

Optical Waveguide Theory (E)



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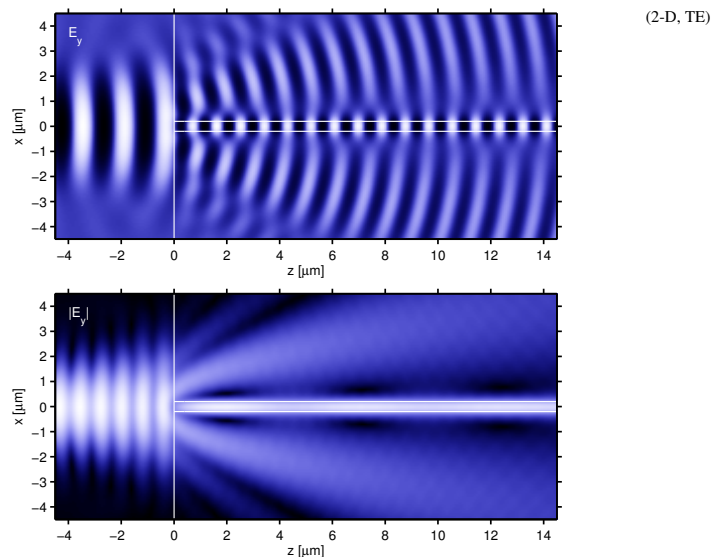
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Course overview

Optical waveguide theory

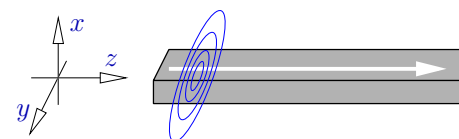
- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
- Hybrid analytical / numerical coupled mode theory.
- Oblique semi-guided waves: 2-D integrated optics.

Context: Relevance of guided modes



Butt-coupling to a waveguide facet.

Waveguides: Mode problems



$$\mu = 1, \epsilon = n^2, \sim \exp(i\omega t) \quad (\text{FD})$$

$$\begin{aligned} \nabla \times \mathbf{E} &= -i\omega\mu_0\mathbf{H}, \\ \nabla \times \mathbf{H} &= i\omega\epsilon_0\epsilon\mathbf{E}. \end{aligned}$$

- Waveguide: a system that is homogeneous along its axis z ,
 $\partial_z\epsilon = 0, \partial_z n = 0$.

- Look for solutions (modes) that vary harmonically with z :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad \begin{array}{l} \text{mode profile } \bar{\mathbf{E}}, \bar{\mathbf{H}}, \\ \text{propagation constant } \beta, \\ \text{effective index } n_{\text{eff}} = \beta/k. \end{array}$$

$$\partial_z \longrightarrow -i\beta, \quad (\& \text{ boundary conditions})$$

↔ Eigenvalue problem with eigenvalue β , eigenfunction $\bar{\mathbf{E}}, \bar{\mathbf{H}}$,
“M(β) (profile) = 0”.

Mode equations

(drop $\bar{}$)

$$\hookrightarrow \begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express E_x, E_y, E_z, H_z through principal components H_x, H_y :

$$\hookrightarrow \begin{aligned} \partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x &= 0, \\ \epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y &= 0, \end{aligned}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{1}{\omega\epsilon_0\epsilon} \begin{pmatrix} \beta H_y - \beta^{-1}(\partial_{yx} H_x + \partial_y^2 H_y) \\ -\beta H_x + \beta^{-1}(\partial_{xy} H_y + \partial_x^2 H_x) \\ -i(\partial_x H_y - \partial_y H_x) \end{pmatrix}, \quad \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} H_x \\ H_y \\ -i\beta^{-1}(\partial_x H_x + \partial_y H_y) \end{pmatrix}.$$

(H_x, H_y are continuous for all x, y .)

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Mode equations

(drop $\bar{}$)

$$\hookrightarrow \begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express E_x, E_y, H_x, H_y through principal components E_z, H_z :

$$\hookrightarrow (\dots).$$

(E_z, H_z are usually small components.)

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Mode equations

(drop $\bar{}$)

$$\hookrightarrow \begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express H_x, H_y, H_z, E_z through principal components E_x, E_y :

$$\hookrightarrow (\dots).$$

(E_x, E_y are discontinuous at specific interfaces.)

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Plane mode profiles

- Modes are eigenfunctions
 \longleftrightarrow profiles are determined up to a complex constant only.

