Optical Waveguide Theory

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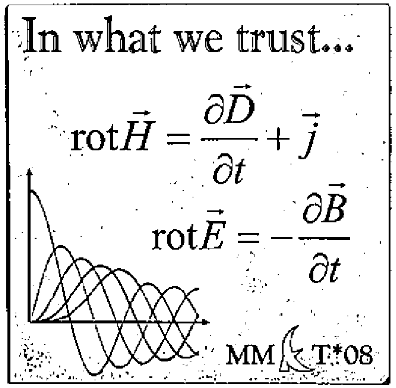
Motto



MMET'08, Mathematical Methods in Electromagnetic Theory
Odesa, Ukraine, June 29 – July 2, 2008

Maxwell equations

SI, in matter, time domain, differential form:

$$\begin{aligned} \nabla\cdot\boldsymbol{D} &= \rho_\mathrm{f}, \\ \nabla\times\boldsymbol{E} &= -\dot{\boldsymbol{B}}, \\ \nabla\cdot\boldsymbol{B} &= 0, \\ \nabla\times\boldsymbol{H} &= \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}, \\ \\ \boldsymbol{D} &= \epsilon_0\boldsymbol{E} + \boldsymbol{P}, \\ \boldsymbol{B} &= \mu_0(\boldsymbol{H} + \boldsymbol{M}). \end{aligned}$$

(+ constitutive relations)

- $\boldsymbol{E}(\boldsymbol{r},t)$: electric field,
- $\boldsymbol{D}(\boldsymbol{r},t)$: (di-)electric displacement,
- $\boldsymbol{B}(\boldsymbol{r},t)$: magnetic induction (field, flux density),
- $\boldsymbol{H}(\boldsymbol{r},t)$: magnetic field (...),
- $\rho_\mathrm{f}(\boldsymbol{r},t)$: density of free charges,
- $\boldsymbol{J}_\mathrm{f}(\boldsymbol{r},t)$: density of free currents,
- $\boldsymbol{P}(\boldsymbol{r},t)$: polarization,
- $\boldsymbol{M}(\boldsymbol{r},t)$: magnetization,
- ϵ_0: free space permittivity,
- μ_0: free space permeability.

Valid for more than a century, firm basis for further considerations.

Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

Formalities

Organization of the course:

- Lectures ($\approx 14\times$)
- Homework ($7\times$)
- Tutorials, Exercises ($13\times$)
- Exam

Related textbooks (examples):

C. Vassallo, *Optical Waveguide Concepts*, Elsevier, Amsterdam (1991),
K. Okamoto, *Fundamentals of Optical Waveguides*, Academic Press, San Diego, USA (2000),
R. März, *Integrated Optics: Design and Modeling*, Artech House, Norwood, USA (1995),
A.W. Snyder, J.D. Love, *Optical Waveguide Theory*, Chapman and Hall, London, UK (1983);

& general introductory texts on classical electrodynamics.

Optical waveguides: phenomena, examples

- Beam propagation in free space
- Guided light propagation
- Waveguide end facet
- Crossing of two waveguides
- Modes of 1-D multilayer slab waveguides
- Modes of 2-D channel waveguides
- Circular step-index optical fibers
- Evanescent coupling between waveguides
- Bent waveguides
- Circular microring-resonator
- Microdisk resonator
- CROW
- Waveguide corner
- Photonic crystal waveguide
- Exciting TET !

Optical waveguide "theory"

Task: solve

$$\begin{array}{llll} \nabla\times\boldsymbol{E} = -\dot{\boldsymbol{B}}, & \nabla\cdot\boldsymbol{D} = \rho_{\mathrm{f}}, & \boldsymbol{D} = \epsilon_0\boldsymbol{E}+\boldsymbol{P}, & \\ \nabla\times\boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}}+\dot{\boldsymbol{D}}, & \nabla\cdot\boldsymbol{B} = 0, & \boldsymbol{B} = \mu_0(\boldsymbol{H}+\boldsymbol{M}), & (\& \ \ldots). \end{array}$$

In this course:

- specialization to problems relevant for integrated optics,
- theoretical basis for the — mostly — numerical solution,
- approximate concepts,
- examples.

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Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z),\; t.$
- Scalar and vector fields: $\phi(\boldsymbol{r}, t),\; \boldsymbol{A}(\boldsymbol{r}, t), \qquad \boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$
- Inner product: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z.$
- Vector product: $\boldsymbol{A} \times \boldsymbol{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}.$
- Time derivatives: $\dfrac{\partial \phi}{\partial t},\; \partial_t \phi,\; \dot{\phi},\; \cancel{\nabla_t \phi}.$

Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$
- Gradient: $\mathrm{grad}\phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix}.$
- Divergence: $\mathrm{div}\boldsymbol{A} = \boldsymbol{\nabla} \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z.$
- Curl: $\mathrm{curl}\boldsymbol{A} = \mathrm{rot}\boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}.$
- Laplacian: $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2,$

  $$\Delta\phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_x^2 \phi, \qquad \Delta\boldsymbol{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$$

Dirac delta

A linear functional
that extracts the value of a function at one point:

1-D: $$\int_a^b f(x)\,\delta(x - x_0)\,\mathrm{d}x = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x - x_0) = 0, \text{ if } x \neq x_0.$$

3-D: $$\int_{\mathcal{V}} f(\boldsymbol{r})\,\delta(\boldsymbol{r} - \boldsymbol{r}_0)\,\mathrm{d}\mathcal{V} = \begin{cases} f(\boldsymbol{r}_0), & \text{if } \boldsymbol{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = 0, \text{ if } \boldsymbol{r} \neq \boldsymbol{r}_0.$$

Implications: manifold.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2.$
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k).$
- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k).$
- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k).$
- $\delta(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k.$

Fourier transform

3-D: A field $\phi(\boldsymbol{r})$:

$$\phi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\boldsymbol{k})\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k, \qquad \tilde{\phi}(\boldsymbol{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r.$$

4-D: A field $\phi(\boldsymbol{r}, t)$:

$$\phi(\boldsymbol{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iint \tilde{\phi}(\boldsymbol{k}, \omega)\ \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 k\, \mathrm{d}\omega,$$

$$\tilde{\phi}(\boldsymbol{k}, \omega) = \frac{1}{\sqrt{2\pi}^4} \iint \phi(\boldsymbol{r}, t)\ \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 r\, \mathrm{d}t.$$

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x, y) = 0,$$

coefficients $A(x, y), \ldots, F(x, y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,$,

- write ψ as $\qquad \psi(x, y) = \int \tilde{\psi}(k, y)\ \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k\,.$

$$\int \left(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2 A)\right) \tilde{\psi}(k, y)\ \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k = 0,$$

$$\left(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2 A)\right) \tilde{\psi}(k, y) = 0, \ \text{(for all } k\text{)},$$

... a set of DEs in one unknown.

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x, y) = 0,$$

coefficients $A(x, y), \ldots, F(x, y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,$,

- use an ansatz $\qquad \psi(x, y) = \tilde{\psi}(y)\ \mathrm{e}^{\mathrm{i}kx}\,.$

$$\left(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2 A)\right) \tilde{\psi}(y) = 0,$$

... a DE in one unknown, with parameter k.

(& boundary conditions, ...)

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right) \psi(\boldsymbol{r}, t) = 0, \quad \psi(\boldsymbol{r}, 0) = \psi_0(\boldsymbol{r}), \quad \partial_t \psi(\boldsymbol{r}, 0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r}, t) = \frac{1}{(2\pi)^2} \iint \tilde{\psi}(\boldsymbol{k}, \omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3 k,$$

$$\left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}^2}\right) \tilde{\psi}(\boldsymbol{k}, \omega) = 0,$$

$$\tilde{\psi}(\boldsymbol{k}, \omega) = a_{\mathrm{f}}(\boldsymbol{k})\,\delta(\omega - \omega_k) + a_{\mathrm{b}}(\boldsymbol{k})\,\delta(\omega + \omega_k), \qquad \omega_k = \mathrm{c}\,|\boldsymbol{k}|,$$

$$\psi(\boldsymbol{r}, t) = \frac{1}{(2\pi)^2} \int \left(a_{\mathrm{f}}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} + a_{\mathrm{b}}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} + \omega_k t)}\right) \mathrm{d}^3 k,$$

- $\psi(\boldsymbol{r}, 0) = \psi_0(\boldsymbol{r}),\ \partial_t \psi(\boldsymbol{r}, 0) = \phi_0(\boldsymbol{r}) \rightsquigarrow \ldots \rightsquigarrow a_{\mathrm{f}}(\boldsymbol{k}), a_{\mathrm{b}}(\boldsymbol{k}).$

A touch of variational calculus

- Functional: $\begin{array}{rcl} \mathcal{L}: U & \longrightarrow & \mathbb{R}, \mathbb{C}, \\ u & \longrightarrow & \mathcal{L}(u), \end{array}$

  a map from a space U of functions to real / complex numbers.

