

Optical Waveguide Theory (H)



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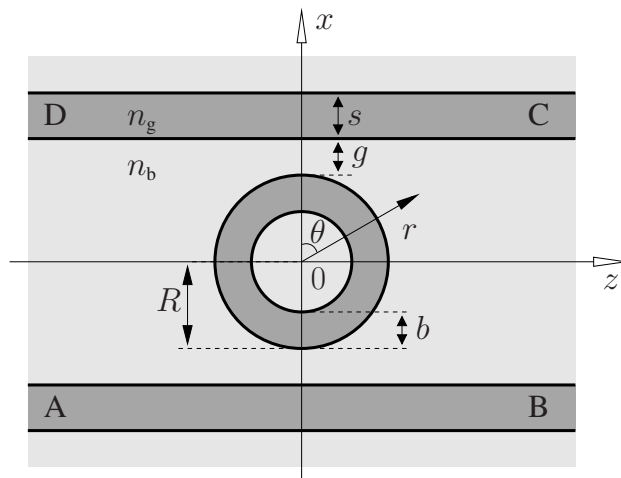
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Course overview

Optical waveguide theory

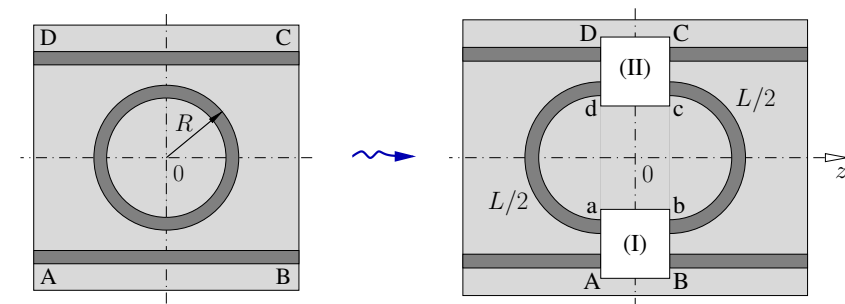
- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
 - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Oblique semi-guided waves: 2-D integrated optics.
 - Summary, concluding remarks.

Circular traveling wave resonators



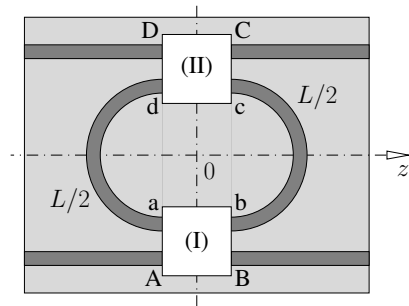
Integrated optical **mirco-ring** or **micro-disk resonators**.

Ringresonator: Abstract model



- Ringresonator \approx 2 couplers + 2 cavity segments
- CW description: $\mathbf{E}, \mathbf{H} \sim e^{i\omega t}$, $\omega = kc$, $k = 2\pi/\lambda$.

Couplers: Scattering matrices



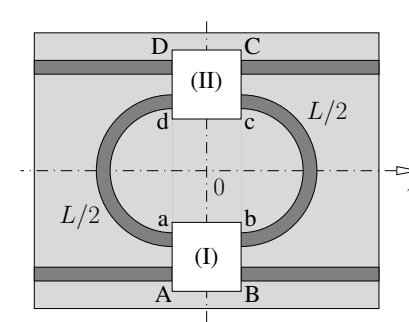
- Uniform polarization, single mode waveguides.
- Linear, nonmagnetic (attenuating) elements.
- Backreflections are negligible.
- Interaction restricted to the couplers \leftrightarrow "port" definition.

\hookrightarrow Symmetric coupler scattering matrices:

$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \chi & \tau \\ \rho & \chi & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

$A_{\pm}, B_{\pm}, a_{\pm}, b_{\pm}$: Amplitudes of waves traveling in $\pm z$ -direction.

Coupler symmetries



Symmetry $z \rightarrow -z$:

$$A_+ \rightarrow b_+ \stackrel{!}{=} B_- \rightarrow a_-$$

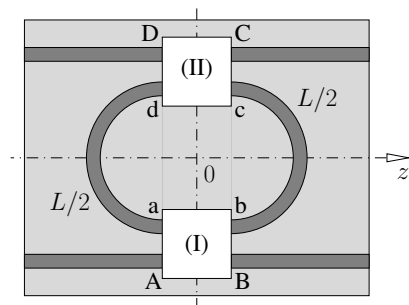
$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \kappa & \tau \\ \rho & \kappa & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} A_- \\ a_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} B_- \\ b_- \end{pmatrix}, \quad \begin{pmatrix} B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \end{pmatrix}.$$

Symmetry $x \rightarrow -x$, (I) = (II):

$$\hookrightarrow \begin{pmatrix} D_- \\ d_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} C_- \\ c_- \end{pmatrix}, \quad \begin{pmatrix} C_+ \\ c_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} D_+ \\ d_+ \end{pmatrix}.$$

Cavity segments



Field evolution $\sim e^{-i\gamma s}$ along the cavity core, propagation distance s .

$$\gamma = \beta - i\alpha,$$

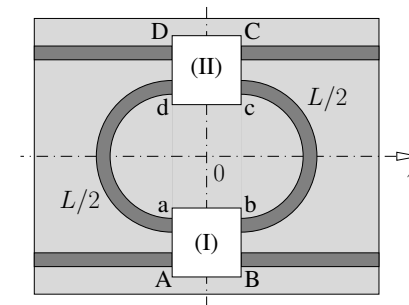
β : phase propagation constant,
 α : attenuation constant.

(\leftrightarrow bend modes, to come.)