- Propagating modes, $\beta \in \mathbb{R}$, lossless structures, $\epsilon \in \mathbb{R}$:

$$E_z := iE'_z, \quad H_z := iH'_z \quad \rightsquigarrow \text{real PDE for } E_x, E_y, E'_z, H_x, H_y, H'_z:$$

$$\begin{pmatrix} \partial_y E'_z + \beta E_y \\ -\beta E_x - \partial_x E'_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ -H'_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H'_z + \beta H_y \\ -\beta H_x - \partial_x H'_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ -E'_z \end{pmatrix};$$

it is possible to choose a phase such that

E_x, E_y, H_x, H_y are real,

E_z, H_z are imaginary

\longleftrightarrow plane mode profiles.

(It makes sense to prepare real plots of mode profile components.)

(That requires a suitable adjustment of the global phase.)

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6

Guided modes

- Guided modes: profiles located “around” the waveguide core

↔ discrete $\beta \in \mathbb{R}$, $\iint S_z dx dy < \infty$.

- In general: Hybrid modes, all six field components present. Planar-like waveguides → adapt 2-D naming scheme; “TE-like” / “TM-like” modes.

(↔ 5-component **semivectorial** approximations, plane \perp x -axis:
 quasi-TE: tiny E_x , dominant E_y , small E_z ; major H_x , small H_y , minor H_z ,
 quasi-TM: tiny H_x , dominant H_y , small H_z ; major E_x , small E_y , minor E_z .)

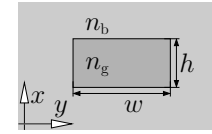
- Mode indices mostly relate to numbers of nodal lines in the dominant electric or magnetic field component.

(Naming schemes are highly context dependent.)



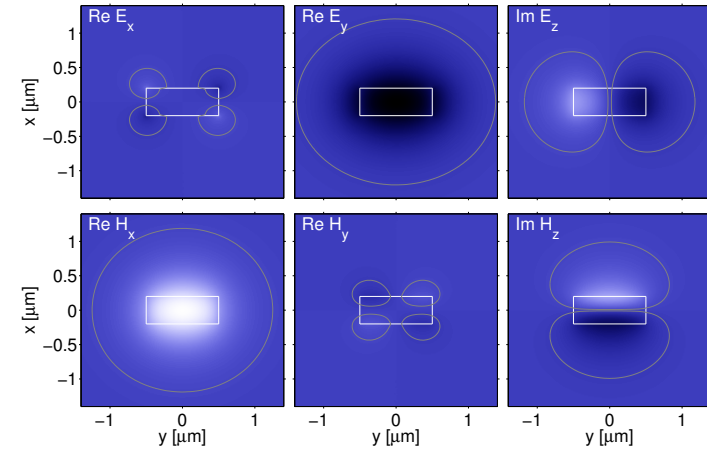
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A rectangular strip waveguide, fundamental mode profiles



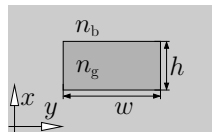
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $w = 1.0 \mu\text{m}$,
 $h = 0.4 \mu\text{m}$;

$x \in [-2, 2] \mu\text{m}$,
 $y \in [-2, 2] \mu\text{m}$;
 $n_{\text{eff}} = 1.63554$
 (q-)TE₀₀ [JCMwave].



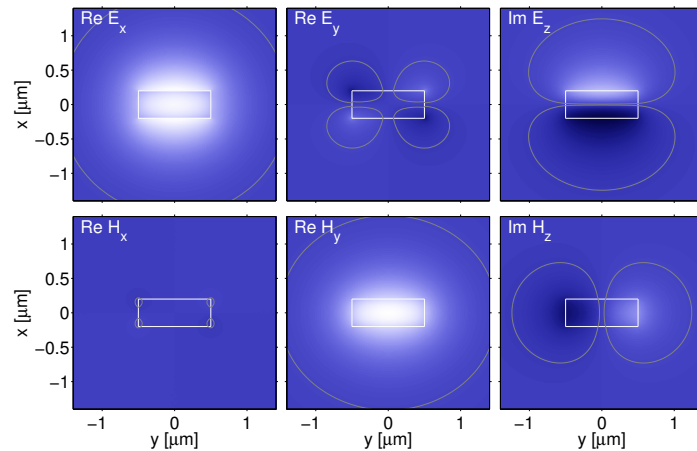
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A rectangular strip waveguide, fundamental mode profiles



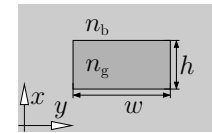
$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $w = 1.0 \mu\text{m}$,
 $h = 0.4 \mu\text{m}$;

$x \in [-2, 2] \mu\text{m}$,
 $y \in [-2, 2] \mu\text{m}$;
 $n_{\text{eff}} = 1.56809$
 (q-)TM₀₀ [JCMwave].



