rho_f, \quad \rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, t_0) \exp\left(-\frac{\sigma}{\epsilon_0 \epsilon}(t - t_0)\right),$$

assume  $\rho_f(\mathbf{r}, t_0) = 0 \rightsquigarrow \rho_f(\mathbf{r}, t) = 0 \quad \forall t.$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}.$$

## Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (. . .) linear media:

$$w = \frac{1}{2}(\epsilon_0 \mathbf{E} \cdot \hat{\epsilon} \mathbf{E} + \mu_0 \mathbf{H} \cdot \hat{\mu} \mathbf{H}), \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad \dot{w} + \nabla \cdot \mathbf{S} = 0,$$

$$\mathbf{E}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \text{Re } \tilde{\mathbf{H}}(\mathbf{r}) e^{i\omega t}$$

$\hookrightarrow \mathbf{S}, w$  oscillate in time.

Consider time-averaged quantities:  $\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t') dt'$  (FD)

$$\hookrightarrow \bar{w} = \frac{1}{4} \text{Re} \left( \epsilon_0 \tilde{\mathbf{E}}^* \cdot \hat{\epsilon} \tilde{\mathbf{E}} + \mu_0 \tilde{\mathbf{H}}^* \cdot \hat{\mu} \tilde{\mathbf{H}} \right), \quad \bar{\mathbf{S}} = \frac{1}{2} \text{Re} \left( \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}} \right).$$

$$\bar{\dot{w}} = \dot{\bar{w}} = 0, \quad \overline{\nabla \cdot \mathbf{S}} = \nabla \cdot \bar{\mathbf{S}} \rightsquigarrow \nabla \cdot \bar{\mathbf{S}} = 0, \quad \oint_V \bar{\mathbf{S}} \cdot d\mathbf{a} = 0;$$

“power balance”, conservation of energy.

## Telegrapher equation

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\mu_0 \mu \dot{\mathbf{H}}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_0 \epsilon \dot{\mathbf{E}}$$

$$\hookrightarrow \Delta \mathbf{E} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{E}} - \mu_0 \mu \sigma \dot{\mathbf{E}} = 0, \quad \Delta \mathbf{H} - \epsilon_0 \mu_0 \epsilon \mu \ddot{\mathbf{H}} - \mu_0 \mu \sigma \dot{\mathbf{H}} = 0,$$

Telegrapher equation.

Frequency domain:  $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}}(\mathbf{r}) e^{i\omega t}$ ,

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma \right) \tilde{\mathbf{E}} = 0.$$

Nonconducting media  $\sigma = 0$ ,  $\Delta \tilde{\mathbf{E}} + \left( \frac{\omega^2}{c^2} \epsilon \mu \right) \tilde{\mathbf{E}} = 0.$

Define  $\bar{\epsilon}$  such that  $\frac{\omega^2}{c^2} \bar{\epsilon} \mu = \frac{\omega^2}{c^2} \epsilon \mu - i\omega \mu_0 \mu \sigma$ , i.e.  $\bar{\epsilon} = \epsilon - i \frac{\sigma}{\epsilon_0 \omega}$

$$\hookrightarrow \Delta \tilde{\mathbf{E}} + k^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \text{Helmholtz equation, } \bar{\epsilon} \in \mathbb{C}, \quad k = \frac{\omega}{c}.$$

## Wave attenuation

$$\Delta \tilde{\mathbf{E}} + k^2 \bar{\epsilon} \mu \tilde{\mathbf{E}} = 0, \quad \bar{\epsilon} \in \mathbb{C}$$

↪ solutions  $\sim e^{-i(k\bar{n}z - \omega t)}$

with refractive index  $\bar{n} = n' - in'' = \sqrt{\bar{\epsilon}\mu} \in \mathbb{C}$ , (!)

$$e^{-i(k\bar{n}z - \omega t)} = e^{-i(kn'z - \omega t)} e^{-kn''z},$$

damped plane wave solutions

↔ sign of  $n''$ , choice of  $\exp(\pm i\omega)$ .

Issues:

- penetration depth,
- $S$  and  $w$  decay with  $z$ ,
- still transverse waves,
- $E, H$  no longer in phase,
- notions of wavenumber, wavelength, phase velocity  $\in \mathbb{C}$ .

## Simulations in integrated optics

A typical setting:

- “uncharged dielectric medium”:  $\rho_f, \mathbf{J}_f$ .
- “linear medium”:  $\mathbf{D} = \epsilon_0 \hat{\epsilon} \mathbf{E}$ ,  $\mathbf{B} = \mu_0 \hat{\mu} \mathbf{H}$ .
- “isotropic medium”:  $\hat{\epsilon} = \epsilon \hat{1}$ ,  $\hat{\mu} = \mu \hat{1}$ .
- “nonmagnetic medium”:  $\hat{\mu} = \hat{1}$ .
- “lossless medium”:  $\hat{\epsilon}^\dagger = \hat{\epsilon}$ ,  $\hat{\mu}^\dagger = \hat{\mu}$ , ( $\epsilon, \mu \in \mathbb{R}$ ).
- “piecewise constant” → “dependent on position”.
- “electric and magnetic field”: eliminate  $\mathbf{D}$  and  $\mathbf{B}$ , retain  $\mathbf{E}$  and  $\mathbf{H}$ .
- “governed by the curl equations”: divergence eqns. are satisfied.
- “frequency domain, time harmonic fields, frequency, wavelength”:  
... as discussed.

## Upcoming

Next lectures:

- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.
- Examples for dielectric optical waveguides.