- Stationary functional: $\left.\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(u+s\,v)\right|_{s=0} = 0$ for all v,

  the variation of $\mathcal{L}$ at u vanishes for arbitrary directions v.

- Restriction of a functional:

  . . . to a parametrized family of functions;

  $\leftrightsquigarrow$ extremization with respect to these parameters,
  $\leftrightsquigarrow$ approximations of stationary points of the functional.

A touch of variational calculus

Example:

$U = \{u : [0,\pi] \to \mathbb{R} \mid u(0) = u(\pi) = 0\}$,

$\mathcal{L} : U \to \mathbb{R}$,

$$\mathcal{L}(u) = \frac{\int_0^\pi (\partial_x u)^2 \mathrm{d}x}{\int_0^\pi u^2 \mathrm{d}x}.$$

(. . .)

$\mathcal{L}$ stationary at u, $\left.\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(u+s\,v)\right|_{s=0} = 0 \quad \forall\, v.$ $\leftrightsquigarrow$ u satisfies DE & b.c., $\partial_x^2 u = -\lambda u, \quad \lambda = \mathcal{L}(u)$, $u(0) = u(\pi) = 0$.

$\downarrow$ Restrict $\mathcal{L}$, $L(\boldsymbol{a}) = \mathcal{L}(u|_{\boldsymbol{a}})$.

L stationary at $\boldsymbol{a}$, $\boldsymbol{\nabla}_{\boldsymbol{a}} L = 0.$ $\leftrightsquigarrow$ Approximate solution of DE / eigenproblem.

Course overview

Optical waveguide theory

. . . ?

"This concerns time harmonic fields . . . with angular frequency . . . , for vacuum wavenumber . . . , speed of light . . . , and wavelength . . . ."

"The problem is governed by the Maxwell curl equations in the frequency domain for the electric field . . . and magnetic field . . . , for (lossless) uncharged dielectric, nonmagnetic linear (isotropic) media with (piecewise constant) relative permittivity . . . :

. . . (.) "

[M. Hammer, A. Hildebrandt, J. Förstner, *Journal of Lightwave Technology* **34**(3), 997 (2016)]

Maxwell equations, Fourier transform

$$\nabla\cdot\boldsymbol{D}=\rho_{\mathrm f},\quad \nabla\times\boldsymbol{E}=-\dot{\boldsymbol{B}},\quad \nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{H}=\boldsymbol{J}_{\mathrm f}+\dot{\boldsymbol{D}}$$

$$\&\quad \boldsymbol{F}(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,\quad \tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\frac{1}{\sqrt{2\pi}}\int\boldsymbol{F}(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

$$\hookrightarrow\quad \boldsymbol{E}(\boldsymbol{r},t),\ \boldsymbol{D}(\boldsymbol{r},t),\ \boldsymbol{B}(\boldsymbol{r},t),\ \boldsymbol{H}(\boldsymbol{r},t),\ \rho_{\mathrm f}(\boldsymbol{r},t),\ \boldsymbol{J}_{\mathrm f}(\boldsymbol{r},t)$$
$$\leftrightsquigarrow\quad \tilde{\boldsymbol{E}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{D}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{H}}(\boldsymbol{r},\omega),\ \tilde\rho_{\mathrm f}(\boldsymbol{r},\omega),\ \tilde{\boldsymbol{J}}_{\mathrm f}(\boldsymbol{r},\omega),$$

$$\nabla\cdot\tilde{\boldsymbol{D}}=\tilde\rho_{\mathrm f},\quad \nabla\times\tilde{\boldsymbol{E}}=-\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \nabla\cdot\tilde{\boldsymbol{B}}=0,\quad \nabla\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm f}+\mathrm{i}\omega\tilde{\boldsymbol{D}}$$

(Caution: arbitrary choice of $\sim\exp(\pm\mathrm{i}\,\omega t)$!).

Maxwell equations, frequency domain

$$\nabla\cdot\tilde{\boldsymbol{D}}=\tilde\rho_{\mathrm f},\quad \nabla\times\tilde{\boldsymbol{E}}=-\mathrm{i}\omega\tilde{\boldsymbol{B}},\quad \nabla\cdot\tilde{\boldsymbol{B}}=0,\quad \nabla\times\tilde{\boldsymbol{H}}=\tilde{\boldsymbol{J}}_{\mathrm f}+\mathrm{i}\omega\tilde{\boldsymbol{D}}.$$

$$\boldsymbol{F}(\boldsymbol{r},t)\in\mathbb{R}\ \rightsquigarrow\ \tilde{\boldsymbol{F}}(\boldsymbol{r},-\omega)=(\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega))^*$$

"at frequency ω_0": $\tilde{\boldsymbol{F}}(\boldsymbol{r},\omega)=\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\delta(\omega-\omega_0)+\sqrt{\frac{\pi}{2}}\,\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\delta(\omega+\omega_0)$

$$\hookrightarrow\quad \boldsymbol{F}(\boldsymbol{r},t)=\frac12\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}+\bar{\boldsymbol{F}}^*(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\right\},$$
$$\boldsymbol{F}(\boldsymbol{r},t)=\mathrm{Re}\left\{\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}\right\},$$
$$\text{“}\boldsymbol{F}(\boldsymbol{r},t)=\frac12\,\bar{\boldsymbol{F}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega_0 t}+\text{c.c.”}.$$

$$\hookrightarrow\quad \bar{\boldsymbol{E}}(\boldsymbol{r}),\ \bar{\boldsymbol{D}}(\boldsymbol{r}),\ \bar{\boldsymbol{B}}(\boldsymbol{r}),\ \bar{\boldsymbol{H}}(\boldsymbol{r}),\ \bar\rho_{\mathrm f}(\boldsymbol{r}),\ \bar{\boldsymbol{J}}_{\mathrm f}(\boldsymbol{r}),\quad \sim\exp(\mathrm{i}\omega_0 t),$$
$$\nabla\cdot\bar{\boldsymbol{D}}=\bar\rho_{\mathrm f},\quad \nabla\times\bar{\boldsymbol{E}}=-\mathrm{i}\omega_0\bar{\boldsymbol{B}},\quad \nabla\cdot\bar{\boldsymbol{B}}=0,\quad \nabla\times\bar{\boldsymbol{H}}=\bar{\boldsymbol{J}}_{\mathrm f}+\mathrm{i}\omega_0\bar{\boldsymbol{D}}.$$

Caution: Decorations $\tilde{}$, $\bar{}$, $_0$ are ususally omitted; context determines interpretation of symbols.

Polarization

$\tilde{\boldsymbol{P}}$: density of electric dipole moment (bound charges).

$$\tilde{\boldsymbol{D}}=\epsilon_0\tilde{\boldsymbol{E}}+\tilde{\boldsymbol{P}},\qquad [\tilde{\boldsymbol{D}}]=[\tilde{\boldsymbol{P}}]=\frac{\mathrm{As\,m}}{\mathrm{m}^3},\ [\tilde{\boldsymbol{E}}]=\frac{\mathrm{V}}{\mathrm{m}},$$

vacuum permittivity $\epsilon_0=8.854187817\ldots\cdot10^{-12}\left[\frac{\mathrm F}{\mathrm m}=\frac{\mathrm{As}}{\mathrm{Vm}}\right]$.

- Local dipoles induced by $\tilde{\boldsymbol{E}}$ $\rightsquigarrow$ $\tilde{\boldsymbol{P}}(\tilde{\boldsymbol{E}})$.
- Linear dielectrica:

$$\tilde{\boldsymbol{P}}=\epsilon_0\hat\chi_{\mathrm e}\tilde{\boldsymbol{E}},\qquad \hat\chi_{\mathrm e}\text{: dielectric susceptibility, } [\hat\chi_{\mathrm e}]=\hat1.$$
$$\hookrightarrow\quad \tilde{\boldsymbol{D}}=\epsilon_0(\hat1+\hat\chi_{\mathrm e})\tilde{\boldsymbol{E}}=\epsilon_0\hat\epsilon\tilde{\boldsymbol{E}},\qquad \hat\epsilon\text{: relative permittivity, } [\hat\epsilon]=\hat1.$$

- $\hat\chi_{\mathrm e}(\boldsymbol{r},\omega)$, $\hat\epsilon(\boldsymbol{r},\omega)$ are determined in the frequency domain.
- Complications: $\mathrm{Im}\,\epsilon$, $\hat\epsilon(T)$, $\hat\epsilon(\boldsymbol{F})$, $\chi^{(2)}_{jkl}E_kE_l$, $\chi^{(3)}_{jklm}E_kE_lE_m$, …
- Simpler cases: $\hat\epsilon(\cancel{\boldsymbol{r}})$, $\hat\epsilon=\epsilon\hat1$.