8

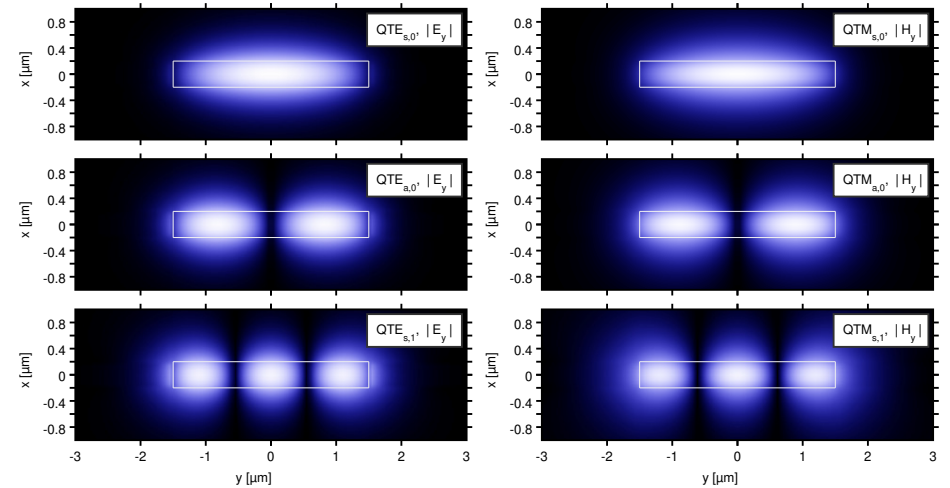
A rectangular strip waveguide, guided modes of higher order



$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $w = 3.0 \mu\text{m}$,
 $h = 0.4 \mu\text{m}$;

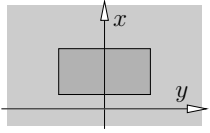
n_{eff}	QTE	QTM
(0, 0)	1.724	1.628
(0, 1)	1.676	1.585
(0, 2)	1.595	1.516

[WMMS] ▶



9

Symmetric waveguides



Waveguide with mirror symmetry $y \rightarrow -y$:
 modes have a definite parity.

$$\begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Equal parity of H_x, E_y, H_z , reversed parity of E_x, H_y, E_z .

10

Modal power

E.m. power density: $\mathbf{S} = \frac{1}{2} \text{Re} (\mathbf{E}^* \times \mathbf{H})$. (FD) $\sim \exp(i\omega t)$

$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}$,
 $\bar{\mathbf{E}} = a(\bar{E}_x, \bar{E}_y, i\bar{E}'_z)$,
 $\bar{\mathbf{H}} = a(\bar{H}_x, \bar{H}_y, i\bar{H}'_z)$,
 $a \in \mathbb{C}, \bar{E}_x, \dots, \bar{H}'_z \in \mathbb{R}$,
 a guided mode, $\beta \in \mathbb{R}$.

$$\mathbf{S} = \frac{|a|^2}{2} \begin{pmatrix} 0 \\ 0 \\ \bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x \end{pmatrix}$$

or $S_x = 0, S_y = 0, S_z = \frac{1}{2} \text{Re} (E_x^* H_y - E_y^* H_x)$. ($S_z(x, y)$)

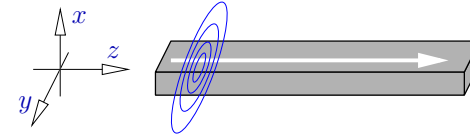
Power carried by the mode:

$$P = \iint S_z dx dy = \frac{1}{4} \iint (E_x^* H_y - E_y^* H_x + E_x H_y^* - E_y H_x^*) dx dy$$

(backward mode, $E_x \rightarrow E_x, E_y \rightarrow E_y, H_x \rightarrow -H_x, H_y \rightarrow -H_y$: $P \rightarrow -P$)