\hookrightarrow Relations of amplitudes at the ends of the cavity segments:

$$\begin{aligned} c_- &= b_+ e^{-i\beta L/2} e^{-\alpha L/2}, & a_+ &= d_- e^{-i\beta L/2} e^{-\alpha L/2}, \\ b_- &= c_+ e^{-i\beta L/2} e^{-\alpha L/2}, & d_+ &= a_- e^{-i\beta L/2} e^{-\alpha L/2}. \end{aligned}$$

Output amplitudes



Coupler scattering matrices
+ Cavity field evolution
+ External input amplitudes

$$A_+ = \sqrt{P_{in}},$$

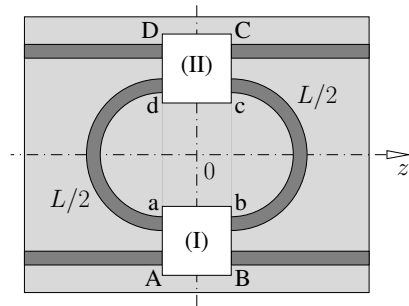
$$B_- = C_- = D_+ = 0$$

External output amplitudes:

$$A_- = 0, \quad C_+ = 0, \quad D_- = \frac{\kappa^2 p}{1 - \tau^2 p^2} A_+, \quad B_+ = \left(\rho + \frac{\kappa^2 \tau p^2}{1 - \tau^2 p^2} \right) A_+,$$

$$p = e^{-i\beta L/2} e^{-\alpha L/2}.$$

Power transfer



Power drop: $P_D = |D_-|^2$,
 Transmission: $P_T = |B_+|^2$.

$$P_D = P_{in} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$P_T = P_{in} \frac{|\rho|^2 (1 + |\tau|^2 d^2 e^{-2\alpha L} - 2|\tau| d e^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$\tau =: |\tau| e^{i\varphi}, \quad d e^{i\psi} := \tau - \kappa^2/\rho, \quad L \neq 2\pi R.$$

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Resonances

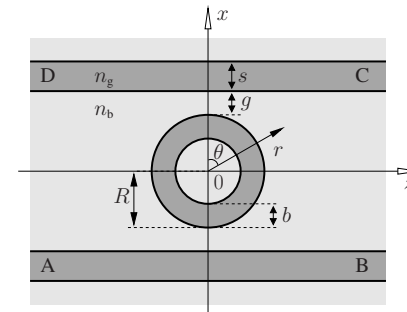
$$P_D = P_{in} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)} (\lambda)$$

$$P_T = P_{in} \frac{|\rho|^2 (1 + |\tau|^2 d^2 e^{-2\alpha L} - 2|\tau| d e^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)} (\lambda)$$

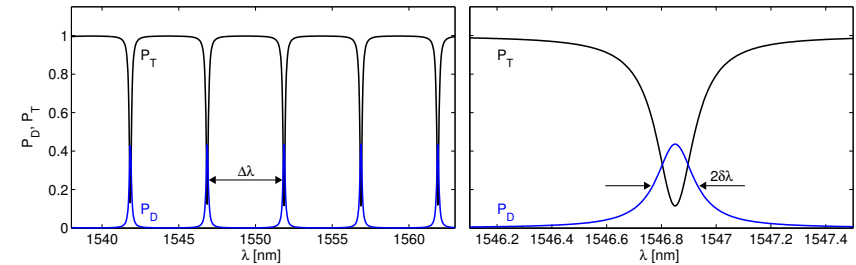
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Spectral response



$R = 50 \mu\text{m}$, $b = s = 1.0 \mu\text{m}$, $g = 0.9 \mu\text{m}$,
 $n_b = 1.45$, $n_g = 1.60$; 2-D, TE.
 $\Delta\lambda = 5.0 \text{ nm}$, $2\delta\lambda = 0.17 \text{ nm}$,
 $F = 30$, $Q = 9400$, $P_{D,\text{res}} = 0.44$.



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Resonances

$$P_D = P_{in} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$P_T = P_{in} \frac{|\rho|^2 (1 + |\tau|^2 d^2 e^{-2\alpha L} - 2|\tau| d e^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L - 2\varphi)}$$

- Resonances:
 \approx Singularities in the denominators of P_D , P_T , origin: $\beta(\lambda)$.
- Correction for finite coupler length l :
 $\beta L - 2\varphi = \beta L_{\text{cav}} - \phi$, $\phi = 2\beta l + 2\varphi$, $L_{\text{cav}} = 2\pi R$, $\partial_\lambda \phi \approx 0$.
- Resonance condition: $\cos(\beta L_{\text{cav}} - \phi) = 1$, or

$$\beta = \frac{2m\pi + \phi}{L_{\text{cav}}} =: \beta_m \quad \text{integer } m; \quad P_D|_{\beta=\beta_m} = P_{in} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - |\tau|^2 e^{-\alpha L})^2}.$$

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Free spectral range

- Resonance next to β_m :

$$\beta_{m-1} = \frac{2(m-1)\pi + \phi}{L_{\text{cav}}} = \beta_m - \frac{2\pi}{L_{\text{cav}}} \approx \beta_m + \left. \frac{\partial \beta}{\partial \lambda} \right|_m \Delta \lambda$$

- $\partial_\lambda \beta = ?$

q_j : waveguide parameters with dimension length,

$$\beta(a\lambda, aq_j) = \beta(\lambda, q_j)/a, \quad \partial_a|_{a=1}$$

$$\hookrightarrow \frac{\partial \beta}{\partial \lambda} = -\frac{1}{\lambda} \left(\beta + \sum_j q_j \frac{\partial \beta}{\partial q_j} \right) \approx -\frac{\beta}{\lambda}.$$

$$\text{FSR:} \quad \Delta \lambda = -\frac{2\pi}{L_{\text{cav}}} \left(\left. \frac{\partial \beta}{\partial \lambda} \right|_m \right)^{-1} \approx \frac{\lambda^2}{n_{\text{eff}} L_{\text{cav}}} \Big|_m, \quad n_{\text{eff}} = \beta/k.$$

(Free spectral range, the spectral distance (here: wavelength) between the drop peaks / the transmission dips.)

Finesse & Q-factor

$$\text{Finesse:} \quad F = \frac{\Delta \lambda}{2\delta \lambda} = \pi \frac{|\tau| e^{-\alpha L/2}}{1 - |\tau|^2 e^{-\alpha L}}.$$

$$\text{Q-factor:} \quad Q = \frac{\lambda}{2\delta \lambda} = \pi \frac{n_{\text{eff}} L_{\text{cav}}}{\lambda} \frac{|\tau| e^{-\alpha L/2}}{1 - |\tau|^2 e^{-\alpha L}} = \frac{n_{\text{eff}} L_{\text{cav}}}{\lambda} F.$$

$$\text{or} \quad Q = k R n_{\text{eff}} F \quad \text{for} \quad L_{\text{cav}} = 2\pi R.$$

