

# Optical Waveguide Theory (F)



Manfred Hammer\*

Theoretical Electrical Engineering  
Paderborn University, Paderborn, Germany

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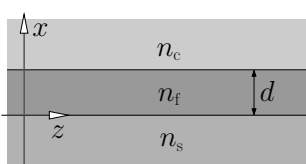
\*Theoretical Electrical Engineering, Paderborn University Phone: +49(0)5251/60-3560  
Wärburger Straße 100, 33098 Paderborn, Germany E-mail: manfred.hammer@uni-paderborn.de

## Course overview

### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F **Examples for dielectric optical waveguides.**
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
- Hybrid analytical / numerical coupled mode theory.
- Oblique semi-guided waves: 2-D integrated optics.

### 2-D waveguide configurations



$\epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i\omega t)$  (FD)

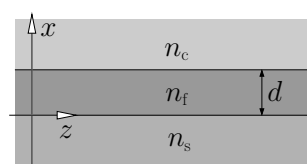
- 2-D waveguide, 1-D cross section.
- Permittivity  $\epsilon = n^2$ , refractive index  $n(x)$ . (1-D waveguide)

- $\partial_y \epsilon = 0 \iff \partial_y \mathbf{E} = 0, \partial_y \mathbf{H} = 0$ , 2-D TE/TM setting.
- $\partial_z \epsilon = 0 \iff$  Modal solutions that vary harmonically with  $z$ :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x) e^{-i\beta z}, \quad \begin{array}{l} \text{mode profile } \bar{\mathbf{E}}, \bar{\mathbf{H}}, \\ \text{propagation constant } \beta, \\ \text{effective index } n_{\text{eff}} = \beta/k. \end{array}$$

(TE): principal component  $\bar{E}_y, \partial_x^2 \bar{E}_y + (k^2 \epsilon - \beta^2) \bar{E}_y = 0,$   
 $\bar{E}_x = 0, \bar{E}_z = 0, \bar{H}_x = \frac{-\beta}{\omega \mu_0} \bar{E}_y, \bar{H}_y = 0, \bar{H}_z = \frac{i}{\omega \mu_0} \partial_x \bar{E}_y,$   
 $\bar{E}_y$  &  $\partial_x \bar{E}_y$  continuous at dielectric interfaces.

### 2-D waveguide configurations



$\epsilon \in \mathbb{R}, \mu = 1, \sim \exp(i\omega t)$  (FD)

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(TM): principal component  $\bar{H}_y, \epsilon \partial_x \frac{1}{\epsilon} \partial_x \bar{H}_y + (k^2 \epsilon - \beta^2) \bar{H}_y = 0,$   
 $\bar{E}_x = \frac{\beta}{\omega \epsilon_0 \epsilon} \bar{H}_y, \bar{E}_y = 0, \bar{E}_z = \frac{-i}{\omega \epsilon_0 \epsilon} \partial_x \bar{H}_y, \bar{H}_x = 0, \bar{H}_z = 0,$   
 $\bar{H}_y$  &  $\epsilon^{-1} \partial_x \bar{H}_y$  continuous at dielectric interfaces.

## Guided 2-D TE/TM modes, orthogonality properties

- A set (index  $m$ ) of guided modes of a 2-D waveguide ( $\epsilon$ ), (→ Exercise.)  
 $\psi_m^p = (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m)$ ,  $p=\text{TE, TM}$  &  $\beta_m$ ,  $\beta_m \neq \beta_l$ , if  $l \neq m$ .
- $(\mathbf{E}_1, \mathbf{H}_1; \mathbf{E}_2, \mathbf{H}_2) := \frac{1}{4} \int (E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}) dx$ .
- Power  $P_m$  per lateral ( $y$ ) unit length carried by mode  $\psi_m^p$ ,  $\beta_m$ :  

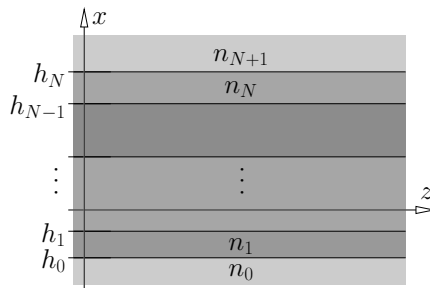
$$P_m := \int S_z dx = (\psi_m^p; \psi_m^p) = \begin{cases} \frac{\beta_m}{2\omega\mu_0} \int |E_{m,y}|^2 dx, & \text{if } p = \text{TE}, \\ \frac{\beta_m}{2\omega\epsilon_0} \int \frac{1}{\epsilon} |H_{m,y}|^2 dx, & \text{if } p = \text{TM}. \end{cases}$$

$$(\psi_l^{\text{TE}}; \psi_m^{\text{TM}}) = 0, \quad (\psi_l^{\text{TE}}; \psi_m^{\text{TE}}) = \frac{\beta_m}{2\omega\mu_0} \int E_{l,y}^* E_{m,y} dx = \delta_{lm} P_m,$$

$$(\psi_l^{\text{TM}}; \psi_m^{\text{TE}}) = 0, \quad (\psi_l^{\text{TM}}; \psi_m^{\text{TM}}) = \frac{\beta_m}{2\omega\epsilon_0} \int \frac{1}{\epsilon} H_{l,y}^* H_{m,y} dx = \delta_{lm} P_m.$$

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## Dielectric multilayer slab waveguide

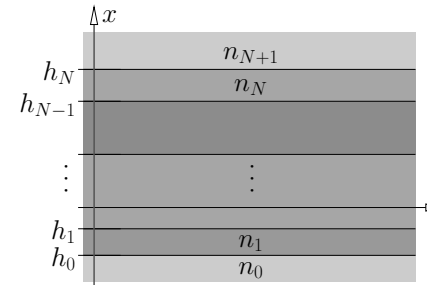


- Interior layer  $l$ ,  
 $h_{l-1} < x < h_l$ ,  
 local refractive index  $n_l$ ,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_l^2) \phi$ .
- Consider a trial value  $\beta^2 \in \mathbb{R}$ .

- $\beta^2 < k^2 n_l^2 \rightsquigarrow \partial_x^2 \phi = -\kappa_l^2 \phi$ ,  $\kappa_l := \sqrt{k^2 n_l^2 - \beta^2}$ ,  
 $\phi(x) = A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x)$ .
- $\beta^2 > k^2 n_l^2 \rightsquigarrow \partial_x^2 \phi = \kappa_l^2 \phi$ ,  $\kappa_l := \sqrt{\beta^2 - k^2 n_l^2}$ ,  
 $\phi(x) = A_l e^{\kappa_l x} + B_l e^{-\kappa_l x}$ .
- Unknowns  $A_l, B_l \in \mathbb{C}$ . (Local coordinate offsets required to cope with the exponentials.)