12

Directional modes



(FD) $\sim \exp(i\omega t)$

Longitudinally homogeneous waveguide: mirror symmetry $z \rightarrow -z$.

$$\begin{pmatrix} \partial_y E_z + i\beta E_y \\ -i\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + i\beta H_y \\ -i\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

forward: $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^f \\ \bar{\mathbf{H}}^f \end{pmatrix} (x, y) e^{-i\beta z}$, $\bar{\mathbf{E}}^f = (E_x, E_y, E_z)$,
 $\bar{\mathbf{H}}^f = (H_x, H_y, H_z)$,

backward: $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^b \\ \bar{\mathbf{H}}^b \end{pmatrix} (x, y) e^{+i\beta z}$, $\bar{\mathbf{E}}^b = (E_x, E_y, -E_z)$,
 $\bar{\mathbf{H}}^b = (-H_x, -H_y, H_z)$.

11

Mode orthogonality

A set of guided modes of the same waveguide (ϵ):

$\beta \in \mathbb{R}$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}, \quad \begin{aligned} \nabla \times \mathbf{E}_m &= -i\omega\mu_0 \mathbf{H}_m, \\ \nabla \times \mathbf{H}_m &= i\omega\epsilon_0 \mathbf{E}_m, \\ \beta_l \neq \beta_m, & \text{ if } l \neq m. \end{aligned}$$

$$P_m = \frac{1}{4} \iint (E_{mx}^* H_{my} - E_{my}^* H_{mx} + E_{mx} H_{my}^* - E_{my} H_{mx}^*) dx dy$$

$\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$ for $x, y \rightarrow \pm\infty$.

$\nabla \cdot (\mathbf{E}_l^* \times \mathbf{H}_m + \mathbf{E}_m \times \mathbf{H}_l^*) = 0$ for all l, m

$$0 = i(\beta_l - \beta_m) \left\{ \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m + \bar{\mathbf{E}}_m \times \bar{\mathbf{H}}_l^*)_z dx dy \right\} e^{i(\beta_l - \beta_m)z}$$

$$(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \iint (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx dy$$

$$(\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m) = \begin{cases} 0, & \text{if } l \neq m, \\ P_m, & \text{otherwise.} \end{cases}$$

(The modes are "power orthogonal".)
 (Statements hold for propagating guided modes.)
 ((, ., ., .) is frequently used for mode normalization.)

13

Power transport by a mode superposition

- A set of guided modes of the same waveguide (ϵ): $\beta \in \mathbb{R}$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}, \quad P_m = (\mathbf{E}_m, \mathbf{H}_m; \mathbf{E}_m, \mathbf{H}_m).$$
- Superposition with amplitudes $a_m \in \mathbb{C}$:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_m a_m \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \sum_m a_m \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}.$$

Power flow along the waveguide:

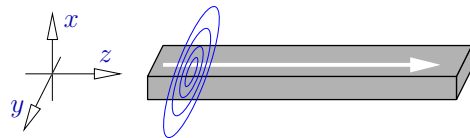
$$\begin{aligned} \iint S_z \, dx \, dy &= (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) \\ &= \sum_l \sum_m a_l^* a_m (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m) \\ &= \sum_m |a_m|^2 P_m. \end{aligned}$$

(Forward / backward modes: $P \geq 0$.)

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14

Polarization of a guided wave field



Unidirectional guided waves in a “long” dielectric channel that supports fundamental TE- and TM-like modes only:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = a_{\text{TE}} \begin{pmatrix} \bar{\mathbf{E}}_{\text{TE}} \\ \bar{\mathbf{H}}_{\text{TE}} \end{pmatrix} (x, y) e^{-i\beta_{\text{TE}} z} + a_{\text{TM}} \begin{pmatrix} \bar{\mathbf{E}}_{\text{TM}} \\ \bar{\mathbf{H}}_{\text{TM}} \end{pmatrix} (x, y) e^{-i\beta_{\text{TM}} z},$$

- $E_{\text{TE}z} \neq 0, E_{\text{TM}z} \neq 0$. amplitudes $a_{\text{TE}}, a_{\text{TM}} \in \mathbb{C}$.
- $\bar{\mathbf{E}}_{\text{TE}}(x, y) \neq \bar{\mathbf{E}}_{\text{TM}}(x, y)$.
- At (x, y) : adjust $\mathbf{E}/|\mathbf{E}|$ via $a_{\text{TE}}, a_{\text{TM}}$.
- $a_{\text{TE}}, a_{\text{TM}}$ fixed: $(\mathbf{E}/|\mathbf{E}|)(x, y)$ varies.