Spectral width of the resonances

$$P_D = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{1 + |\tau|^4 e^{-2\alpha L} - 2|\tau|^2 e^{-\alpha L} \cos(\beta L_{\text{cav}} - \phi)},$$

$$P_D|_{\beta_m} = P_{D,\text{res}}.$$

$$P_D|_{\beta_m + \delta \beta} = P_{D,\text{res}}/2. \quad \delta \beta = ?$$

- Expansion of cos-terms

$$\hookrightarrow \delta \beta = \pm \frac{1}{L_{\text{cav}}} \left(\frac{1}{|\tau|} e^{\alpha L/2} - |\tau| e^{-\alpha L/2} \right) \approx -\frac{\beta_m}{\lambda} \delta \lambda$$

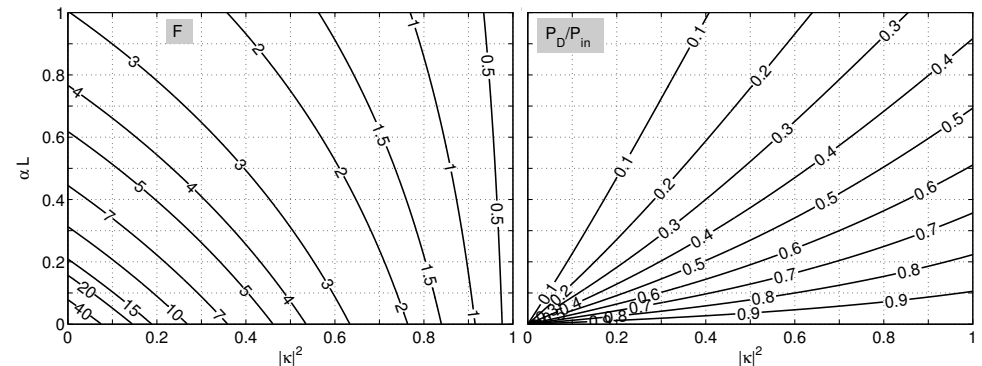
$$\text{FWHM:} \quad 2\delta \lambda = \frac{\lambda^2}{\pi L_{\text{cav}} n_{\text{eff}}} \Big|_m \left(\frac{1}{|\tau|} e^{\alpha L/2} - |\tau| e^{-\alpha L/2} \right).$$

(Full width at half maximum of the spectral drop peaks / the transmission dips (wavelength).)

Performance versus coupling strength & losses

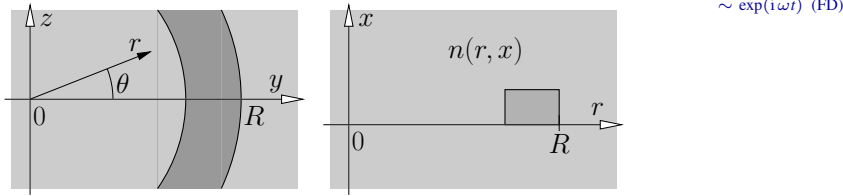
Assumption: Lossless coupler elements, $|\rho|^2 = |\tau|^2 = 1 - |\kappa|^2$.

$$F = \pi \frac{(\sqrt{1 - |\kappa|^2}) e^{-\alpha L/2}}{1 - (1 - |\kappa|^2) e^{-\alpha L}}, \quad P_D|_{\text{res}} = P_{\text{in}} \frac{|\kappa|^4 e^{-\alpha L}}{(1 - (1 - |\kappa|^2) e^{-\alpha L})^2}.$$



$\alpha, \kappa = ?$

Modes of bent waveguides



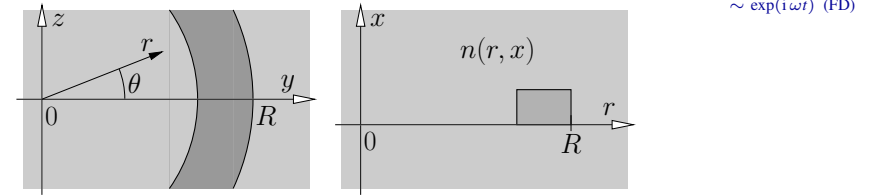
- Constant curvature \leftrightarrow cylindrical coordinates r, θ, x .
- Bend radius R , $\partial_\theta \epsilon = 0$, $\partial_\theta n = 0$

$$\hookrightarrow \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (r, \theta, x) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (r, x) e^{-i\gamma R\theta}, \quad \text{bend modes,}$$

$\bar{\mathbf{E}}, \bar{\mathbf{H}}$: bend mode profile, components $\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x$,
 $\gamma = \beta - i\alpha \in \mathbb{C}$: propagation constant,
 $\beta \in \mathbb{R}$: phase constant,
 $\alpha \in \mathbb{R}$: attenuation constant.

(Exponent $i\gamma R\theta$: a convention, "propagation distance" $R\theta$.)

Modes of bent waveguides



- Piecewise constant $n(r, x)$, $\psi \in \{\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x\}$,

$$\hookrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \psi = 0, \quad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),

& boundary conditions:

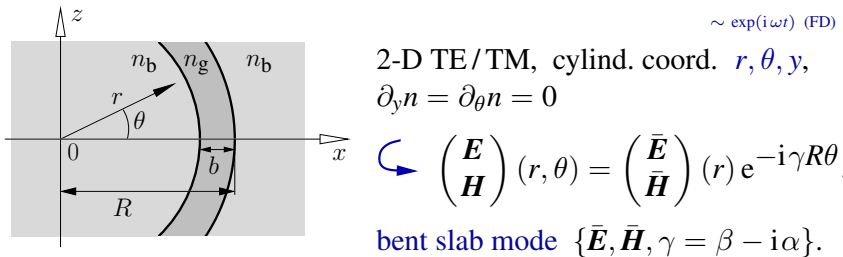
regularity at $r = 0$, outgoing waves at $x = \pm\infty$, $r = \infty$.

(or: normalizability versus x .)

Vectorial 3-D bend mode eigenvalue problem.

(Practical setting: computational domain $r_1 < r < r_0$, $x_b < x < x_t$, PML boundary conditions / $\psi = 0$ at $r = r_1$.)

Modes of bent slab waveguides



2-D TE/TM, cylind. coord. r, θ, y ,
 $\partial_y n = \partial_\theta n = 0$

$$\hookrightarrow \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (r, \theta) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix} (r) e^{-i\gamma R\theta},$$

bent slab mode $\{\bar{\mathbf{E}}, \bar{\mathbf{H}}, \gamma = \beta - i\alpha\}$.