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## Dielectric multilayer slab waveguide



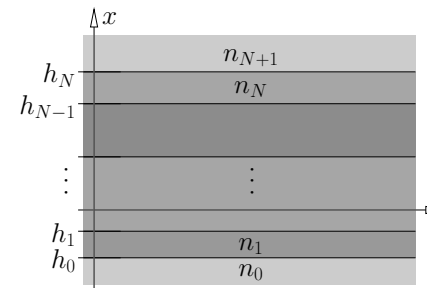
$\epsilon \in \mathbb{R}$ ,  $\mu = 1$ ,  $\sim \exp(i\omega t)$  (2-D, FD)

- $N$  interior layers,  
 piecewise constant  $\epsilon = n^2$ :  

$$n(x) = \begin{cases} n_{N+1} & \text{if } h_N < x, \\ n_l & \text{if } h_{l-1} < x < h_l, \\ n_0 & \text{if } x < h_0. \end{cases}$$
- Principal component  $\phi(x)$  (TE:  $\phi = \bar{E}_y$ , TM:  $\phi = \bar{H}_y$ ).
- $\partial_x^2 \phi + (k^2 n_l^2 - \beta^2) \phi = 0$ ,  $x \in \text{layer } l$ ,  $l = 0, \dots, N+1$   
(Half-infinite substrate ( $l = 0$ ) and cover ( $l = N+1$ ) layers.)
- $\phi$  &  $\eta \partial_x \phi$  continuous at  $x = h_l$ , (TE:  $\eta = 1$ , TM:  $\eta = n^{-2}$ ).

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## Dielectric multilayer slab waveguide, guided modes

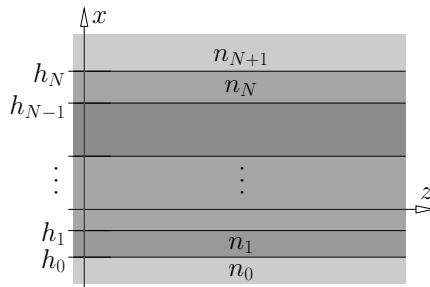


- Substrate region,  
 $x < h_0$ ,  
 local refractive index  $n_0$ ,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_0^2) \phi$ .
- Consider a trial value  $\beta^2 \in \mathbb{R}$ .

- $\beta^2 < k^2 n_0^2 \rightsquigarrow \partial_x^2 \phi = -\kappa_0^2 \phi$ ,  $\kappa_0 := \sqrt{k^2 n_0^2 - \beta^2}$ ,  
 $\phi(x) = A_0 \sin(\kappa_0 x) + B_0 \cos(-\kappa_0 x)$ .
- $\beta^2 > k^2 n_0^2 \rightsquigarrow \partial_x^2 \phi = \kappa_0^2 \phi$ ,  $\kappa_0 := \sqrt{\beta^2 - k^2 n_0^2}$ ,  
 $\phi(x) = A_0 e^{\kappa_0 x} + B_0 e^{-\kappa_0 x}$ .
- Unknown  $A_0 \in \mathbb{C}$ . **Guided modes:  $n_{\text{eff}} = \beta/k > n_0$ .**

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## Dielectric multilayer slab waveguide, guided modes



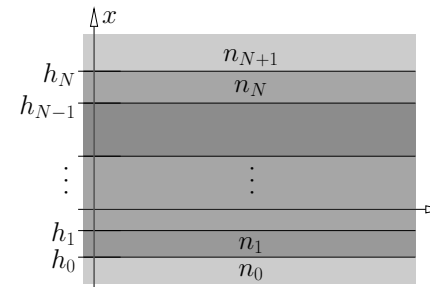
- Cover region,  $h_N < x$ , local refractive index  $n_{N+1}$ ,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_{N+1}^2) \phi$ .
- Consider a trial value  $\beta^2 \in \mathbb{R}$ .

- $\beta^2 < k^2 n_{N+1}^2 \rightsquigarrow \partial_x^2 \phi = -\kappa_{N+1}^2 \phi$ ,  $\kappa_{N+1} := \sqrt{k^2 n_{N+1}^2 - \beta^2}$ ,  
 $\phi(x) = A_{N+1} \sin(\kappa_{N+1} x) + B_{N+1} \cos(-\kappa_{N+1} x)$ .
- $\beta^2 > k^2 n_{N+1}^2 \rightsquigarrow \partial_x^2 \phi = \kappa_{N+1}^2 \phi$ ,  $\kappa_{N+1} := \sqrt{\beta^2 - k^2 n_{N+1}^2}$ ,  
 $\phi(x) = A_{N+1} e^{\kappa_{N+1} x} + B_{N+1} e^{-\kappa_{N+1} x}$ .
- Unknown  $B_{N+1} \in \mathbb{C}$ . Guided modes:  $n_{\text{eff}} = \beta/k > n_{N+1}$ .

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## Dielectric multilayer slab waveguide



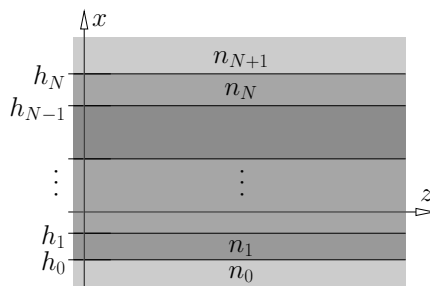
- Trial value  $\beta^2 \in \mathbb{R}$ ,  
 $\beta/k > n_0, n_{N+1}$ ,
- $\rightsquigarrow \kappa_l$ ,  $l = 0, \dots, N + 1$ .

$$\phi(x) = \begin{cases} B_{N+1} e^{-\kappa_{N+1} x}, & \text{for } h_N < x, \\ \begin{cases} A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x), & \text{if } \beta^2 < k^2 n_l^2, \\ A_l e^{\kappa_l x} + B_l e^{-\kappa_l x}, & \text{if } \beta^2 > k^2 n_l^2, \end{cases} & \text{for } h_{l-1} < x < h_l, \\ A_0 e^{\kappa_0 x}, & \text{for } x < h_0. \end{cases}$$

- $2N + 2$  unknowns  $A_0, A_1, B_1, \dots, A_N, B_N, B_{N+1}$ .
- Continuity of  $\phi, \eta \partial_x \phi$  at  $N + 1$  interfaces  $\rightsquigarrow 2N + 2$  equations.

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## Dielectric multilayer slab waveguide



- Trial value  $\beta^2 \in \mathbb{R}$ ,  
 $\beta/k > n_0, n_{N+1}$ .