“Polarization” frequently indicates the presence of only one mode.

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16

Mode interference

- Two modes $m = 1, 2$: $\beta \in \mathbb{R}$

$$\begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z}.$$
- Superposition with amplitudes a_1, a_2 :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = a_1 \begin{pmatrix} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{H}}_1 \end{pmatrix} (x, y) e^{-i\beta_1 z} + a_2 \begin{pmatrix} \bar{\mathbf{E}}_2 \\ \bar{\mathbf{H}}_2 \end{pmatrix} (x, y) e^{-i\beta_2 z}.$$
- Fix a position x, y and component F : $\text{Omit } (x, y)$.

$$F(z) = a_1 \bar{F}_1 e^{-i\beta_1 z} + a_2 \bar{F}_2 e^{-i\beta_2 z}, \quad r e^{-i\phi} := a_1^* a_2 \bar{F}_1^* \bar{F}_2,$$

$$\curvearrowright |F|^2(z) = |a_1|^2 |\bar{F}_1|^2 + |a_2|^2 |\bar{F}_2|^2 + 2r \cos((\beta_1 - \beta_2)z + \phi).$$

Periodic beating pattern with half-beat-length $L_c = \frac{\pi}{|\beta_1 - \beta_2|}$.

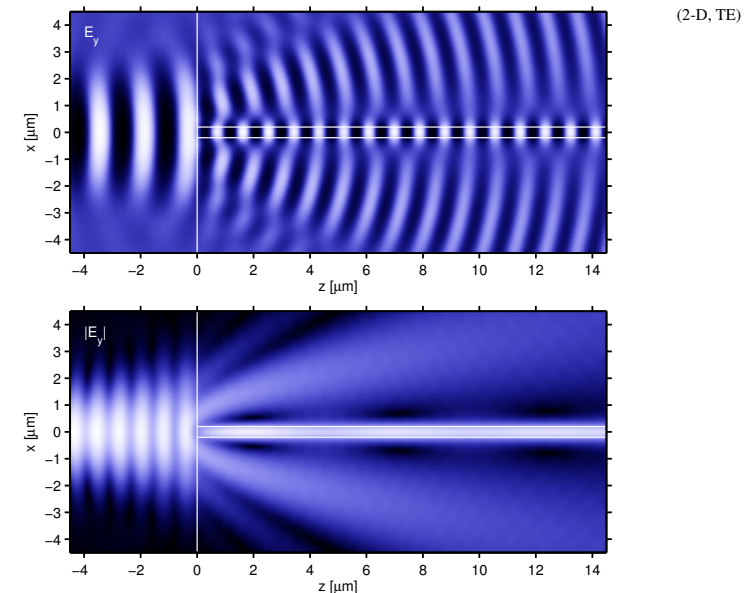
(Supermodes) (Evanescent coupling)

(“Coupling length” L_c .)

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15

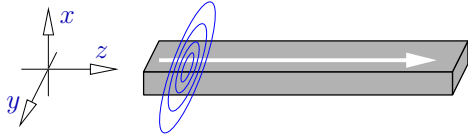
What about non-guided fields?



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17

Normal modes: real mode problems



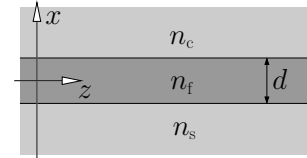
- lossless waveguide, $\epsilon \in \mathbb{R}$,
- “real” boundary conditions at x, y “far away” from the core,
- “real” vectorial mode equations:

$$\partial_x^2 H_x + \epsilon \partial_y \frac{1}{\epsilon} \partial_y H_x + \partial_{xy} H_y - \epsilon \partial_y \frac{1}{\epsilon} \partial_x H_y + (k^2 \epsilon - \beta^2) H_x = 0,$$

$$\epsilon \partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon \partial_x \frac{1}{\epsilon} \partial_y H_x + (k^2 \epsilon - \beta^2) H_y = 0,$$

→ real principal components $H_x(x, y), H_y(x, y)$, $\beta^2 \in \mathbb{R}$.