Magnetization

$\tilde{\boldsymbol{M}}$: density of magnetic dipole moments (bound currents).

$$\tilde{\boldsymbol{H}}=\frac{1}{\mu_0}\tilde{\boldsymbol{B}}-\tilde{\boldsymbol{M}},\qquad [\tilde{\boldsymbol{H}}]=[\boldsymbol{M}]=\frac{\mathrm{A\,m^2}}{\mathrm{m}^3},\ [\tilde{\boldsymbol{B}}]=\mathrm{T}=\frac{\mathrm{Vs}}{\mathrm{m}^2},$$

vacuum permeability $\mu_0=4\pi\cdot10^{-7}\left[\frac{\mathrm N}{\mathrm A^2}=\frac{\mathrm{Vs}}{\mathrm{Am}}\right]$.

- Local dipoles induced by $\tilde{\boldsymbol{H}}$ $\rightsquigarrow$ $\tilde{\boldsymbol{M}}(\tilde{\boldsymbol{H}})$.
- Linear magnetic media:

$$\tilde{\boldsymbol{M}}=\hat\chi_{\mathrm m}\tilde{\boldsymbol{H}},\qquad \hat\chi_{\mathrm m}\text{: magnetic susceptibility, } [\hat\chi_{\mathrm m}]=\hat1.$$
$$\hookrightarrow\quad \tilde{\boldsymbol{B}}=\mu_0(\hat1+\hat\chi_{\mathrm m})\tilde{\boldsymbol{H}}=\mu_0\hat\mu\tilde{\boldsymbol{H}},\quad \hat\mu\text{: relative permeability, } [\hat\mu]=\hat1.$$

- $\hat\chi_{\mathrm m}(\boldsymbol{r},\omega)$, $\hat\mu(\boldsymbol{r},\omega)$ are determined in the frequency domain.
- Complications: manifold.
- Traditional integrated optics (frequencies, media): $\hat\mu(\cancel{\boldsymbol{r}})=\hat1$.

Maxwell equations, dispersion

(Material) dispersion: $\hat{\epsilon}(\boldsymbol{r},\omega)$, $\hat{\mu}(\boldsymbol{r},\omega)$ are frequency dependent.

$$\tilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \epsilon_0\hat{\epsilon}(\boldsymbol{r},\omega)\tilde{\boldsymbol{E}}(\boldsymbol{r},\omega), \quad \tilde{\boldsymbol{B}}(\boldsymbol{r},\omega) = \mu_0\hat{\mu}(\boldsymbol{r},\omega)\tilde{\boldsymbol{H}}(\boldsymbol{r},\omega)$$

$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \int \hat{\epsilon}_{\mathrm{TD}}(\boldsymbol{r},t-t')\boldsymbol{E}(\boldsymbol{r},t')\,\mathrm{d}t',$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \int \hat{\mu}_{\mathrm{TD}}(\boldsymbol{r},t-t')\boldsymbol{H}(\boldsymbol{r},t')\,\mathrm{d}t'.$$

Helmholtz equations

Linear dielectric media without free charges or currents,
time dependence $\sim \exp(\mathrm{i}\omega t)$, fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{D}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$,
material properties $\hat{\epsilon}(\boldsymbol{r})$, $\hat{\mu}(\boldsymbol{r})$:

$$\nabla\cdot\boldsymbol{D} = 0, \quad \nabla\times\boldsymbol{E} = -\mathrm{i}\omega\boldsymbol{B}, \quad \nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\boldsymbol{H} = \mathrm{i}\omega\boldsymbol{D},$$
$$\boldsymbol{D} = \epsilon_0\hat{\epsilon}\boldsymbol{E}, \quad \boldsymbol{B} = \mu_0\hat{\mu}\boldsymbol{H}.$$

$$\nabla\times\boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H}, \quad \nabla\times\boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}, \qquad \nabla\cdot\hat{\epsilon}\boldsymbol{E} = 0, \quad \nabla\cdot\hat{\mu}\boldsymbol{H} = 0.$$

$$\nabla\times(\hat{\mu}^{-1}\nabla\times\boldsymbol{E}) = \omega^2\epsilon_0\mu_0\hat{\epsilon}\boldsymbol{E} \quad \text{or} \quad \nabla\times(\hat{\epsilon}^{-1}\nabla\times\boldsymbol{H}) = \omega^2\epsilon_0\mu_0\hat{\mu}\boldsymbol{H}.$$

Where $\hat{\epsilon} = \epsilon\hat{1}$, $\nabla\epsilon = 0$, $\hat{\mu} = \mu\hat{1}$, $\nabla\mu = 0$: (!)

$$\Delta\boldsymbol{E} + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\boldsymbol{E} = 0 \quad \text{or} \quad \Delta\boldsymbol{H} + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\boldsymbol{H} = 0, \qquad \mathrm{c} = \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$

Plane harmonic waves

Where $\hat{\epsilon} = \epsilon\hat{1}$, $\nabla\epsilon = 0$, $\hat{\mu} = \mu\hat{1}$, $\nabla\mu = 0$: (!)
Components of $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy $\Delta\psi + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\,\psi = 0.$

$$\psi(\boldsymbol{r},t) = \psi_0\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_\mathrm{m}\cdot\boldsymbol{r}-\omega t)}, \qquad -\boldsymbol{k}_\mathrm{m}^2 + \frac{\omega^2}{\mathrm{c}^2}\epsilon\mu = 0.$$

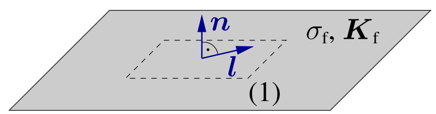
(Mixture of TD and FD expressions; ˜, ¯, Re , 1/2, c.c. omitted; sloppy, but common.)

- Medium: refractive index: $n = \sqrt{\epsilon\mu}$
- Periodicity in time: angular frequency: ω,
  frequency: $f = \omega/(2\pi)$,
  period: $T = 1/f = 2\pi/\omega$,
- Spatial periodicity: wave vector: $\boldsymbol{k}_\mathrm{m}$, $k_\mathrm{m} = |\boldsymbol{k}_\mathrm{m}|$,
  wavenumber: $k_\mathrm{m} = \omega/\mathrm{c}_\mathrm{m} = (\omega/\mathrm{c})n = k\,n$,
  vacuum wavenumber: $k = \omega/\mathrm{c}$,
  vacuum wavelength: $\lambda = 2\pi/k = 2\pi\mathrm{c}/\omega$,
  wavelength in the medium: $\lambda_\mathrm{m} = 2\pi/k_\mathrm{m} = 2\pi/(kn) = \lambda/n$.
- Phase velocity: speed of light in vacuum: $\mathrm{c} = 1/\sqrt{\epsilon_0\mu_0} = \lambda f$,
  in the medium: $\mathrm{c}_\mathrm{m} = \mathrm{c}/n = \lambda_\mathrm{m} f$.

(Use of symbols depends highly on context.)

Electromagnetic spectrum

Interface conditions

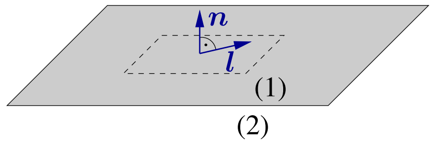


(2)

Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$,
surface charge density σ_f, surface current density $\boldsymbol{K}_\mathrm{f}$:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2) = \sigma_\mathrm{f}, \quad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2) = 0,$$
$$\boldsymbol{n}\cdot(\boldsymbol{B}_1-\boldsymbol{B}_2) = 0, \quad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2) = \boldsymbol{l}\cdot(\boldsymbol{K}_\mathrm{f}\times\boldsymbol{n}).$$

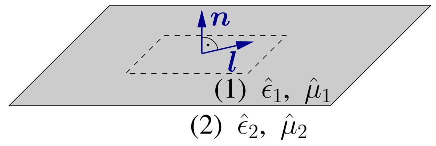
Interface conditions



Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, surface without free charges or currents:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2)=0, \quad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$
$$\boldsymbol{n}\cdot(\boldsymbol{B}_1-\boldsymbol{B}_2)=0, \quad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=0\ .$$