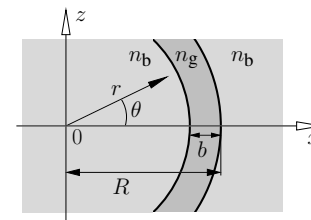
- Piecewise constant $n(r)$, $\phi = \bar{E}_y$ (TE), $\phi = \bar{H}_y$ (TM)

$$\hookrightarrow \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \left(k^2 n^2 - \frac{\gamma^2 R^2}{r^2} \right) \phi = 0,$$

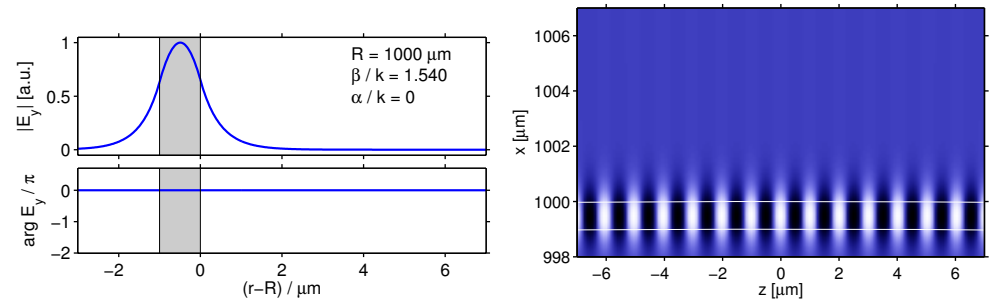
(Bessel differential equation with (complex) order γR .)

- Nonzero solutions,
- bounded at the origin, $\sim J_{\gamma R}(nkr)$ for $r < R - b$,
- outgoing exterior fields, $\sim H_{\gamma R}^{(2)}(nkr)$ for $r > R$, ($\sim \exp(i\omega t)$),
- continuity at interfaces: ϕ , $\partial_r \phi$ (TE), ϕ , $(\partial_r \phi)/n^2$ (TM).

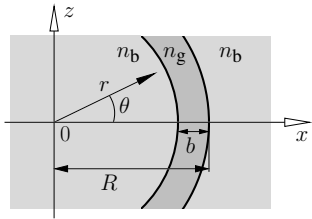
Bend modes, 2-D example



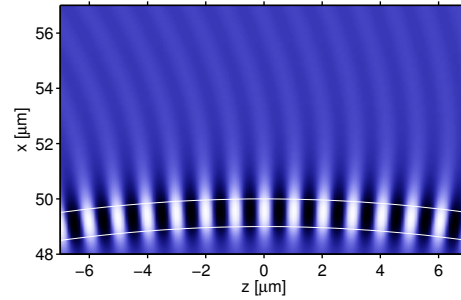
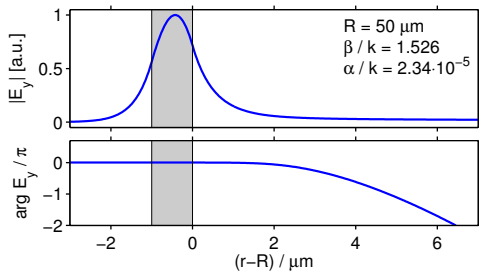
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 1000 \mu\text{m}$.



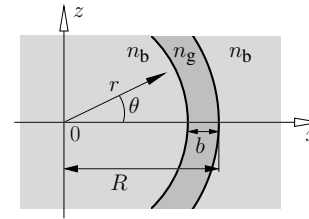
Bend modes, 2-D example



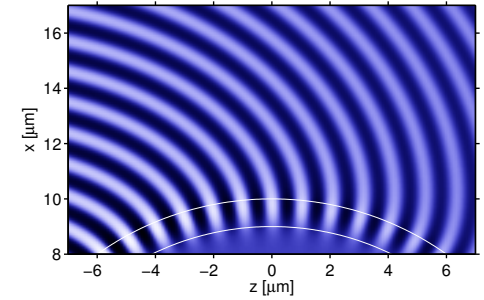
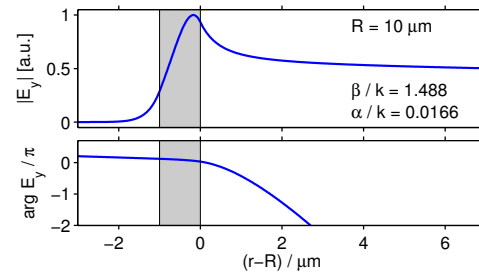
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 50 \mu\text{m}$.



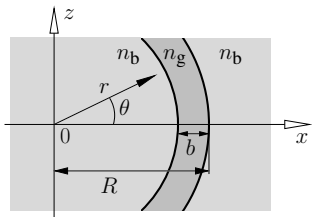
Bend modes, 2-D examples



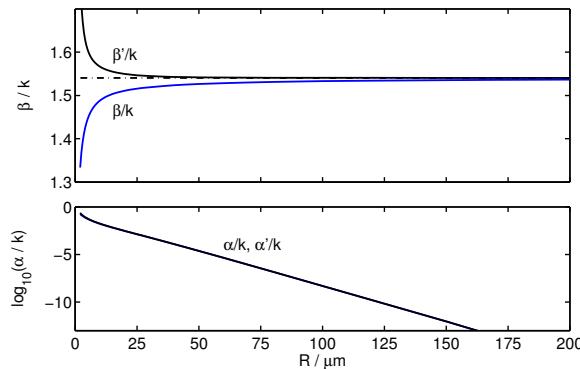
2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R = 10 \mu\text{m}$.



Propagation constant vs. bend radius



2-D, TE,
 $n_b = 1.45$, $n_g = 1.60$, $b = 1.0 \mu\text{m}$, $\lambda = 1.55 \mu\text{m}$,
 $R \in [2, 200] \mu\text{m}$.



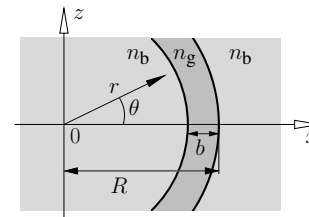
Alternative definition:
 $R' = R - b/2$.