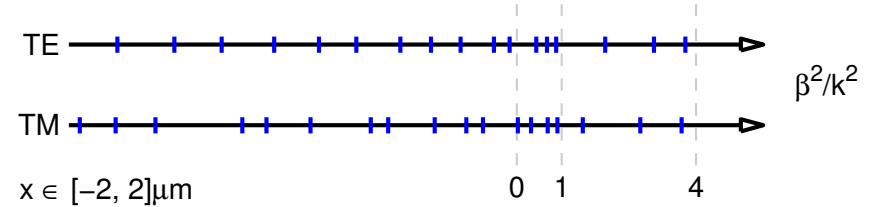
2-D slab waveguide, normal mode spectrum



$$n_s = n_c = 1.0, n_f = 2.0,$$

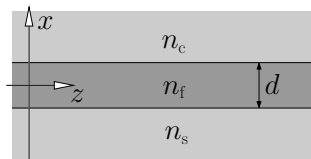
$$d = 1.3 \mu\text{m}, \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm 2 \mu\text{m}.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided modes**.
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation modes**.
- $\beta^2/k^2 < 0$: **evanescent radiation modes**.

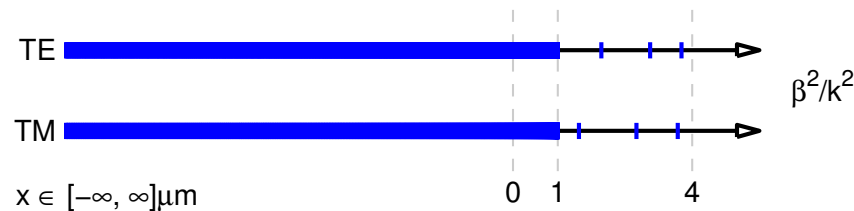
2-D slab waveguide, normal mode spectrum



$$n_s = n_c = 1.0, n_f = 2.0,$$

$$d = 1.3 \mu\text{m}, \lambda = 1.55 \mu\text{m},$$

$$E_y = 0, H_y = 0 \text{ at } x = \pm \infty.$$



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: **guided modes** (discrete spectrum).
- $0 < \beta^2/k^2 < n_s^2$: **propagating radiation modes** (continuous spec.).
- $\beta^2/k^2 < 0$: **evanescent radiation modes** (continuous spec.).

Propagating & evanescent modes

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}^{f,b} \\ \bar{\mathbf{H}}^{f,b} \end{pmatrix} (x, y) e^{\mp i \beta z}. \quad \sim \exp(i \omega t) \text{ (FD)}$$

- $\beta^2 > 0 \iff \beta = \sqrt{\beta^2}, \beta \in \mathbb{R}, \beta > 0,$
 $\sim e^{\mp i \beta z}$, a forward/backward **propagating mode**.
(Physical relevance of individual modes.)
- $\beta^2 < 0 \iff \beta = -i \sqrt{|\beta^2|} = -i \alpha, \alpha = \sqrt{|\beta^2|} \in \mathbb{R}, \alpha > 0,$
 $\sim e^{\mp \alpha z}$, a forward/backward traveling **evanescent mode**.
“forward”: $\sim e^{-\alpha z}$, field decays with z ,
“backward”: $\sim e^{+\alpha z}$, field grows with z .
(Relevant for purposes of field expansions.)
- {forward & backward, propagating & evanescent modes}
= the set of **normal modes**.

Evanescent modes

$$\beta = -i\alpha, \alpha \in \mathbb{R} \quad \epsilon \in \mathbb{R}$$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -i\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = i\omega\epsilon_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- **Plane mode profiles:** real PDE for $E_x, E_y, E_z, iH_x, iH_y, iH_z$; common phase with real E_x, E_y, E_z , imaginary H_x, H_y, H_z .

- **Directional evanescent modes:**

$$\{E_x, E_y, E_z, H_x, H_y, H_z; \alpha\}^f \rightsquigarrow \{E_x, E_y, -E_z, -H_x, -H_y, H_z; -\alpha\}^b.$$

- **Modal power:**

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-\alpha z}, \quad \begin{aligned} \bar{\mathbf{E}} &= a(E'_x, E'_y, E'_z), \\ \bar{\mathbf{H}} &= ia(H'_x, H'_y, H'_z), \\ E'_x, \dots, H'_z &\in \mathbb{R}, \quad a \in \mathbb{C} \end{aligned}$$

$$\hookrightarrow S_z = \frac{1}{2} \text{Re} (E_x^* H_y - E_y^* H_x) = 0, \quad \iint S_z dx dy = 0.$$

Completeness of normal modes

Stronger statement:

“any” transverse 2-component field on a cross sectional plane can be expanded into alternatively

- the transverse electric components of forward normal modes,
- the transverse magnetic components of forward normal modes,
- the transverse electric components of backward normal modes,
- the transverse magnetic components of backward normal modes.