Interface conditions

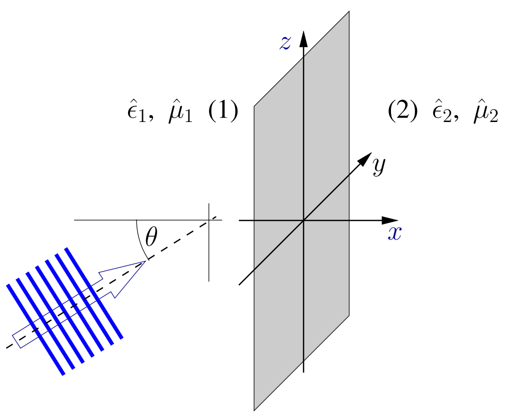


Surface between media (1) and (2), surface normal $\boldsymbol{n}$, tangents $\boldsymbol{l}$, linear media with permittivities $\hat{\epsilon}_1$, $\hat{\epsilon}_2$, and permeabilities $\hat{\mu}_1$, $\hat{\mu}_2$:

$$\boldsymbol{n}\cdot(\hat{\epsilon}_1\boldsymbol{E}_1-\hat{\epsilon}_2\boldsymbol{E}_2)=0, \quad \boldsymbol{l}\cdot(\boldsymbol{E}_1-\boldsymbol{E}_2)=0,$$
$$\boldsymbol{n}\cdot(\hat{\mu}_1\boldsymbol{H}_1-\hat{\mu}_2\boldsymbol{H}_2)=0, \quad \boldsymbol{l}\cdot(\boldsymbol{H}_1-\boldsymbol{H}_2)=0.$$

Reflection and transmission of plane waves at dielectric interfaces

(FD)



- $\nabla\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}$, $\nabla\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}$
- $\hat{\epsilon}(\boldsymbol{r})$ and $\hat{\mu}(\boldsymbol{r})$ are constant along y, z

$$\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{E}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)},\quad \boldsymbol{H}(\boldsymbol{r})=\boldsymbol{H}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)}$$

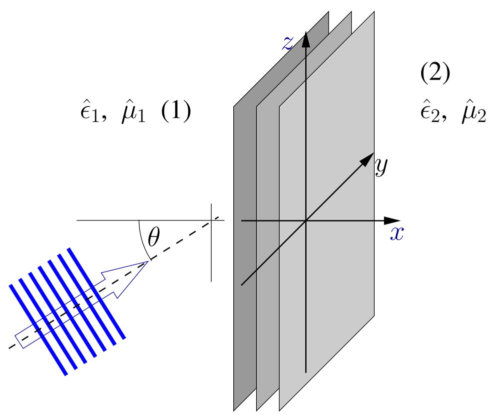
1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

(incoming plane wave at angle θ)
(orient coordinates ($k_y = 0$), plane of incidence, distinguish polarizations)
(write ansatz functions for incoming, reflected, and transmitted waves)
(interface conditions determine the amplitudes)

Fresnel equations.

Dielectric multilayer structures

(FD)



- $\nabla\times\boldsymbol{E}=-\mathrm{i}\omega\mu_0\hat{\mu}\,\boldsymbol{H}$, $\nabla\times\boldsymbol{H}=\mathrm{i}\omega\epsilon_0\hat{\epsilon}\,\boldsymbol{E}$
- $\hat{\epsilon}(\boldsymbol{r})$ and $\hat{\mu}(\boldsymbol{r})$ are constant along y, z

$$\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{E}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)},\quad \boldsymbol{H}(\boldsymbol{r})=\boldsymbol{H}'(x)\,\mathrm{e}^{-\mathrm{i}(k_y y+k_z z)}$$

1-D problem for $\boldsymbol{E}'$, $\boldsymbol{H}'$.

(. . .)
(. . .)
(. . .)
(. . .)

Reflectance and transmittance properties.

Energy of electromagnetic fields

(TD)

- Force on a particle with charge q, velocity $\boldsymbol{v}$, in a field $\boldsymbol{E}, \boldsymbol{B}$:
  $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,
- work for shifting the particle by $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$:
  $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\,\mathrm{d}t = q\boldsymbol{E} \cdot \boldsymbol{v}\,\mathrm{d}t$,
- respective power: $\dfrac{\mathrm{d}W}{\mathrm{d}t} = q\boldsymbol{E} \cdot \boldsymbol{v}$.

For a charge density $\rho_\mathrm{f}(\boldsymbol{r}, t)$:

force density $\boldsymbol{f} = \rho_\mathrm{f}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$,

$\hookrightarrow$ power density $\boldsymbol{f} \cdot \boldsymbol{v} = \rho_\mathrm{f}\boldsymbol{E} \cdot \boldsymbol{v} = \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}$,

total work per time unit done in $\mathcal{V}$: $\dfrac{\mathrm{d}W_\mathcal{V}}{\mathrm{d}t} = \displaystyle\int_\mathcal{V} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}\,\mathrm{d}\mathcal{V}$.

Power & energy density, Poynting theorem

(TD)

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dot{\boldsymbol{D}}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$$

$$\hookrightarrow \frac{\mathrm{d}}{\mathrm{d}t} W_\mathcal{V}^\mathrm{mech} = \int_\mathcal{V} \boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}\,\mathrm{d}\mathcal{V} = -\int_\mathcal{V} (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})\,\mathrm{d}\mathcal{V} - \int_\mathcal{V} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H})\,\mathrm{d}\mathcal{V},$$

- Poynting vector: $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, (energy flux density, power density)
- energy density: $w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})$, $\quad W_\mathcal{V}^\mathrm{field} = \displaystyle\int_\mathcal{V} w\,\mathrm{d}\mathcal{V}$,
- $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\epsilon}\,\cancel{(\omega)}$, $\boldsymbol{D} = \epsilon_0\hat{\epsilon}\boldsymbol{E}$, $\hat{\mu}^\dagger = \hat{\mu}$, $\hat{\mu}\,\cancel{(\omega)}$, $\boldsymbol{B} = \mu_0\hat{\mu}\boldsymbol{H}$ (!)
  $\leadsto \quad \dot{w} = (\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}})$

$\mathcal{V}$ arbitrary

$$\hookrightarrow \quad \dot{w} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J}_\mathrm{f} \cdot \boldsymbol{E}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(W_\mathcal{V}^\mathrm{mech} + W_\mathcal{V}^\mathrm{field}\right) = -\oint_{\partial\mathcal{V}} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a}.$$

Electromagnetic energy, frequency domain

Lossless uncharged nondispersive (...) linear media:

$$w = \frac{1}{2}(\epsilon_0\,\boldsymbol{E} \cdot \hat{\epsilon}\boldsymbol{E} + \mu_0\,\boldsymbol{H} \cdot \hat{\mu}\boldsymbol{H}), \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \quad \dot{w} + \nabla \cdot \boldsymbol{S} = 0,$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\ \tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \boldsymbol{H}(\boldsymbol{r}, t) = \mathrm{Re}\ \tilde{\boldsymbol{H}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}$$

$\hookrightarrow$ $\boldsymbol{S}$, w oscillate in time.

(FD)

Consider time-averaged quantities: $\quad \bar{f}(t) = \dfrac{1}{T}\displaystyle\int_t^{t+T} f(t')\,\mathrm{d}t'$

(exercise)

$$\hookrightarrow \quad \overline{w} = \frac{1}{4}\,\mathrm{Re}\,\left(\epsilon_0\,\tilde{\boldsymbol{E}}^* \cdot \hat{\epsilon}\,\tilde{\boldsymbol{E}} + \mu_0\,\tilde{\boldsymbol{H}}^* \cdot \hat{\mu}\,\tilde{\boldsymbol{H}}\right), \quad \overline{\boldsymbol{S}} = \frac{1}{2}\,\mathrm{Re}\,\left(\tilde{\boldsymbol{E}}^* \times \tilde{\boldsymbol{H}}\right).$$

$$\overline{\dot{w}} = \dot{\overline{w}} = 0, \quad \overline{\nabla \cdot \boldsymbol{S}} = \nabla \cdot \overline{\boldsymbol{S}} \quad \leadsto \quad \nabla \cdot \overline{\boldsymbol{S}} = 0, \quad \oint_\mathcal{V} \overline{\boldsymbol{S}} \cdot \mathrm{d}\boldsymbol{a} = 0;$$

"power balance", conservation of energy.

Wave propagation in attenuating media

Specifically: homogeneous isotropic conductors, linear media.

Electric field drives the free currents:
Ohm's law $\quad \boldsymbol{J}_\mathrm{f} = \sigma\boldsymbol{E}$, $\quad \sigma$: conductivity of the material.

$$\nabla \cdot \boldsymbol{D} = \rho_\mathrm{f}, \quad \nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \sigma\boldsymbol{E} + \dot{\boldsymbol{D}}.$$

$$\hookrightarrow \quad \dot{\rho}_\mathrm{f} = -\frac{\sigma}{\epsilon_0\epsilon}\rho_\mathrm{f}, \quad \rho_\mathrm{f}(\boldsymbol{r}, t) = \rho_\mathrm{f}(\boldsymbol{r}, t_0)\exp\left(-\frac{\sigma}{\epsilon_0\epsilon}(t - t_0)\right),$$

assume $\rho_\mathrm{f}(\boldsymbol{r}, t_0) = 0 \quad \leadsto \quad \rho_\mathrm{f}(\boldsymbol{r}, t) = 0 \quad \forall t$.