Identical physical fields

$$\gamma'R' = \gamma R,$$

$$\gamma' = \gamma \frac{R}{R - b/2}.$$

Power & orthogonality

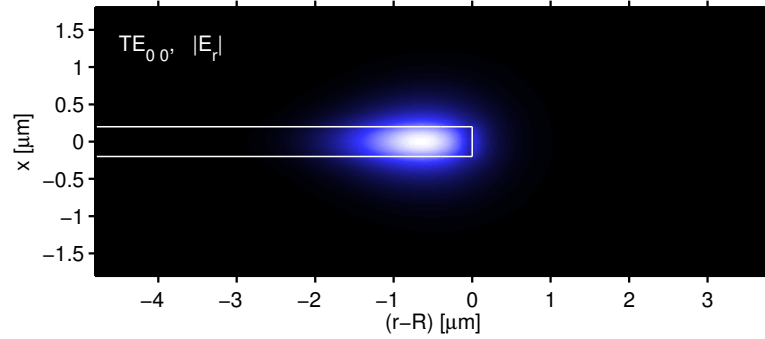
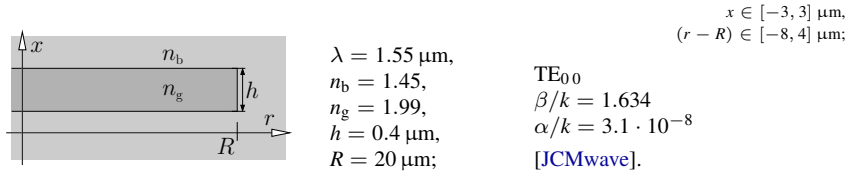


2-D TE/TM bend modes:

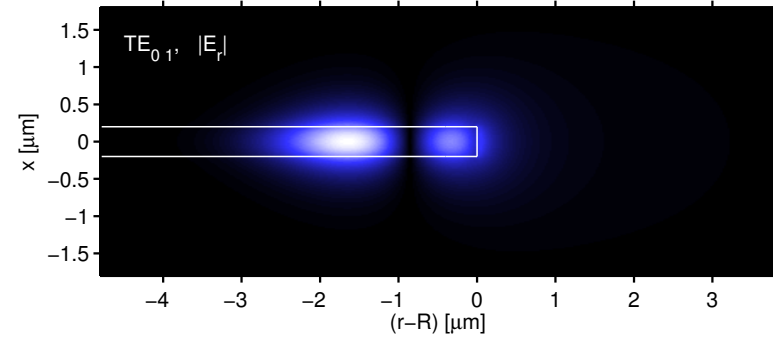
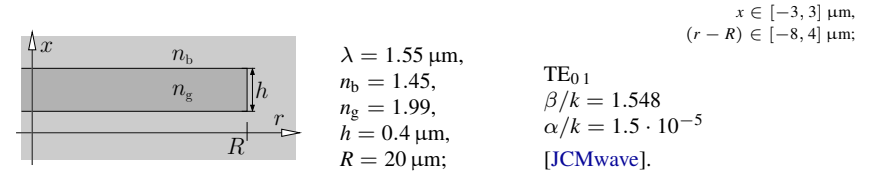
- Power flow: $S_r \neq 0$, $S_r, S_\theta \sim e^{-2\alpha R\theta}$, $S_\theta \sim |\phi|^2/r$
 $\int_0^\infty S_\theta(r) dr < \infty$ \leftrightarrow power normalization.
- Orthogonality of nondegenerate bend modes, product
 $[E_1, H_1; E_2, H_2] = \int_0^\infty (E_1 \times H_2 + E_2 \times H_1) \cdot e_\theta dr.$

(Here $[; ;]$ is complex valued.)
 (Expressions $\sim \phi^2/r \leftrightarrow$ convergence of the integrals.)

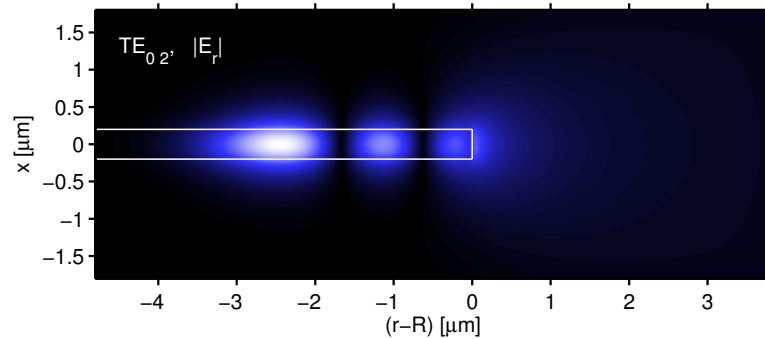
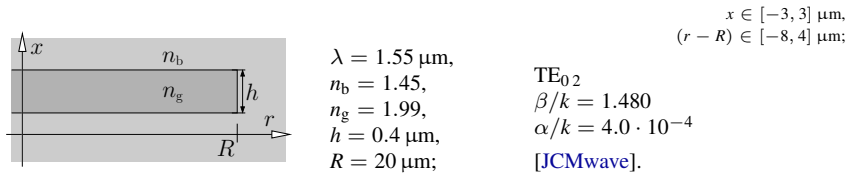
Bend modes supported by an angular disc segment



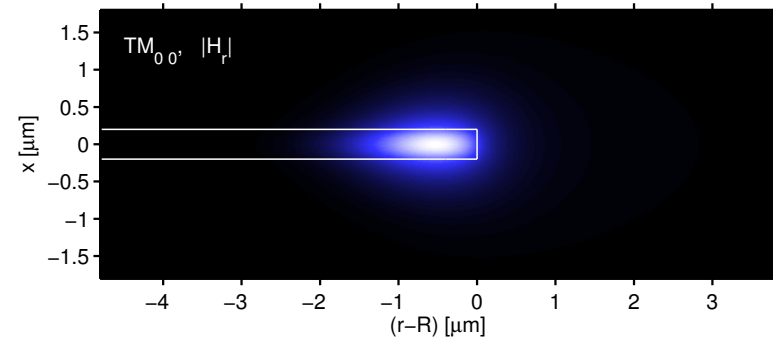
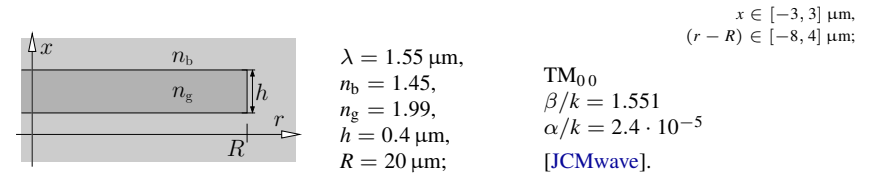
Bend modes supported by an angular disc segment