Completeness of normal modes



$$\epsilon \in \mathbb{R}, \sim \exp(i\omega t) \text{ (FD)}$$

A lossless, z -homogeneous waveguide configuration; **general solution** of the Maxwell equations between cross sectional planes A and B:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_{m \in \mathcal{N}} F_m \begin{pmatrix} \bar{\mathbf{E}}_m^f \\ \bar{\mathbf{H}}_m^f \end{pmatrix} (x, y) e^{-i\beta_m z} + \sum_{m \in \mathcal{N}} B_m \begin{pmatrix} \bar{\mathbf{E}}_m^b \\ \bar{\mathbf{H}}_m^b \end{pmatrix} (x, y) e^{+i\beta_m z}, \quad \Sigma \rightarrow \not\neq$$

\mathcal{N} : the set of forward **normal modes** supported by the waveguide.

(“Solution”: obvious; “general”: without proof.)

Orthogonality of normal modes

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad \sim \exp(i\omega t) \text{ (FD)}$$

	$\bar{\mathbf{E}}$	$\bar{\mathbf{H}}$	β
[prop., f]	(E'_x, E'_y, iE'_z)	(H'_x, H'_y, iH'_z)	$\beta > 0$
[prop., b]	$(E'_x, E'_y, -iE'_z)$	$(-H'_x, -H'_y, iH'_z)$	$\beta < 0$
[evan., f]	(E'_x, E'_y, E'_z)	(iH'_x, iH'_y, iH'_z)	$\beta = -i\alpha, \alpha > 0$
[evan., b]	$(E'_x, E'_y, -E'_z)$	$(-iH'_x, -iH'_y, iH'_z)$	$\beta = i\alpha, \alpha > 0$

individual $E'_x, \dots, H'_z \in \mathbb{R}$.

$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) := \frac{1}{4} \iint (E_{ax}^* H_{by} - E_{ay}^* H_{bx} + H_{ay}^* E_{bx} - H_{ax}^* E_{by}) dx dy$$

$$\begin{pmatrix} \mathbf{E}_{1,2} \\ \mathbf{H}_{1,2} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_{1,2} \\ \bar{\mathbf{H}}_{1,2} \end{pmatrix} (x, y) e^{-i\beta_{1,2} z}, \quad \begin{aligned} \nabla \times \mathbf{E}_{1,2} &= -i\omega\mu_0 \mathbf{H}_{1,2}, \\ \nabla \times \mathbf{H}_{1,2} &= i\omega\epsilon_0 \mathbf{E}_{1,2}, \end{aligned}$$

$$\nabla \cdot (\mathbf{E}_1^* \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{H}_1^*) = 0 \rightsquigarrow 0 = (\beta_1^* - \beta_2) (\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2).$$

...

Orthogonality of normal modes

Nondegenerate directional normal modes of the same waveguide (ϵ):

$$\begin{pmatrix} \mathbf{E}_m^{\text{f,b}} \\ \mathbf{H}_m^{\text{f,b}} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m^{\text{f,b}} \\ \bar{\mathbf{H}}_m^{\text{f,b}} \end{pmatrix} (x, y) e^{-i\beta_m^{\text{f,b}} z}, \quad \begin{aligned} \nabla \times \mathbf{E}_m &= -i\omega\mu_0\mathbf{H}_m, \\ \nabla \times \mathbf{H}_m &= i\omega\epsilon_0\epsilon\mathbf{E}_m, \\ \beta_l &\neq \beta_m, \text{ if } l \neq m. \end{aligned}$$

- A propagating mode m :

$$\begin{aligned} (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) &=: P_m, & (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) &= -P_m, & P_m &\in \mathbb{R}, \\ (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) &= (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) &= 0, \\ (\bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}; \bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}) &= (\bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}; \bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}) &= 0 \text{ for all } l \neq m, \text{ d,r = f,b.} \end{aligned}$$

- An evanescent mode m :

$$\begin{aligned} (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) &= (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) &= 0, \\ (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}) &=: P_m, & (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}) &= -P_m, & P_m &\notin \mathbb{R}, \\ (\bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}; \bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}) &= (\bar{\mathbf{E}}_l^{\text{r}}, \bar{\mathbf{H}}_l^{\text{r}}; \bar{\mathbf{E}}_m^{\text{d}}, \bar{\mathbf{H}}_m^{\text{d}}) &= 0 \text{ for all } l \neq m, \text{ d,r = f,b.} \end{aligned}$$

(This implies orthogonality of propagating and evanescent modes.)
($1/\sqrt{|P_m|}$ is frequently used for mode normalization.)