- $2N + 2$  unknowns  $A_0, A_1, B_1, \dots, A_N, B_N, B_{N+1}$ .
- Continuity of  $\phi, \eta \partial_x \phi$  at  $N + 1$  interfaces  $\rightsquigarrow 2N + 2$  equations.
- Arrange as linear system of equations  $\mathbf{M}(\beta^2) (A_0, \dots, B_{N+1})^T = 0$ .
- Identify propagation constants where  $\mathbf{M}(\beta^2)$  becomes singular.

(Equations relate to the series of interfaces  $\leftrightarrow$  A transfer-matrix technique can be applied.)

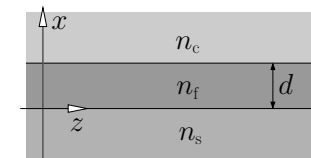
- Choose e.g.  $A_0 = 1$ , fill  $A_1, \dots, B_{N+1}$ , normalize.  $(\dots, \dots)$

Guided modes  $\{\beta_m, (\bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m)\}$ .

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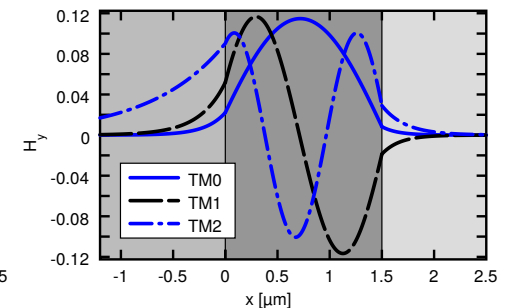
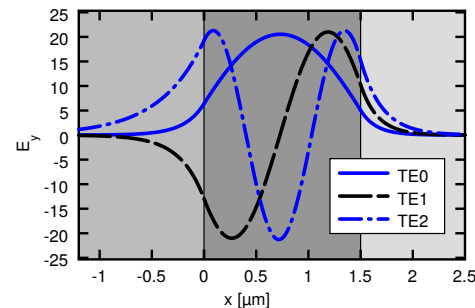
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## A nonsymmetric 3-layer slab waveguide



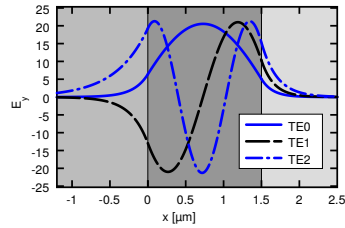
$n_s = 1.45$ ,  $n_f = 1.99$ ,  $n_c = 1.0$ ,  
 $d = 1.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .

TE<sub>0</sub>:  $n_{\text{eff}} = 1.944$ , TM<sub>0</sub>:  $n_{\text{eff}} = 1.933$ ,  
 TE<sub>1</sub>:  $n_{\text{eff}} = 1.804$ , TM<sub>1</sub>:  $n_{\text{eff}} = 1.759$ ,  
 TE<sub>2</sub>:  $n_{\text{eff}} = 1.562$ , TM<sub>2</sub>:  $n_{\text{eff}} = 1.490$ .



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### Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE/TM.)

$$\partial_x(\partial_x\phi) = -(k^2 n^2 - \beta^2)\phi.$$

$k^2 n^2 - \beta^2$  determines the rate of change of the slope of  $\phi$ .

Imagine a numerical ODE algorithm of "shooting-type".



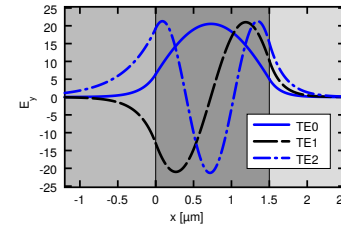
- Guided modes with a growing number of nodes ( $x$  with  $\phi(x) = 0$ ) with decreasing effective indices

↔ mode indices = number of nodes in  $\phi$ .

"Quantum numbers".

- A **fundamental mode** with zero nodes and highest effective index.
- Modes of the same polarization are **non-degenerate**.

### Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE/TM.)

$$\partial_x(\partial_x\phi) = -(k^2 n^2 - \beta^2)\phi.$$

$k^2 n^2 - \beta^2$  determines the rate of change of the slope of  $\phi$ .

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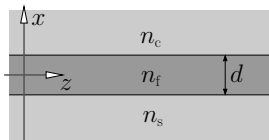


- A sign change of  $\partial_x\phi$  is required to form a guided mode  
 ↳ There must be some region (layer) with  $k^2 n^2 - \beta^2 > 0$ .

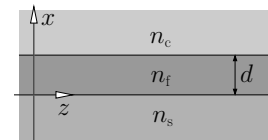
Interval for effective indices  $n_{\text{eff}}$  of guided modes:

$$\max\{n_0, n_{N+1}\} < n_{\text{eff}} < \max\{n_l\}.$$

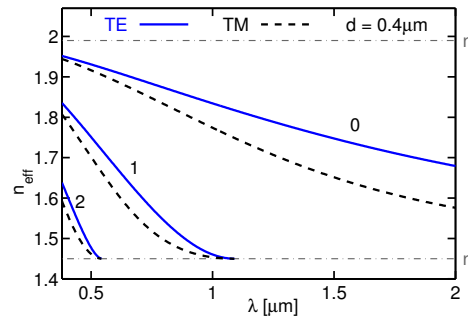
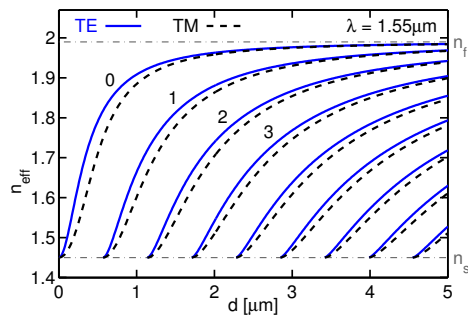
### 3-layer slab waveguide, dispersion curves



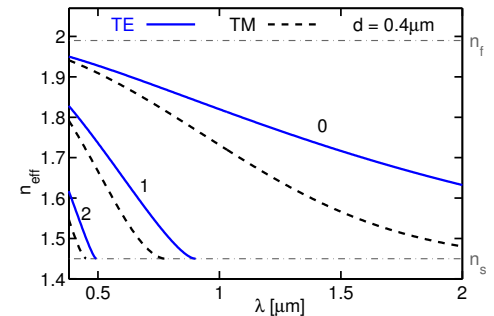
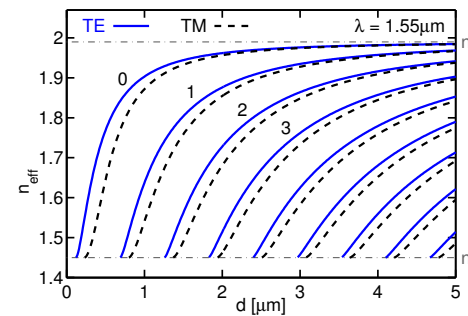
Symmetric waveguide, moderate refractive index contrast,  $n_s = 1.45$ ,  $n_f = 1.99$ ,  $n_c = 1.45$ .