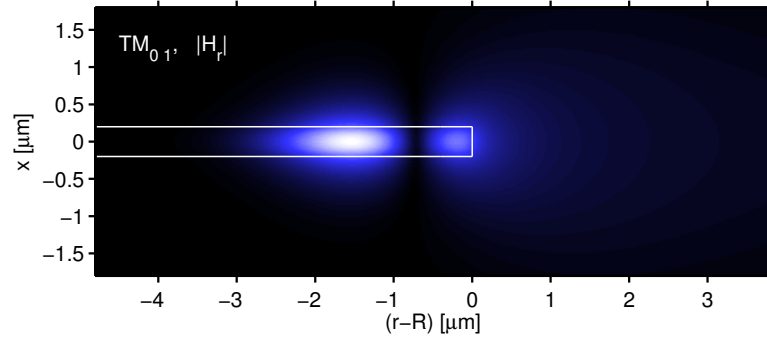
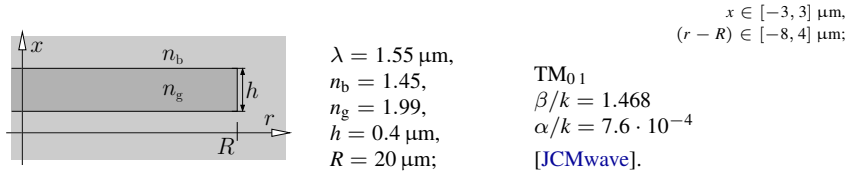
Bend modes supported by an angular disc segment



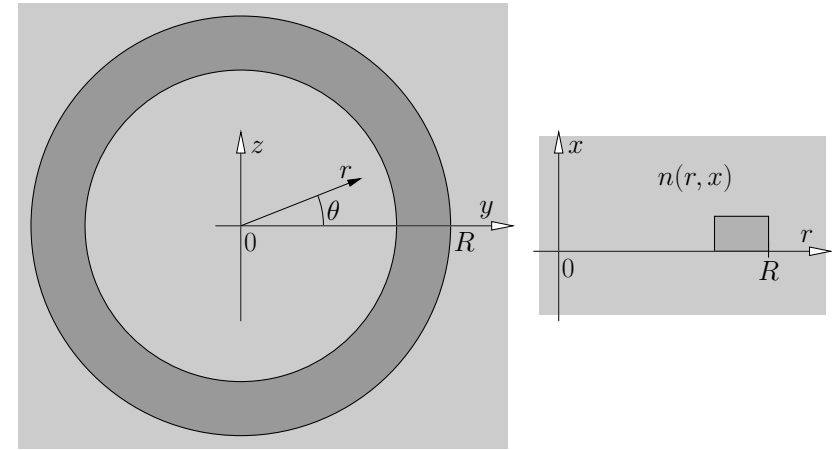
Bend modes supported by an angular disc segment



Bend modes supported by an angular disc segment



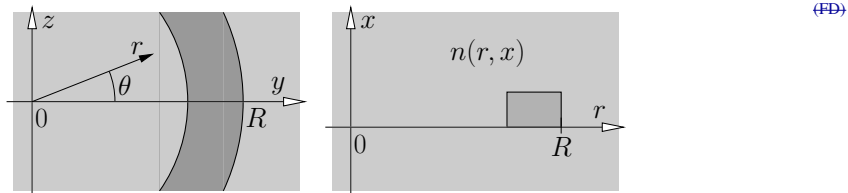
Circular microcavity



Bend modes \longleftrightarrow Whispering gallery resonances.

(Terms not always clearly distinguished.)

Whispering gallery resonances



- Full cavity, $\theta \in [0, 2\pi]$:
Look for resonances in the form of **whispering gallery modes**

$$\left(\begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right) (r, \theta, x, t) = \left(\begin{matrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{matrix} \right) (r, x) e^{i\omega_c t - im\theta}, \quad \text{+c.c.}$$

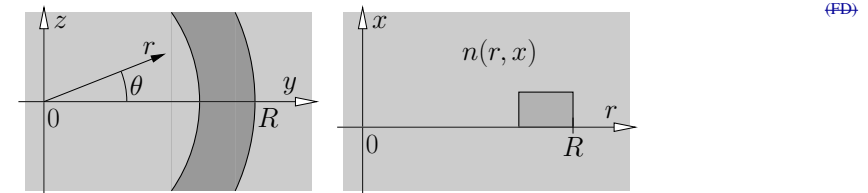
$\tilde{\mathbf{E}}, \tilde{\mathbf{H}}$: **WGM profile**, components $\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x$,

$m \in \mathbb{Z}$: **angular order**,

$\omega_c = \omega'_c + i\omega''_c \in \mathbb{C}$: **eigenfrequency**, $\omega'_c, \omega''_c \in \mathbb{R}$.

Q-factor $Q = \omega'_c / (2\omega''_c)$, resonance wavelength $\lambda_r = 2\pi c / \omega'_c$, outgoing radiation, FWHM: $2\delta\lambda = \lambda_r / Q$.

Whispering gallery resonances



- Piecewise constant $n(r, x)$, $\psi \in \{\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x\}$,
(Dispersion ?)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \left(\frac{\omega_c^2}{c^2} n^2 - \frac{m^2}{r^2} \right) \psi = 0, \quad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),

& boundary conditions:

regularity at $r = 0$, outgoing waves at $x = \pm\infty$, $r = \infty$.

(or: normalizability versus x .)

Vectorial eigenproblem for whispering gallery resonances.

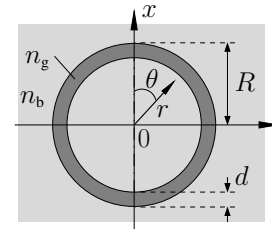
(Practical setting: computational domain $r_1 < r < r_0$, $x_b < x < x_t$, PML boundary conditions / $\psi = 0$ at $r = r_1$)

2-D whispering gallery resonances

... as discussed for the 2-D TE/TM bend modes.

(WGMs: Bessel differential equation of integer order.)
 (Notation: $\text{WGM}(\rho, m)$ — mode of radial order ρ and angular order m .)