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \times \boldsymbol{E} = -\mu_0\mu\dot{\boldsymbol{H}}, \quad \nabla \cdot \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} = \sigma\boldsymbol{E} + \epsilon_0\epsilon\dot{\boldsymbol{E}}.$$

Telegrapher equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\mu_0\mu\dot{\boldsymbol{H}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{H} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \sigma\boldsymbol{E}+\epsilon_0\epsilon\dot{\boldsymbol{E}}$$

$$\Delta\boldsymbol{E} - \epsilon_0\mu_0\epsilon\mu\ddot{\boldsymbol{E}} - \mu_0\mu\sigma\dot{\boldsymbol{E}} = 0, \quad \Delta\boldsymbol{H} - \epsilon_0\mu_0\epsilon\mu\ddot{\boldsymbol{H}} - \mu_0\mu\sigma\dot{\boldsymbol{H}} = 0,$$

Telegrapher equation.

Frequency domain: $\boldsymbol{E}(\boldsymbol{r},t) = \tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}$,

$$\Delta\tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu - \mathrm{i}\omega\mu_0\mu\sigma\right)\tilde{\boldsymbol{E}} = 0.$$

Nonconducting media $\sigma = 0$, $\Delta\tilde{\boldsymbol{E}} + \left(\frac{\omega^2}{\mathrm{c}^2}\epsilon\mu\right)\tilde{\boldsymbol{E}} = 0.$

Define $\bar{\epsilon}$ such that $\dfrac{\omega^2}{\mathrm{c}^2}\bar{\epsilon}\mu = \dfrac{\omega^2}{\mathrm{c}^2}\epsilon\mu - \mathrm{i}\omega\mu_0\mu\sigma$, i.e. $\bar{\epsilon} = \epsilon - \mathrm{i}\dfrac{\sigma}{\epsilon_0\omega}$

$\Delta\tilde{\boldsymbol{E}} + k^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}} = 0$, Helmholtz equation, $\bar{\epsilon}\in\mathbb{C}$, $\quad k = \dfrac{\omega}{\mathrm{c}}$.

Wave attenuation

$$\Delta\tilde{\boldsymbol{E}} + k^2\bar{\epsilon}\mu\tilde{\boldsymbol{E}} = 0, \quad \bar{\epsilon}\in\mathbb{C}$$

solutions $\sim \mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)}$

with refractive index $\bar{n} = n' - \mathrm{i}n'' = \sqrt{\bar{\epsilon}\mu} \in \mathbb{C}$, (!)

$$\mathrm{e}^{-\mathrm{i}(k\bar{n}z - \omega t)} = \mathrm{e}^{-\mathrm{i}(kn'z - \omega t)}\,\mathrm{e}^{-kn''z},$$

damped plane wave solutions

$\leftrightsquigarrow$ sign of n'', choice of $\exp(\pm\mathrm{i}\omega)$.

Issues:

- penetration depth,
- $\boldsymbol{S}$ and w decay with z,
- still transverse waves,
- $\boldsymbol{E}$, $\boldsymbol{H}$ no longer in phase,
- notions of wavenumber, wavelength, phase velocity $\in\mathbb{C}$.

Simulations in integrated optics

A typical setting:

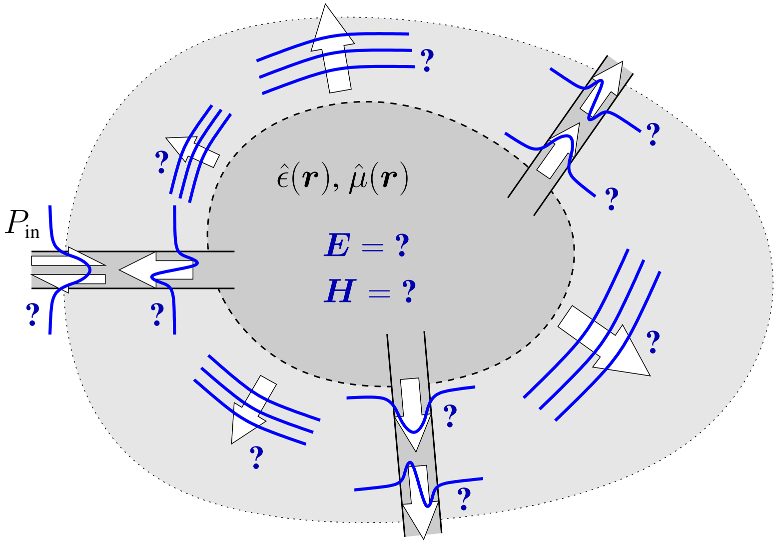
- "uncharged dielectric medium": $\cancel{q_\mathrm{f}}$, $\cancel{\boldsymbol{J}_\mathrm{f}}$.
- "linear medium": $\boldsymbol{D} = \epsilon_0\hat{\epsilon}\boldsymbol{E}$, $\boldsymbol{B} = \mu_0\hat{\mu}\boldsymbol{H}$.
- "isotropic medium": $\hat{\epsilon} = \epsilon\,\hat{1}$, $\hat{\mu} = \mu\,\hat{1}$.
- "nonmagnetic medium": $\hat{\mu} = \hat{1}$.
- "lossless medium": $\hat{\epsilon}^\dagger = \hat{\epsilon}$, $\hat{\mu}^\dagger = \hat{\mu}$, $(\epsilon, \mu \in \mathbb{R})$.
- "piecewise constant" $\rightarrow$ "dependent on position".
- "electric and magnetic field": eliminate $\boldsymbol{D}$ and $\boldsymbol{B}$, retain $\boldsymbol{E}$ and $\boldsymbol{H}$.
- "governed by the curl equations": divergence eqns. are satisfied.
- "frequency domain, time harmonic fields, frequency, wavelength": ... as discussed.

Course overview

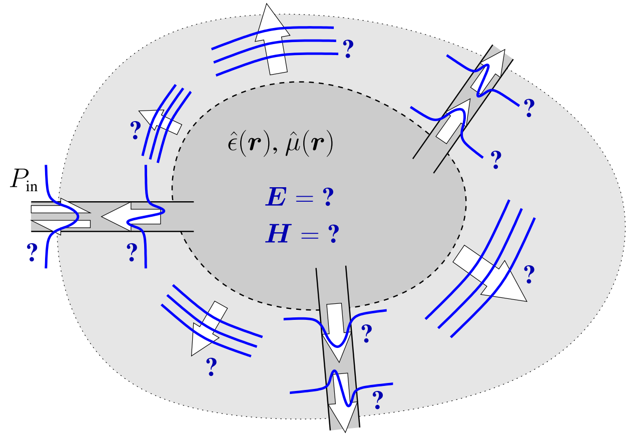
Optical waveguide theory

A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
B Brush up on mathematical tools.
C Maxwell equations, different formulations, interfaces, energy and power flow.
D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
E Normal modes of dielectric optical waveguides, mode interference.
F Examples for dielectric optical waveguides.
G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
H Bent optical waveguides; whispering gallery resonances; circular microresonators.
I Coupled mode theory, perturbation theory.
• Hybrid analytical / numerical coupled mode theory.
J A touch of photonic crystals; a touch of plasmonics.
• Oblique semi-guided waves: 2-D integrated optics.
• Summary, concluding remarks.

Guided wave scattering problems, schematically



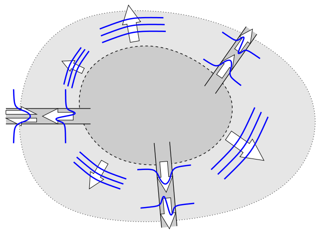
Guided wave scattering problems, schematically



Given $\hat{\epsilon}(\boldsymbol{r}), \hat{\mu}(\boldsymbol{r})$ & external excitation (incoming guided mode),
determine $\boldsymbol{E}, \boldsymbol{H}$ within the computational domain
& determine the optical power carried by outgoing waves.

Scattering problems, time domain

(TD)

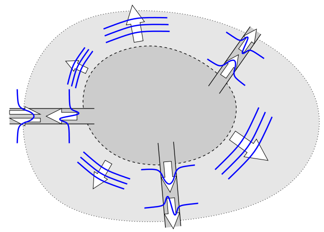


$\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{H}(\boldsymbol{r}, t)$,
$\nabla \times \boldsymbol{E} = -\mu_0 \hat{\mu} \dot{\boldsymbol{H}}$,
$\nabla \times \boldsymbol{H} = \epsilon_0 \hat{\epsilon} \dot{\boldsymbol{E}}$.

- $\begin{pmatrix} \text{3-D} \\ \text{2-D} \\ \text{1-D} \end{pmatrix}$ computational domain $\times$ time interval.
- Initial & boundary conditions $\longleftrightarrow$ incident waves.
- "Local" time-explicit iterative schemes possible (e.g. FDTD).
- Time evolution available; direct modeling of pulse propagation.
- Dispersion (. . . ?).
- Guided wave excitation (. . . ?).
- Fourier transform $\longrightarrow$ spectral information.