Nonsymmetric waveguide, moderate refractive index contrast,  $n_s = 1.45$ ,  $n_f = 1.99$ ,  $n_c = 1.0$ .

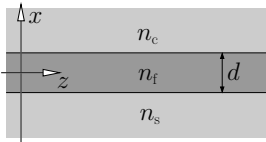


(Caution:  $\partial_\lambda \epsilon = 0$  assumed!)

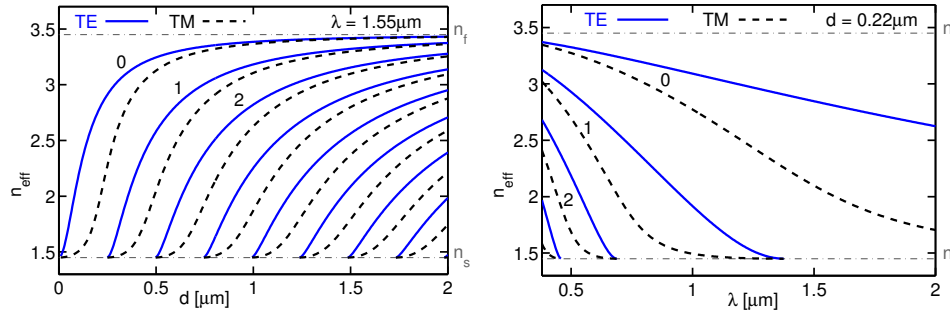


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### 3-layer slab waveguide, dispersion curves

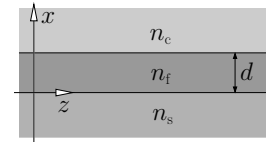


Symmetric waveguide,  
high refractive index contrast,  
 $n_s = 1.45, n_f = 3.45, n_c = 1.45$ .

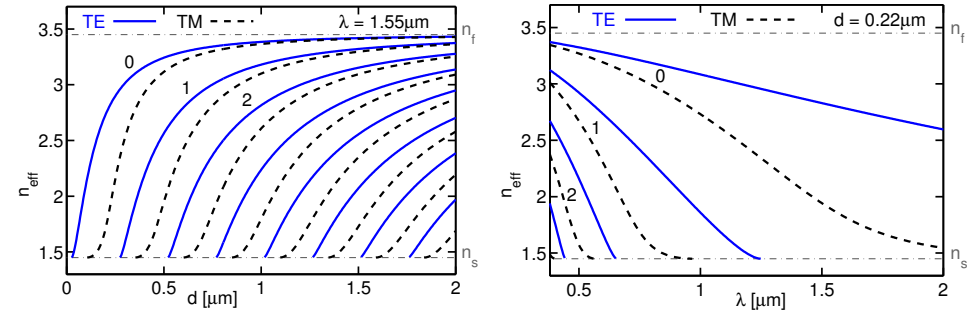


(Caution:  $\partial_\lambda \epsilon = 0$  assumed!)

### 3-layer slab waveguide, dispersion curves



Nonsymmetric waveguide,  
high refractive index contrast,  
 $n_s = 1.45, n_f = 3.45, n_c = 1.0$ .



(Caution:  $\partial_\lambda \epsilon = 0$  assumed!)

### 3-layer slab waveguide, dispersion curves

Remarks / observations:

- At large core thicknesses, or short wavelengths, for all modes:  $n_{\text{eff}}$  approaches the level  $n_f$  of bulk waves in the core material.
- Modes of higher order at the same  $n_{\text{eff}}$  supported by waveguides with thickness increased by specific distances.

Guided mode, layer  $l$  with  $\kappa_l^2 = (k^2 n_l^2 - \beta^2) > 0$ , field  $\phi(x) \sim \cos(\kappa_l x + \chi)$  for  $x \in$  layer  $l$ ;  
increase layer thickness by  $\Delta x = \pi / \kappa_l$ , such that  $\kappa_l(x + \Delta x) = \kappa_l x + \pi$   
→ the thicker waveguide supports a mode of order  $+1$  with the same propagation constant.

- Cutoff thicknesses at fixed wavelength.

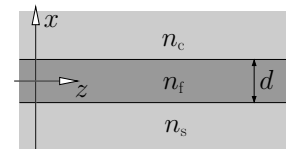
Nonsymmetric 3-layer waveguide  $n_s \neq n_c$ : There exist cutoff thicknesses for all modes.  
Symmetric 3-layer waveguide  $n_s = n_c$ : Cutoff thicknesses exist for all modes of order  $\geq 1$ ,  
no cutoff thickness for the fundamental TE/TM modes.

- $\lambda$  is the “length-defining” quantity; wavelength scaling, factor  $a$ :  
 $n_{\text{eff}}(a\lambda, ad) = n_{\text{eff}}(\lambda, d), \beta(a\lambda, ad) = a^{-1} \beta(\lambda, d)$ .

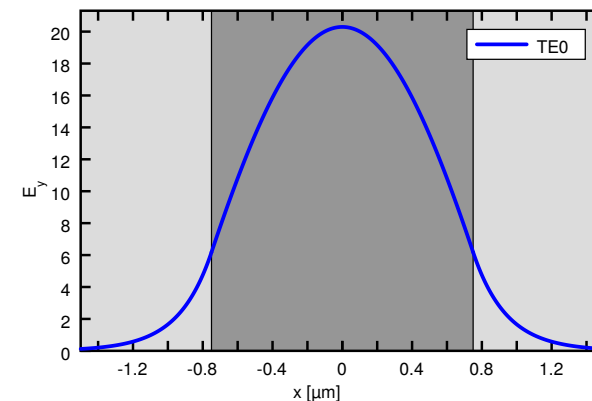
- Cutoff wavelengths for waveguides with fixed thickness.

For all modes; exception: no cutoff wavelength for the fundamental TE/TM modes in a symmetric 3-layer waveguide.

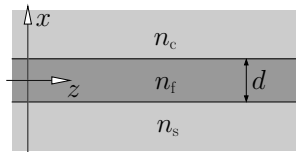
### 3-layer slab waveguide, mode confinement



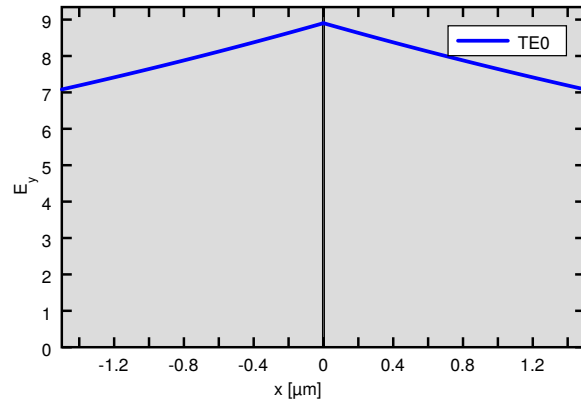
Symmetric waveguide,  
moderate refractive index contrast,  
 $n_s = 1.45, n_f = 1.99, n_c = 1.45, \lambda = 1.55 \mu\text{m}$ ,  
 $d = 1.50 \mu\text{m}$ , TE<sub>0</sub>:  $n_{\text{eff}} = 1.946$ .