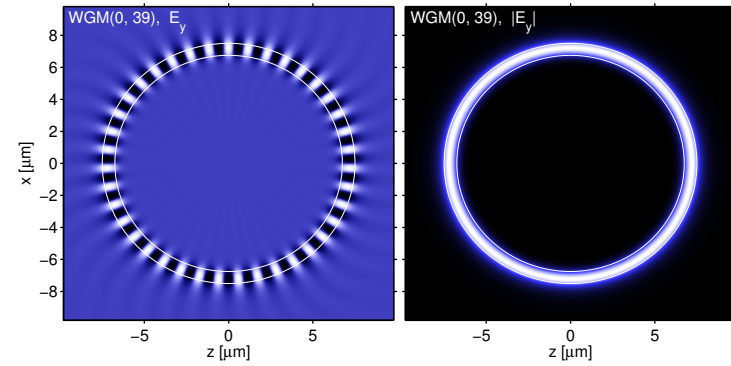
2-D whispering gallery resonances



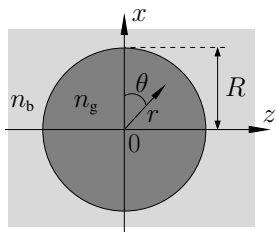
TE, $R = 7.5 \mu\text{m}$, $d = 0.75 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.5637 \mu\text{m}$, $Q = 1.1 \cdot 10^5$, $2\delta\lambda = 1.4 \cdot 10^{-5} \mu\text{m}$.



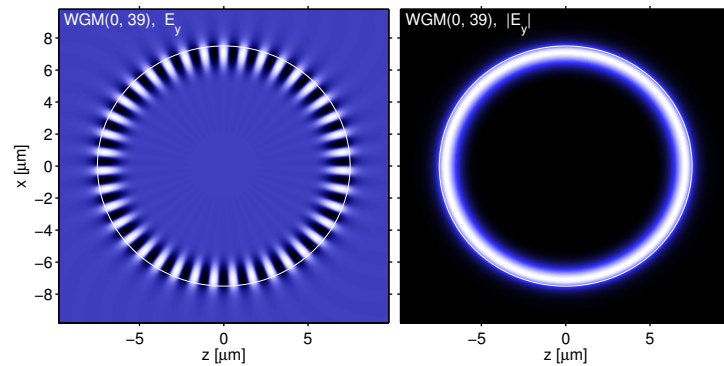
2-D whispering gallery resonances



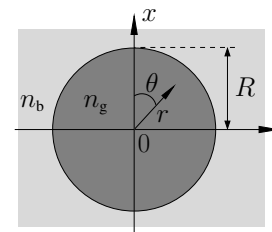
TE, $R = 7.5 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.6025 \mu\text{m}$, $Q = 5.7 \cdot 10^5$, $2\delta\lambda = 2.8 \cdot 10^{-6} \mu\text{m}$.



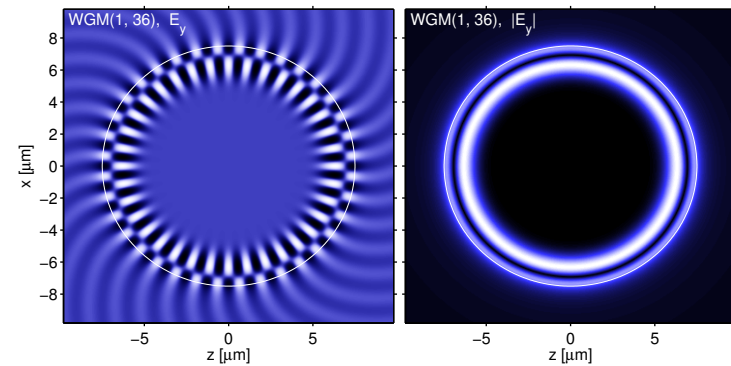
2-D whispering gallery resonances



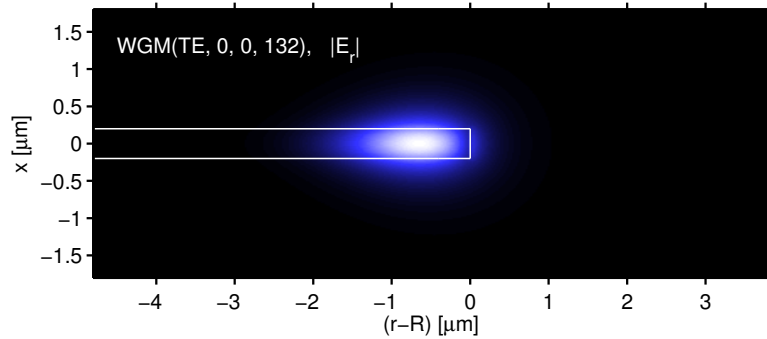
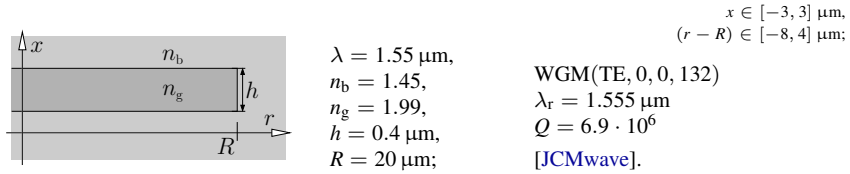
TE, $R = 7.5 \mu\text{m}$, $n_g = 1.5$, $n_b = 1.0$.

WGM(1, 36):

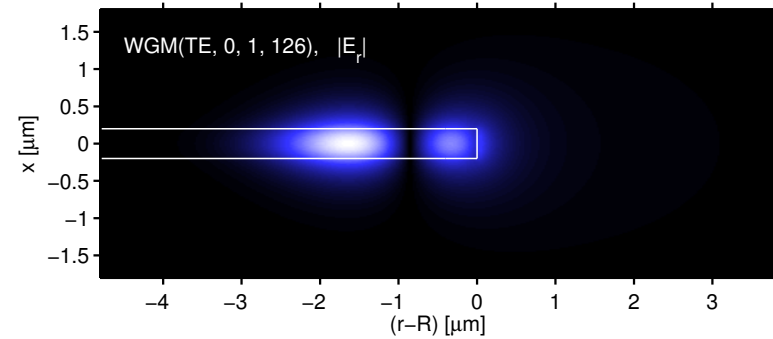
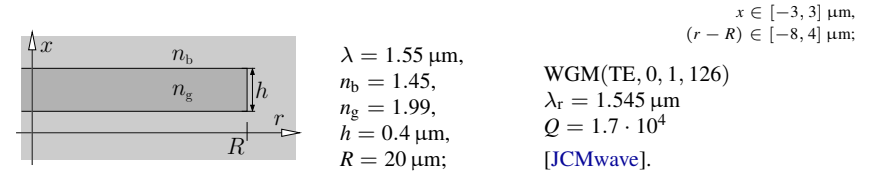
$\lambda_r = 1.5367 \mu\text{m}$, $Q = 2.2 \cdot 10^4$, $2\delta\lambda = 7.0 \cdot 10^{-4} \mu\text{m}$.