Scattering problems, frequency domain

(FD)

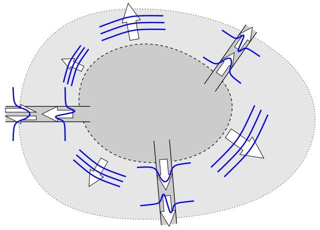


$\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$, $\sim \exp(\mathrm{i}\omega t)$,
$\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \hat{\mu} \boldsymbol{H}$,
$\nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0 \hat{\epsilon} \boldsymbol{E}$.

- $\begin{pmatrix} \text{3-D} \\ \text{2-D} \\ \text{1-D} \end{pmatrix}$ computational domain.
- "$\mathsf{M}(\overrightarrow{\text{field}}) = (\overrightarrow{\text{excitation}})$";
  matrix needs to be determined, stored; system needs to be solved.
- Spectral information directly available.
- Dispersion — straightforward.
- Guided wave excitation — straightforward.
- Fourier transform $\longrightarrow$ time evolution / pulse propagation.

Open problems

(TD & FD)



"Open" spatial computational domain
⇝ boundary conditions need to

- permit outgoing radiated fields
  & outgoing (reflected) guided modes to exit the domain,
- launch the incoming external excitation.

⇜⇝ simulate a nonexisting boundary, an unlimited domain.

Keywords:
- transparent-influx boundary conditions,
- absorbing boundary conditions,
- perfectly matched layers (PMLs).

2-D problems

$\hat{\epsilon} = \epsilon \hat{1}, \ \hat{\mu} = \mu \hat{1}, \ \sim \exp(\mathrm{i}\omega t)$ (FD)

Assume $\partial_y \epsilon = 0, \ \partial_y \mu = 0$; consider solutions $\partial_y \boldsymbol{E} = 0, \ \partial_y \boldsymbol{H} = 0$:

$$\begin{pmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{pmatrix} = -\mathrm{i}\omega\mu_0\mu \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \qquad \begin{pmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

↪ Two decoupled sets of equations:

- $\{E_y, H_x, H_z\}$: transverse electric (TE) fields, $\boldsymbol{E} \perp x$-z-plane.
- $\{H_y, E_x, E_z\}$: transverse magnetic (TM) fields, $\boldsymbol{H} \perp x$-z-plane.

(Different conventions on the use of TE, TM.)

(Applies also to the TD.)

2-D TE waves

$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0$ (FD)

- Principal component E_y,

  $$H_x = \frac{-\mathrm{i}}{\omega\mu_0\mu}\,\partial_z E_y, \quad H_z = \frac{\mathrm{i}}{\omega\mu_0\mu}\,\partial_x E_y, \quad \mathrm{i}\omega\epsilon_0\epsilon E_y = \partial_z H_x - \partial_x H_z$$

  ↪ $$\partial_x \frac{1}{\mu} \partial_x E_y + \partial_z \frac{1}{\mu} \partial_z E_y + k^2 \epsilon E_y = 0. \qquad (*)$$

- Continuity of $E_y, \ \frac{1}{\mu}\partial_n E_y$ required at interfaces with normal $\boldsymbol{n}$.

- If $\mu = 1$: $\epsilon(x,z)$ (!)

  ↪ $$\partial_x^2 E_y + \partial_z^2 E_y + k^2 \epsilon E_y = 0, \qquad (**)$$

  scalar 2-D (TE) Helmholtz equation ($E_y, \ \partial_n E_y$ continuous).

(Reflection / transmission problems: s-polarized waves satisfy $(*)$, $(**)$.)

2-D TM waves

$k^2 = \omega^2/c^2 = \omega^2\epsilon_0\mu_0$ (FD)

- Principal component H_y,

  $$E_x = \frac{\mathrm{i}}{\omega\epsilon_0\epsilon}\,\partial_z H_y, \quad E_z = \frac{-\mathrm{i}}{\omega\epsilon_0\epsilon}\,\partial_x H_y, \quad -\mathrm{i}\omega\mu_0\mu H_y = \partial_z E_x - \partial_x E_z$$

  ↪ $$\partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_z \frac{1}{\epsilon} \partial_z H_y + k^2 \mu H_y = 0. \qquad (*)$$

- Continuity of $H_y, \ \frac{1}{\epsilon}\partial_n H_y$ required at interfaces with normal $\boldsymbol{n}$.

- If $\mu = 1$: $\epsilon(x,z)$ (!)

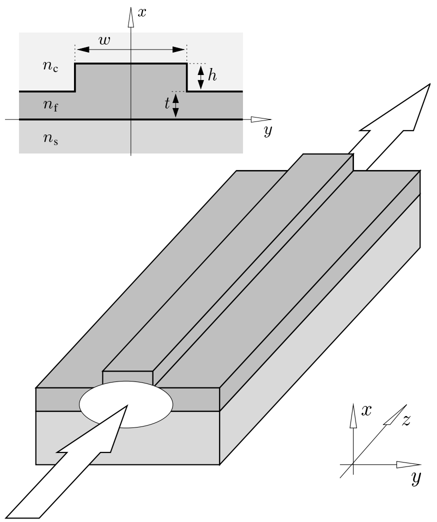
  ↪ $$\partial_x \frac{1}{\epsilon} \partial_x H_y + \partial_z \frac{1}{\epsilon} \partial_z H_y + k^2 H_y = 0, \qquad (**)$$

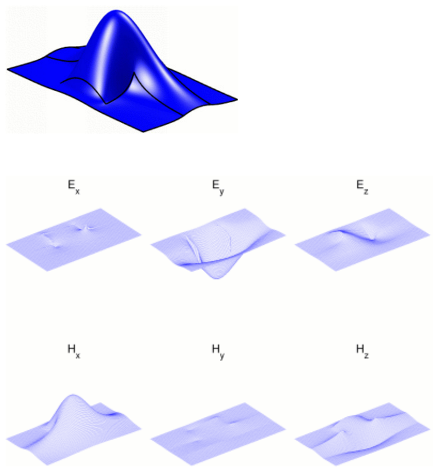
  scalar 2-D (TM) Helmholtz equation ($H_y, \ \frac{1}{\epsilon}\partial_n H_y$ continuous).

(Reflection / transmission problems: p-polarized waves satisfy $(*)$, $(**)$.)

Rib waveguide

. . . variant of an integrated optical waveguide with 2-D confinement





Waveguides: Mode problems

$\nabla\times\boldsymbol{E} = -\mathrm{i}\omega\mu_0\mu\boldsymbol{H},\quad \nabla\times\boldsymbol{H} = \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}.$ $\sim\exp(\mathrm{i}\omega t)$ (FD)

- Waveguide: a system that is homogeneous along its axis z,
  $\partial_z\epsilon = 0,\ \partial_z\mu = 0,\ \partial_z n = 0.$
- Look for solutions (modes) that vary harmonically with z:
  $\boldsymbol{E}(x,y,z) = \bar{\boldsymbol{E}}(x,y)\,\mathrm{e}^{-\mathrm{i}\beta z},\quad \boldsymbol{H}(x,y,z) = \bar{\boldsymbol{H}}(x,y)\,\mathrm{e}^{-\mathrm{i}\beta z},$
  mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$, propagation constant β. (drop $^-$)

$$\begin{pmatrix}\partial_y E_z + \mathrm{i}\beta E_y\\ -\mathrm{i}\beta E_x - \partial_x E_z\\ \partial_x E_y - \partial_y E_x\end{pmatrix} = -\mathrm{i}\omega\mu_0\mu\begin{pmatrix}H_x\\H_y\\H_z\end{pmatrix},\quad \begin{pmatrix}\partial_y H_z + \mathrm{i}\beta H_y\\ -\mathrm{i}\beta H_x - \partial_x H_z\\ \partial_x H_y - \partial_y H_x\end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix},$$

vectorial mode equations, variants. (. . .)