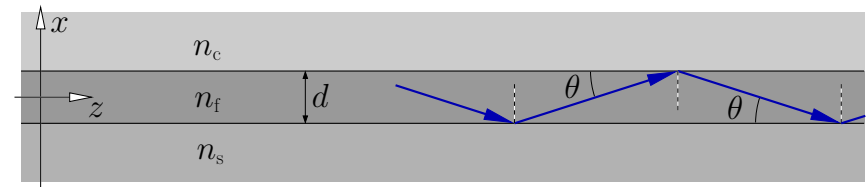
### 3-layer slab waveguide, mode confinement



Symmetric waveguide, moderate refractive index contrast,  $n_s = 1.45$ ,  $n_f = 1.99$ ,  $n_c = 1.45$ ,  $\lambda = 1.55 \mu\text{m}$ ,  $d = 0.01 \mu\text{m}$ ,  $\text{TE}_0$ :  $n_{\text{eff}} = 1.450$ .



### 3-layer slab waveguide, ray model



Field in the core:

$$\sim a_u e^{-i(\kappa x + \beta z)} + a_d e^{-i(-\kappa x + \beta z)}, \quad k^2 n_f^2 = \beta^2 + \kappa^2$$

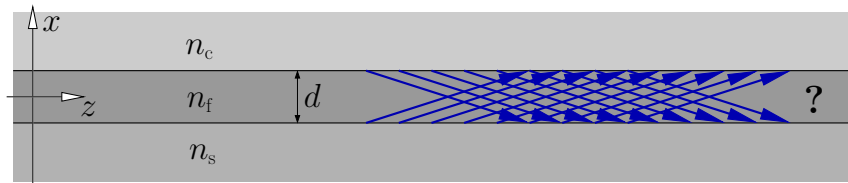
propagation angle  $\theta$  with  $\beta = kn_f \cos \theta$ ,  $\kappa = kn_f \sin \theta$ .

Guided mode formation:

- Repeated total internal reflection of waves in the core at upper and lower interfaces
- Calculate optical phase gain, including phase jumps for reflection at interfaces (polarization dependent).
- Phase gain of  $2\pi$  for one "round trip", "transverse resonance condition"  $\longleftrightarrow$  constructive interference of waves.

(A frequently encountered intuitive model . . . of very limited applicability.)

### 3-layer slab waveguide, ray model



Field in the core:

$$\sim a_u e^{-i(\kappa x + \beta z)} + a_d e^{-i(-\kappa x + \beta z)}, \quad k^2 n_f^2 = \beta^2 + \kappa^2$$

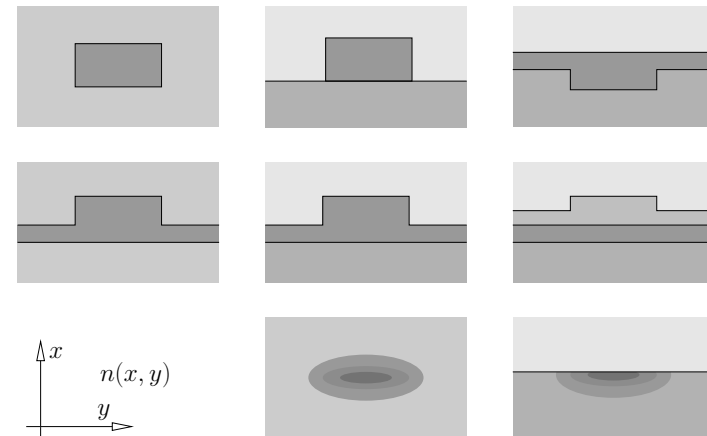
propagation angle  $\theta$  with  $\beta = kn_f \cos \theta$ ,  $\kappa = kn_f \sin \theta$ .

Guided mode formation:

- Repeated total internal reflection of waves in the core at upper and lower interfaces
- Calculate optical phase gain, including phase jumps for reflection at interfaces (polarization dependent).
- Phase gain of  $2\pi$  for one "round trip", "transverse resonance condition"  $\longleftrightarrow$  constructive interference of waves.

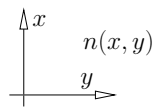
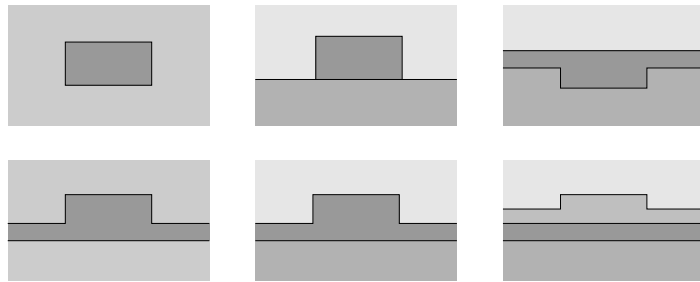
(A frequently encountered intuitive model . . . of very limited applicability.)

### 3-D waveguides



Cross sections (2-D) of typical integrated-optical waveguides.

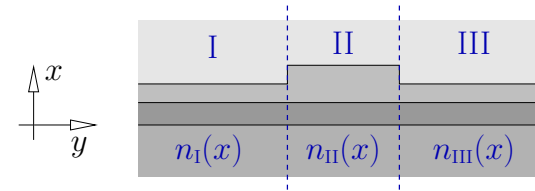
### 3-D rectangular waveguides



No analytical solutions:

- numerical mode solvers.
- approximations.