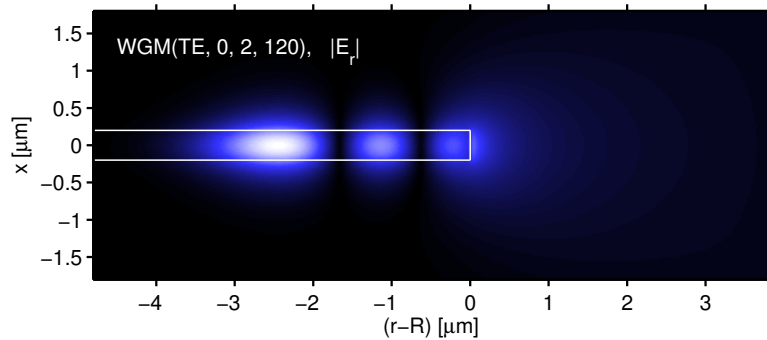
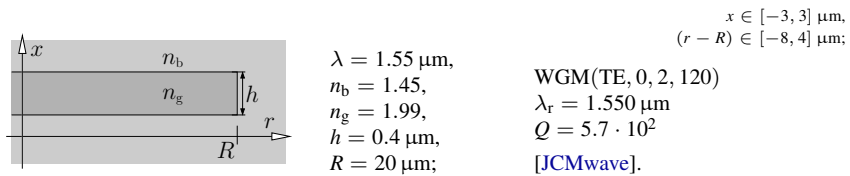
WGMs supported by a circular slab disc



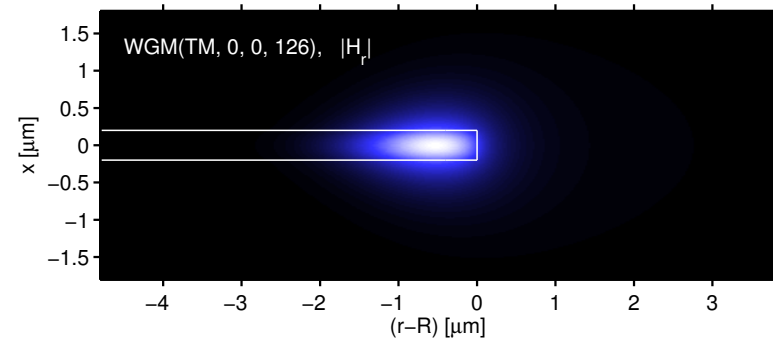
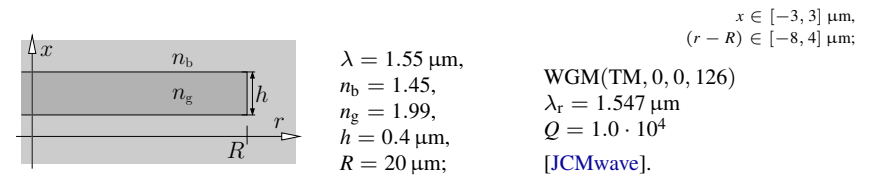
WGMs supported by a circular slab disc



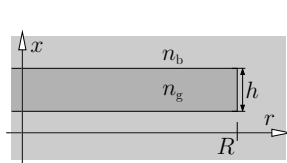
WGMs supported by a circular slab disc



WGMs supported by a circular slab disc



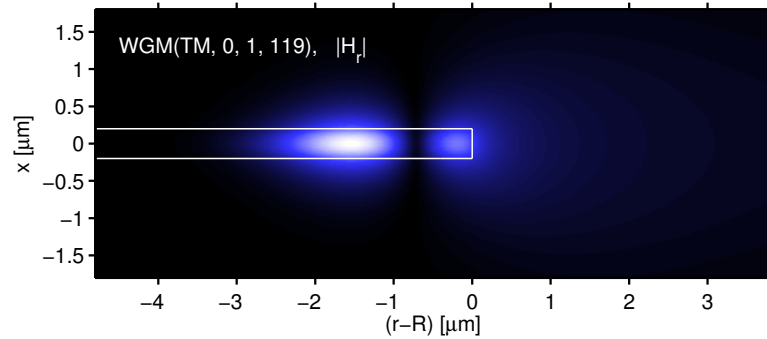
WGMs supported by a circular slab disc



$\lambda = 1.55 \mu\text{m}$,
 $n_b = 1.45$,
 $n_g = 1.99$,
 $h = 0.4 \mu\text{m}$,
 $R = 20 \mu\text{m}$;

WGM(TM, 0, 1, 119)
 $\lambda_r = 1.550 \mu\text{m}$
 $Q = 3.0 \cdot 10^2$
 [JCMwave].

$x \in [-3, 3] \mu\text{m}$,
 $(r - R) \in [-8, 4] \mu\text{m}$;



Bend modes versus whispering gallery resonances

(Field supported by a full circular cavity.)
 (Incompatible models, in principle.)

[BWG] $\omega \in \mathbb{R}$ given, $\gamma = \beta - i\alpha \in \mathbb{C}$ eigenvalue,

$$\Phi(r, \theta, t) = \phi(r) e^{i\omega t - i\beta R\theta} e^{-\alpha R\theta}.$$

[WGM] $\omega_c = \omega_c + i\omega_c'' \in \mathbb{C}$ eigenvalue, $m \in \mathbb{Z}$ given,

$$\Psi(r, \theta, t) = \psi(r) e^{i\omega_c' t - im\theta} e^{-\omega_c'' t}.$$

Look at a resonant low-loss configuration:

- Translate $\omega \approx \omega_c'$, $m \approx \beta R$.
- Equate the power loss during one time period $T = 2\pi/\omega \approx 2\pi/\omega_c'$
 $\rightsquigarrow \beta/\alpha \approx \omega_c'/\omega_c'' = 2Q.$

Upcoming

Next lectures:

- Coupled mode theory, perturbation theory.
- Hybrid analytical / numerical coupled mode theory.
- A touch of photonic crystals; a touch of plasmonics.