Waveguides: Mode equations

- Where $\epsilon\,\cancel{(\boldsymbol{r})},\ \mu\,\cancel{(\boldsymbol{r})}$: $\sim\exp(\mathrm{i}\omega t)$ (FD)

$$\Delta\tilde{\boldsymbol{E}} + k^2\epsilon\mu\tilde{\boldsymbol{E}} = 0,\quad \Delta\tilde{\boldsymbol{H}} + k^2\epsilon\mu\tilde{\boldsymbol{H}} = 0$$

$$\partial_x^2\boldsymbol{E} + \partial_y^2\boldsymbol{E} + (k^2\epsilon\mu - \beta^2)\boldsymbol{E} = 0,$$
$$\partial_x^2\boldsymbol{H} + \partial_y^2\boldsymbol{H} + (k^2\epsilon\mu - \beta^2)\boldsymbol{H} = 0,$$

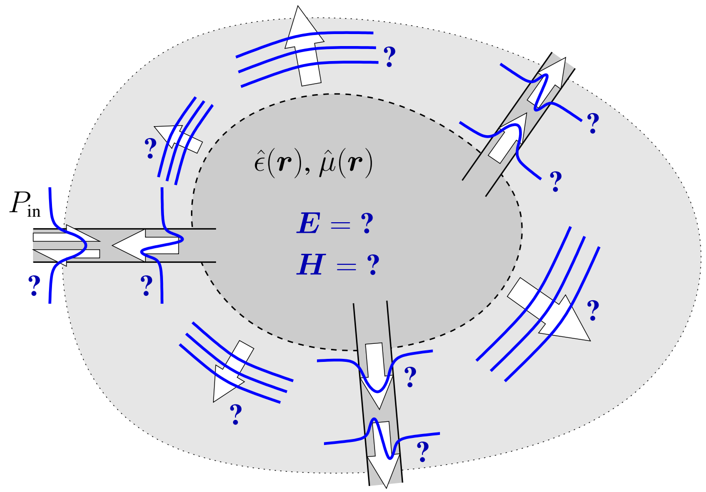
scalar mode equation, valid for all components of $\boldsymbol{E}, \boldsymbol{H}$,
to be supplemented by suitable boundary and interface conditions.

Eigenvalue problem with eigenvalue β, eigenfunction $\boldsymbol{E}, \boldsymbol{H}$,
"$\mathsf{M}(\beta)\,(\overrightarrow{\text{profile}}) = 0$".

- Guided modes: discrete $\beta\in\mathbb{R}$, $\iint S_z\,\mathrm{d}x\mathrm{d}z < \infty$.
- Radiation modes: continuum of $\beta^2\in\mathbb{R}$, oscillating external fields.
- Leaky modes: discrete $\beta\in\mathbb{C}$, outgoing wave boundary conditions.
- (. . .)

Guided wave scattering problems

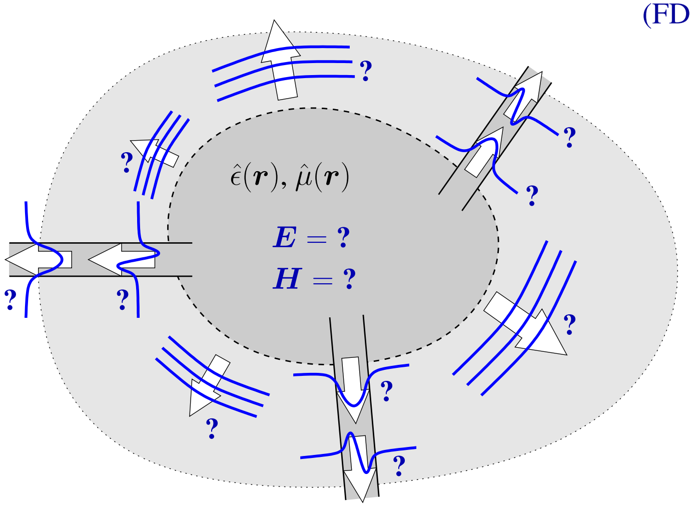
(FD)



Given external excitation $\sim\exp(\mathrm{i}\omega t)$, $\omega\in\mathbb{R}$.

Resonance problems

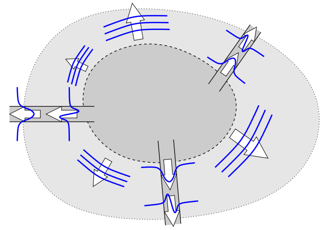
(FD . . .)



Omit excitation, look for nonzero solutions that decay in time.

Resonance problems

(FD . . .)



$\boldsymbol{E}(\boldsymbol{r}),\ \boldsymbol{H}(\boldsymbol{r}),\ \sim \exp(\mathrm{i}\omega t),\ \omega = ?$
$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H},$
$\boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E},$
& outgoing wave boundary conditions.

- Look for nonzero solutions with $\omega \in \mathbb{C}$ that oscillate and decay (slowly . . .) in time.
- "$\mathsf{M}(\omega)\ (\overrightarrow{\text{field}}) = 0$", eigenvalue problem.
- Solutions: discrete eigenfrequencies ω, resonant mode profiles.

Scalar approximation

Linear, isotropic, nonmagnetic media, $\epsilon = n^2$;
a structure with "small" variations in ϵ :

A scalar approximation may be adequate,

$$\boldsymbol{\nabla} \cdot (\epsilon\boldsymbol{E}) \approx \epsilon\boldsymbol{\nabla} \cdot \boldsymbol{E}$$

$$\Delta\psi - \frac{1}{\mathrm{c}^2}\epsilon\,\ddot{\psi} = 0, \qquad \text{(TD)}$$

$$\Delta\psi + k^2\epsilon\,\psi = 0, \qquad \text{(FD)}$$

satisfied by all components ψ of $\boldsymbol{E}, \boldsymbol{H}$.

(Applicable to basically all types of problems.)

Beam propagation method

- Starting point: $\Delta\psi + k^2\epsilon\,\psi = 0,$ $\quad\sim \exp(\mathrm{i}\omega t)$ (FD)
"small" changes in $\epsilon = n^2$ along a propagation coordinate z.
- Ansatz: $\psi(x, y, z) = \psi_0(x, y, z)\ \mathrm{e}^{-\mathrm{i}kn_\mathrm{r}z},$
reference effective index n_r,
assume that ψ_0 varies "slowly" along z $\longleftrightarrow$ neglect $\partial_z^2\psi_0$.
$$-\mathrm{i}2kn_\mathrm{r}\partial_z\psi_0 + (\partial_x^2 + \partial_y^2)\psi_0 + k^2(\epsilon - n_\mathrm{r}^2)\psi_0 = 0,$$
PDE of first order in z, solved as an initial value problem.
- Restriction to unidirectional propagation, reflections are neglected.
- Paraxial propagation, errors for waves with effective indices $\neq n_\mathrm{r}$.

(Many variants (vectorial, wide-angle, bi-directional, ...) have been proposed.)
(Other ways of motivating the approximation exist.)
(Term "BPM" in use also for other types of methods.)

- Keywords: Paraxial approximation,
Slowly-varying-envelope approximation (SVEA),
Beam propagation method (BPM).

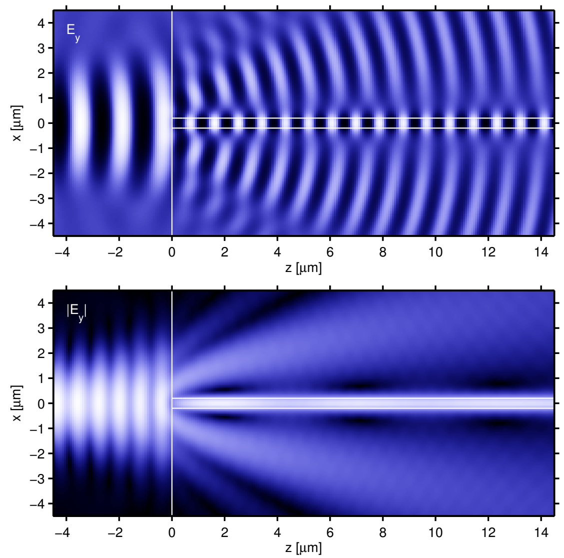
Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

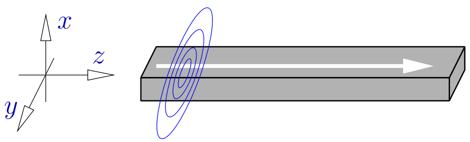
Context: Relevance of guided modes

(2-D, TE)



Butt-coupling to a waveguide facet.

Waveguides: Mode problems



$\mu = 1,\ \epsilon = n^2,\ \sim \exp(i\omega t)$ (FD)

$$\nabla \times \boldsymbol{E} = -i\omega\mu_0 \boldsymbol{H},$$
$$\nabla \times \boldsymbol{H} = i\omega\epsilon_0\epsilon \boldsymbol{E}.$$

- Waveguide: a system that is homogeneous along its axis z, $\partial_z \epsilon = 0,\ \partial_z n = 0.$

- Look for solutions (modes) that vary harmonically with z:

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}\\ \bar{\boldsymbol{H}}\end{pmatrix}(x,y)\ e^{-i\beta z},$$

mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
propagation constant β,
effective index $n_{\text{eff}} = \beta/k$.

$$\partial_z \longrightarrow -i\beta,$$ (& boundary conditions)

Eigenvalue problem with eigenvalue β, eigenfunction $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
"$\mathsf{M}(\beta)\,(\overrightarrow{\text{profile}}) = 0$".