### Effective index method



Outline:

- Divide into slices  $\rho = I, II, III$ :  $n(x, y) = n_\rho(x)$ , if  $y \in \text{slice } \rho$ .
- Compute polarized modes  $X_\rho(x), \beta_\rho$ ,  $X_\rho'' + (k^2 n_\rho^2 - \beta_\rho^2)X_\rho = 0$ ,  $N_\rho = \beta_\rho/k$ .
- Consider a scalar mode equation for the principal component  $\Psi$  of the 3-D waveguide

$$\partial_x^2 \Psi + \partial_y^2 \Psi + (k^2 n^2 - \beta^2)\Psi = 0, \quad \Psi = E_y \text{ (TE)}, \quad \Psi = H_y \text{ (TM)}.$$

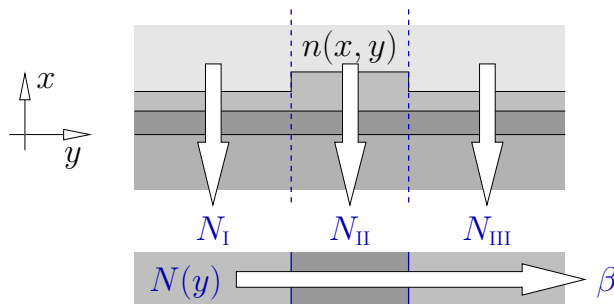
- Ansatz:  $\Psi(x, y) = X_\rho(x) Y(y)$ , if  $y \in \text{slice } \rho$ ; require continuity of  $Y$  and  $Y'$ .
- **Effective index profile:**  $N(y) := N_\rho$ , if  $y \in \text{slice } \rho$ .

$$Y'' + (k^2 N^2 - \beta^2)Y = 0,$$

a 1-D mode equation for  $Y, \beta$  with the effective index profile  $N$  in place of the refractive indices.

(!)

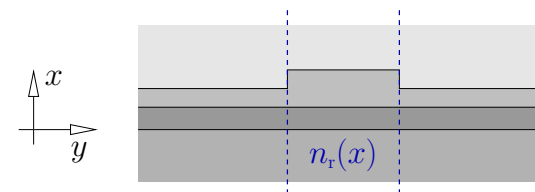
### Effective index method, schematically



Remarks / issues:

- A popular, quite intuitive method.
- Frequently an (often informal) basis for discussion of waveguide properties.
- $\leftrightarrow$  Relevance of the slab waveguide model.
- Manifold variants / ways of improvements exist.
- What if a slice does not support a guided slab mode?
- What about higher order modes?
- How to evaluate modal fields? What about other than principal components?
- ...

### Variational effective index method



Outline:

- Identify a reference slice, refractive index profile  $n_r(x)$ .
- Compute polarized guided slab modes  $(\bar{E}, \bar{H})_r, \beta_r$  for the reference slice.
- For each each reference slab mode: ...
- Choose an ansatz:

$$\begin{pmatrix} E_x, E_y, E_z \\ H_x, H_y, H_z \end{pmatrix} (x, y, z) = \begin{pmatrix} 0, & \bar{E}_{r,y}(x)Y^{E_y}(y), & \bar{E}_{r,y}(x)Y^{E_z}(y) \\ \bar{H}_{r,x}(x)Y^{H_x}(y), & \bar{H}_{r,z}(x)Y^{H_y}(y), & \bar{H}_{r,z}(x)Y^{H_z}(y) \end{pmatrix} \quad \text{(TE)}$$

$$\begin{pmatrix} E_x, E_y, E_z \\ H_x, H_y, H_z \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{E}_{r,x}(x)Y^{E_x}(y), & \bar{E}_{r,z}(x)Y^{E_y}(y), & \bar{E}_{r,z}(x)Y^{E_z}(y) \\ 0, & \bar{H}_{r,y}(x)Y^{H_y}(y), & \bar{H}_{r,y}(x)Y^{H_z}(y) \end{pmatrix} \quad \text{(TM)}$$

$$Y \cdot (y) = ?$$

(!)

## A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

$$\bullet \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\beta z}, \quad \beta \in \mathbb{R},$$

$$\bar{\mathbf{E}}, \bar{\mathbf{H}} \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.$$

$$\bullet (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{E}} = -i\omega\mu_0\bar{\mathbf{H}}, \quad (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{H}} = i\omega\epsilon_0\bar{\mathbf{E}},$$

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

$$\bullet \mathcal{B}(\mathbf{E}, \mathbf{H}) := \frac{\omega\epsilon_0\langle \mathbf{E}, \epsilon\mathbf{E} \rangle + \omega\mu_0\langle \mathbf{H}, \mathbf{H} \rangle + i\langle \mathbf{E}, \mathbf{C}\mathbf{H} \rangle - i\langle \mathbf{H}, \mathbf{C}\mathbf{E} \rangle}{\langle \mathbf{E}, \mathbf{R}\mathbf{H} \rangle - \langle \mathbf{H}, \mathbf{R}\mathbf{E} \rangle},$$

$$\langle \mathbf{F}, \mathbf{G} \rangle = \iint \mathbf{F}^* \cdot \mathbf{G} \, dx \, dy.$$

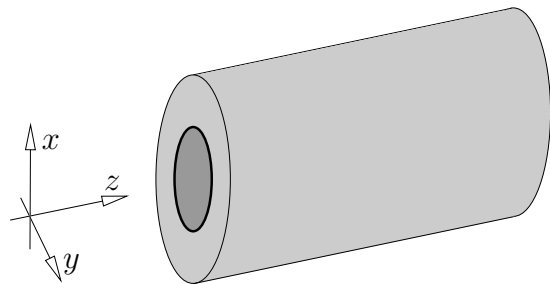
$$\mathcal{B}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta, \quad \left. \frac{d}{ds} \mathcal{B}(\bar{\mathbf{E}} + s\delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + s\delta\bar{\mathbf{H}}) \right|_{s=0} = 0$$

at valid mode fields  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ , for arbitrary  $\delta\bar{\mathbf{E}}, \delta\bar{\mathbf{H}}$ .

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## Optical fibers

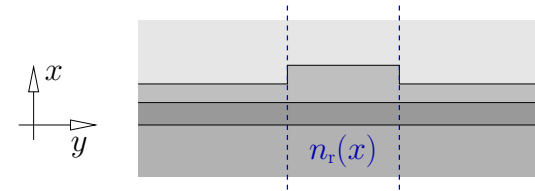


[Optical Communication A-D]

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## Variational effective index method



Outline, continued:

- Restrict  $\mathcal{B}$  to the VEIM ansatz, require stationarity with respect to the  $\{Y\}$ .

↪ 1-D mode ("s-like") equations for principal unknowns  $Y^{H_x}$  (TE) and  $Y^{E_x}$  (TM) with effective quantities in place of refractive indices, all other  $Y$  can be computed.

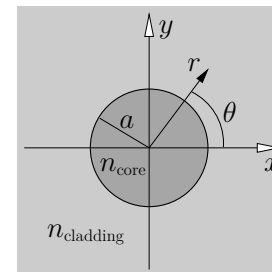
(!)