Projection onto normal modes



\mathbf{E}, \mathbf{H} : a solution of the Maxwell equations for the z -homogeneous waveguide between two cross sectional planes A and B.

Extract local mode amplitudes by **projection onto normal modes**:

- A propagating mode m , $\beta_m > 0$:

$$\begin{aligned} (\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \mathbf{E}, \mathbf{H}) &= F_m P_m e^{-i\beta_m z}, \\ (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \mathbf{E}, \mathbf{H}) &= -B_m P_m e^{i\beta_m z}. \end{aligned} \quad F_m e^{-i\beta_m z} = \frac{(\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \mathbf{E}, \mathbf{H})}{(\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}})}$$

- An evanescent mode m , $\beta_m = -i\alpha_m$, $\alpha_m > 0$:

$$(\bar{\mathbf{E}}_m^{\text{f}}, \bar{\mathbf{H}}_m^{\text{f}}; \mathbf{E}, \mathbf{H}) = B_m P_m e^{\alpha_m z}, \quad (\bar{\mathbf{E}}_m^{\text{b}}, \bar{\mathbf{H}}_m^{\text{b}}; \mathbf{E}, \mathbf{H}) = -F_m P_m e^{-\alpha_m z}.$$

Ports of a photonic integrated circuit.

Power flow associated with a normal mode expansion

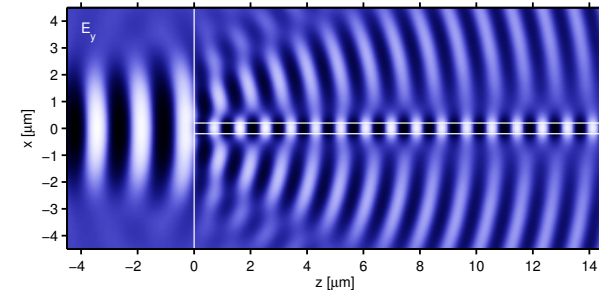
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_{m \in \mathcal{N}} \left\{ F_m \begin{pmatrix} \bar{\mathbf{E}}_m^{\text{f}} \\ \bar{\mathbf{H}}_m^{\text{f}} \end{pmatrix} (x, y) e^{-i\beta_m z} + B_m \begin{pmatrix} \bar{\mathbf{E}}_m^{\text{b}} \\ \bar{\mathbf{H}}_m^{\text{b}} \end{pmatrix} (x, y) e^{+i\beta_m z} \right\}$$

Power carried along z :

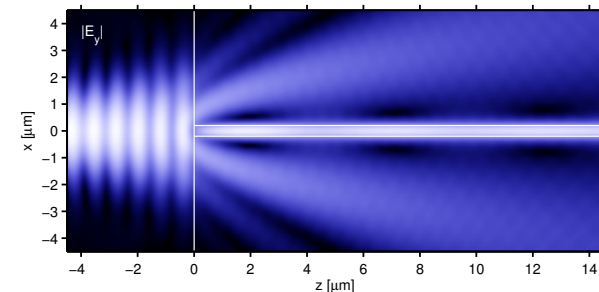
$$\begin{aligned} P &= \iint S_z dx dy = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) \\ &= \sum_{m \text{ propag.}} (|F_m|^2 - |B_m|^2) P_m + \sum_{m \text{ evanesc.}} (F_m^* B_m - B_m^* F_m) P_m. \end{aligned}$$

- P is independent of z .
- Individual contributions from forward and backward propagating modes.
- Contributions from evanescent modes require forward and backward fields to be present.
- Unidirectional field (forward: $B_m = 0$ for all m): Only propagating modes carry power.

Waveguide facet: Port definition



(2-D, TE)



Next lectures:

- Examples for dielectric optical waveguides.
- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- Bent optical waveguides; whispering gallery resonances; circular microresonators.