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## Circular step index optical fibers



(FD)

Circular symmetry

↔ cylindrical coordinates  $r, \theta, z$ .

$$\epsilon = n^2, \quad n(r) = \begin{cases} n_{\text{core}}, & r \leq a, \\ n_{\text{cladding}}, & r > a. \end{cases}$$

Circular and axial symmetry:

$$\hookrightarrow \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (r, \theta, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (r) e^{-il\theta - i\beta z}, \quad l \in \mathbb{Z}, \beta \in \mathbb{R}.$$

( $E_r, E_\theta, E_z, H_r, H_\theta, H_z$ )

Where  $\partial\epsilon = 0$ :  $\Delta\psi + k^2 n^2 \psi = 0, \quad \psi \in \{E_r, \dots, H_z\}$ .

$$\hookrightarrow \partial_r^2 \phi + \frac{1}{r} \partial_r \phi + (k^2 n^2 - \beta^2 - \frac{l^2}{r^2}) \phi = 0, \quad \phi \in \{\bar{E}_r, \dots, \bar{H}_z\}$$

(An ODE of Bessel type.)

& vectorial interface conditions at  $r = a$ . (Alternatively: Scalar theory, LP modes.)

(...)



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## “Complex” waveguides

Attenuating / gain media, leakage

$\sim \exp(i\omega t)$  (FD)

Mode amplitudes change along propagation distance.

$\partial_z \epsilon = 0$ ,  $\partial_z n = 0$ , mode ansatz with complex propagation constant:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (x, y) e^{-i\gamma z},$$

$\bar{\mathbf{E}}, \bar{\mathbf{H}}$ : mode profile,

$\gamma = \beta - i\alpha \in \mathbb{C}$ : propagation constant,

$\beta \in \mathbb{R}$ : phase constant,

$\alpha \in \mathbb{R}$ : attenuation constant,

$$L_p = \frac{1}{2\alpha}: \text{propagation length,}$$

$$n_{\text{eff}} = \gamma/k \in \mathbb{C},$$

$$\psi(z) \sim e^{-i\gamma z} = e^{-i\beta z} e^{-\alpha z}, \quad |\psi(z)|^2 \sim e^{-2\alpha z},$$

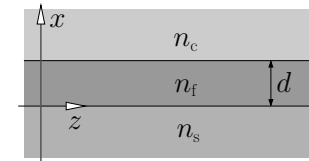
if  $\alpha > 0$ .

Applies to all former examples.

$\gamma \in \mathbb{C}$ : Entire theory needs to be reconsidered, in principle.

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## “Complex” waveguides, loss

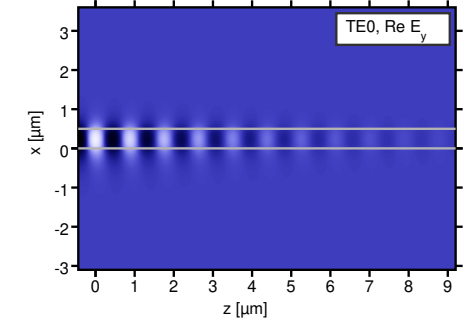
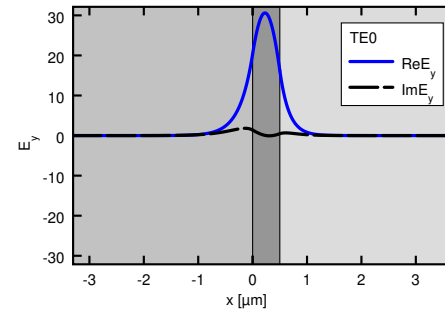


2-D,

$n_s = 1.45$ ,  $n_f = 1.99 - i0.1$ ,  $n_c = 1.0$ ,  
 $d = 0.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .

Bound modes:

TE<sub>0</sub>:  $n_{\text{eff}} = 1.767 - i0.093$ ,  $L_p = 1.32 \mu\text{m}$ .

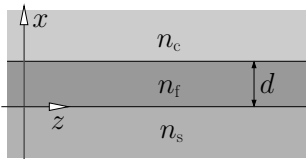


(Mode attenuation, essentially complex non-plane profiles, curved wavefronts,  $S_x \neq 0$ .)

(Analysis: as before (...); boundary conditions: bound fields, integrability.)

Navigation icons and page number 27

## “Complex” waveguides, loss

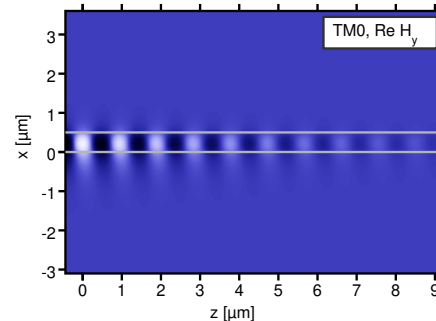
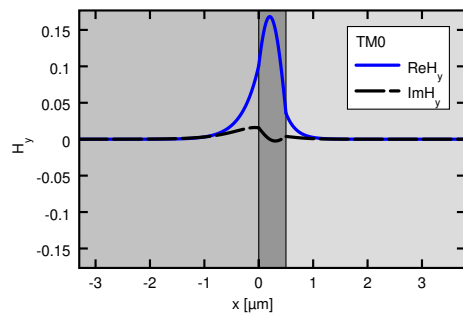


2-D,

$n_s = 1.45$ ,  $n_f = 1.99 - i0.1$ ,  $n_c = 1.0$ ,  
 $d = 0.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .

Bound modes:

TM<sub>0</sub>:  $n_{\text{eff}} = 1.640 - i0.074$ ,  $L_p = 1.66 \mu\text{m}$ .

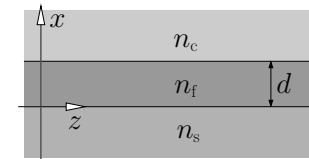


(Mode attenuation, essentially complex non-plane profiles, curved wavefronts,  $S_x \neq 0$ .)

(Analysis: as before (...); boundary conditions: bound fields, integrability.)

Navigation icons and page number 27

## “Complex” waveguides, gain

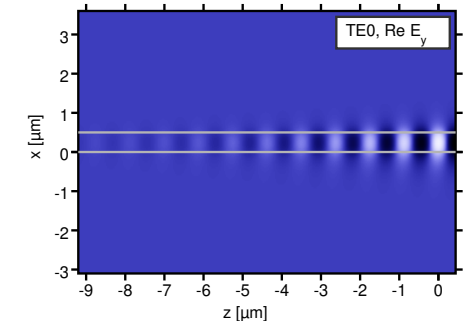
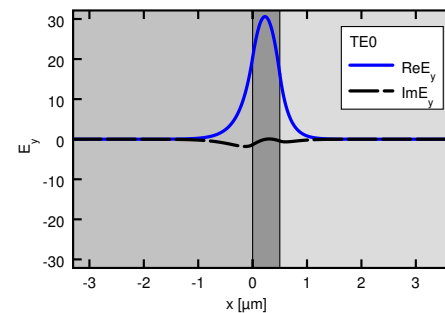


2-D,

$n_s = 1.45$ ,  $n_f = 1.99 + i0.1$ ,  $n_c = 1.0$ ,  
 $d = 0.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .

Bound modes:

TE<sub>0</sub>:  $n_{\text{eff}} = 1.767 + i0.093$ ,  $\frac{1}{2|\alpha|} = 1.32 \mu\text{m}$ .

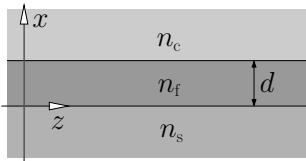


(Modal gain, essentially complex non-plane profiles, curved wavefronts,  $S_x \neq 0$ .)

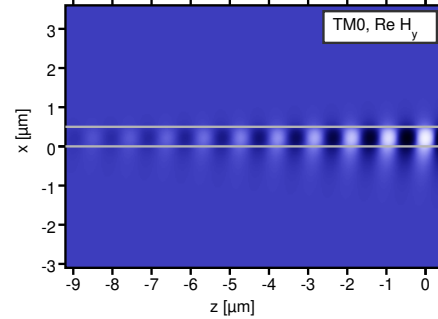
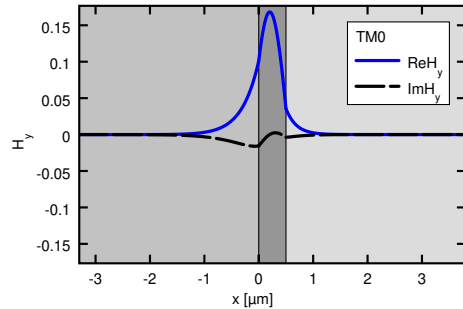
(Analysis: as before (...); boundary conditions: bound fields, integrability.)

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## “Complex” waveguides, gain

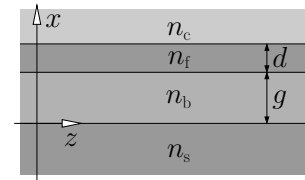


2-D,  
 $n_s = 1.45$ ,  $n_f = 1.99 + i0.1$ ,  $n_c = 1.0$ ,  
 $d = 0.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .  
**Bound modes:**  
 TM<sub>0</sub>:  $n_{\text{eff}} = 1.640 + i0.074$ ,  $\frac{1}{2|\alpha|} = 1.66 \mu\text{m}$ .

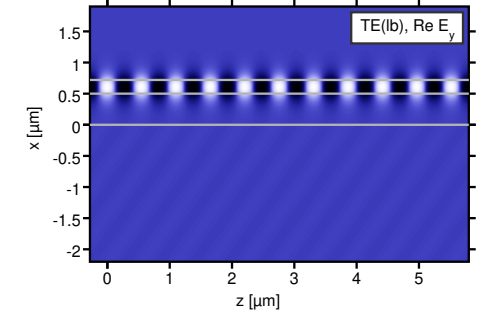
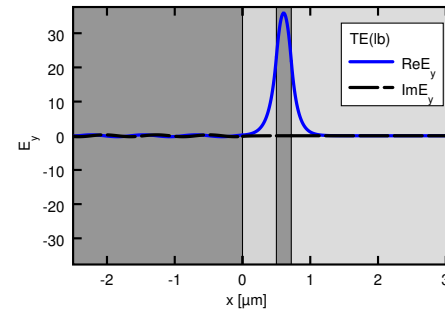


(Modal gain, essentially complex non-plane profiles, curved wavefronts,  $S_x \neq 0$ )  
 (Analysis: as before (...); boundary conditions: bound fields, integrability)

## “Complex” waveguides, leakage

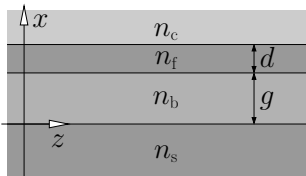


2-D,  
 $n_s = 3.45$ ,  $n_b = 1.45$ ,  $n_f = 3.45$ ,  $n_c = 1.0$ ,  
 $d = 0.22 \mu\text{m}$ ,  $g = 0.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .  
**Leaky modes:**  
 TE<sub>0</sub>:  $n_{\text{eff}} = 2.805 - i2.432 \cdot 10^{-5}$ ,  $L_p = 5073 \mu\text{m}$ .

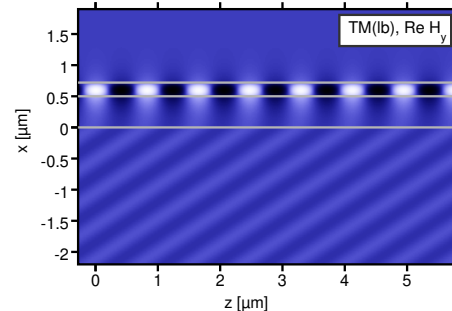
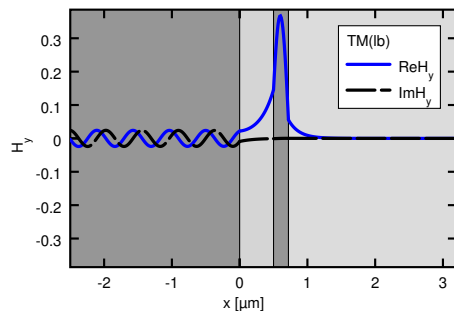


(Radiative loss, essentially complex non-plane profiles, curved wavefronts,  $S_x \neq 0$ , field growth for  $x \rightarrow -\infty$ )  
 (Analysis: as before (...); boundary conditions: outgoing wave for  $x \rightarrow -\infty$ , bound field at  $x \rightarrow \infty$ )

## “Complex” waveguides, leakage



2-D,  
 $n_s = 3.45$ ,  $n_b = 1.45$ ,  $n_f = 3.45$ ,  $n_c = 1.0$ ,  
 $d = 0.22 \mu\text{m}$ ,  $g = 0.5 \mu\text{m}$ ,  $\lambda = 1.55 \mu\text{m}$ .  
**Leaky modes:**  
 TM<sub>0</sub>:  $n_{\text{eff}} = 1.878 - i3.203 \cdot 10^{-3}$ ,  $L_p = 38.51 \mu\text{m}$ .



(Radiative loss, essentially complex non-plane profiles, curved wavefronts,  $S_x \neq 0$ , field growth for  $x \rightarrow -\infty$ )  
 (Analysis: as before (...); boundary conditions: outgoing wave for  $x \rightarrow -\infty$ , bound field at  $x \rightarrow \infty$ )

## Upcoming

Next lectures:

- Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- Bent optical waveguides; whispering gallery resonances; circular microresonators.
- Coupled mode theory, perturbation theory.