Mode equations

(drop $^-$)

$$\begin{pmatrix}\partial_y E_z + i\beta E_y\\ -i\beta E_x - \partial_x E_z\\ \partial_x E_y - \partial_y E_x\end{pmatrix} = -i\omega\mu_0 \begin{pmatrix}H_x\\ H_y\\ H_z\end{pmatrix}, \quad \begin{pmatrix}\partial_y H_z + i\beta H_y\\ -i\beta H_x - \partial_x H_z\\ \partial_x H_y - \partial_y H_x\end{pmatrix} = i\omega\epsilon_0\epsilon \begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}.$$

- Express E_x, E_y, E_z, H_z through principal components H_x, H_y:

$$\partial_x^2 H_x + \epsilon\partial_y \frac{1}{\epsilon}\partial_y H_x + \partial_{xy} H_y - \epsilon\partial_y \frac{1}{\epsilon}\partial_x H_y + (k^2\epsilon - \beta^2) H_x = 0\,,$$
$$\epsilon\partial_x \frac{1}{\epsilon}\partial_x H_y + \partial_y^2 H_y + \partial_{yx} H_x - \epsilon\partial_x \frac{1}{\epsilon}\partial_y H_x + (k^2\epsilon - \beta^2) H_y = 0\,,$$

$$\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix} = \frac{1}{\omega\epsilon_0\epsilon}\begin{pmatrix}\beta H_y - \beta^{-1}(\partial_{yx} H_x + \partial_y^2 H_y)\\ -\beta H_x + \beta^{-1}(\partial_{xy} H_y + \partial_x^2 H_x)\\ -i(\partial_x H_y - \partial_y H_x)\end{pmatrix}, \quad \begin{pmatrix}H_x\\ H_y\\ H_z\end{pmatrix} = \begin{pmatrix}H_x\\ H_y\\ -i\beta^{-1}(\partial_x H_x + \partial_y H_y)\end{pmatrix}.$$

(H_x, H_y are continuous for all x, y.)

Mode equations

(drop $^{-}$)

$$\hookrightarrow \quad \begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express H_x, H_y, H_z, E_z through principal components E_x, E_y :

$\hookrightarrow$ $(\ldots)$.

(E_x, E_y are discontinuous at specific interfaces.)

Mode equations

(drop $^{-}$)

$$\hookrightarrow \quad \begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- Express E_x, E_y, H_x, H_y through principal components E_z, H_z :

$\hookrightarrow$ $(\ldots)$.

(E_z, H_z are usually small components.)

Plane mode profiles

- Modes are eigenfunctions
  $\leftrightsquigarrow$ profiles are determined up to a complex constant only.

- Propagating modes, $\beta \in \mathbb{R}$, lossless structures, $\epsilon \in \mathbb{R}$:
  $E_z := \mathrm{i}E_z'$, $H_z := \mathrm{i}H_z'$ $\rightsquigarrow$ real PDE for $E_x, E_y, E_z', H_x, H_y, H_z'$:

$$\begin{pmatrix} \partial_y E_z' + \beta E_y \\ -\beta E_x - \partial_x E_z' \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ -H_z' \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z' + \beta H_y \\ -\beta H_x - \partial_x H_z' \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ -E_z' \end{pmatrix};$$

  it is possible to choose a phase such that
  E_x, E_y, H_x, H_y are real,
  E_z, H_z are imaginary
  $\leftrightsquigarrow$ plane mode profiles.

(It makes sense to prepare real plots of mode profile components.)
(That requires a suitable adjustment of the global phase.)

Guided modes

- Guided modes: profiles located “around” the waveguide core
  $\leftrightsquigarrow$ discrete $\beta \in \mathbb{R}$, $\iint S_z \,\mathrm{d}x\,\mathrm{d}y < \infty$.

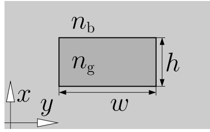
- In general: Hybrid modes, all six field components present.
  Planar-like waveguides $\rightsquigarrow$ adapt 2-D naming scheme;
  “TE-like” / “TM-like” modes.

($\leftrightarrow$ 5-component semivectorial approximations, plane $\perp$ x-axis:
quasi-TE: tiny E_x, dominant E_y, small E_z; major H_x, small H_y, minor H_z,
quasi-TM: tiny H_x, dominant H_y, small H_z; major E_x, small E_y, minor E_z.)

- Mode indices mostly relate to numbers of nodal lines in the dominant electric or magnetic field component.

(Naming schemes are highly context dependent.)

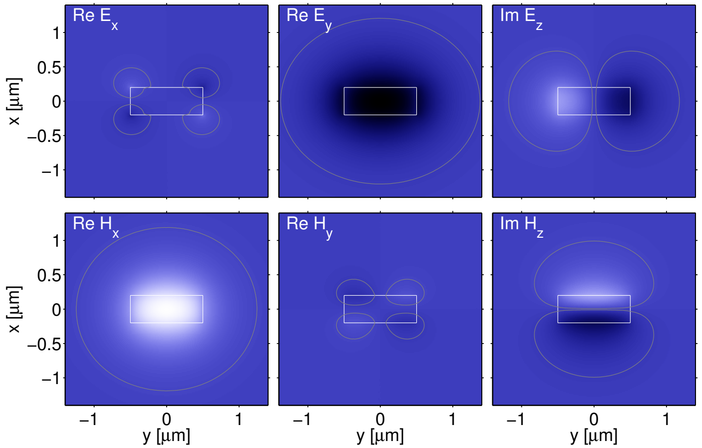
A rectangular strip waveguide, fundamental mode profiles



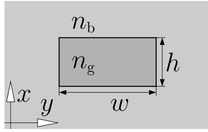
$\lambda = 1.55\,\mu$m,
$n_b = 1.45$,
$n_g = 1.99$,
$w = 1.0\,\mu$m,
$h = 0.4\,\mu$m;

(q-) $\mathrm{TE}_{0\,0}$

$x \in [-2, 2]\,\mu$m,
$y \in [-2, 2]\,\mu$m;
$n_{\mathrm{eff}} = 1.63554$
[JCMwave].



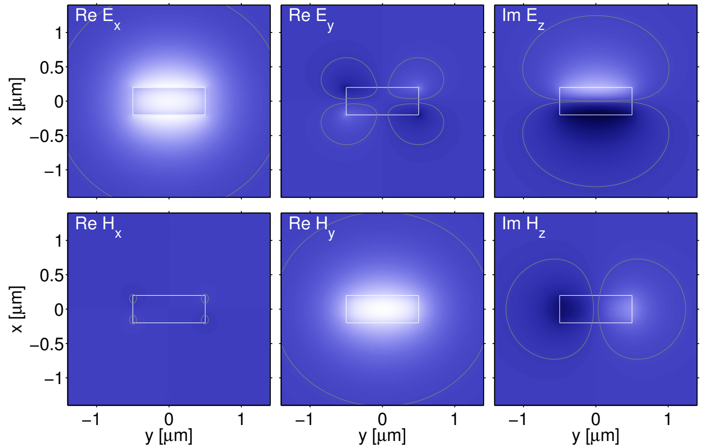
A rectangular strip waveguide, fundamental mode profiles



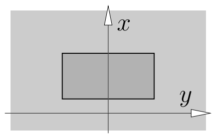
$\lambda = 1.55\,\mu$m,
$n_b = 1.45$,
$n_g = 1.99$,
$w = 1.0\,\mu$m,
$h = 0.4\,\mu$m;

(q-) $\mathrm{TM}_{0\,0}$

$x \in [-2, 2]\,\mu$m,
$y \in [-2, 2]\,\mu$m;
$n_{\mathrm{eff}} = 1.56809$
[JCMwave].



Symmetric waveguides

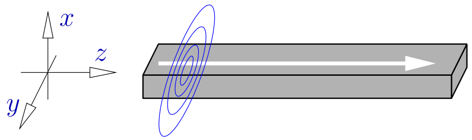


Waveguide with mirror symmetry $y \to -y$:
$\leftrightsquigarrow$ modes have a definite parity.

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$\hookrightarrow$ Equal parity of H_x, E_y, H_z, reversed parity of E_x, H_y, E_z.

Directional modes



(FD) $\sim \exp(\mathrm{i}\omega t)$

Longitudinally homogeneous waveguide: mirror symmetry $z \to -z$.

$$\begin{pmatrix} \partial_y E_z + \mathrm{i}\beta E_y \\ -\mathrm{i}\beta E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \mathrm{i}\beta H_y \\ -\mathrm{i}\beta H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

forward: $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f}} \\ \bar{\boldsymbol{H}}^{\mathrm{f}} \end{pmatrix}(x, y)\ \mathrm{e}^{-\mathrm{i}\beta z}$, $\quad \bar{\boldsymbol{E}}^{\mathrm{f}} = (E_x, E_y, E_z)$, $\bar{\boldsymbol{H}}^{\mathrm{f}} = (H_x, H_y, H_z)$,

$\leftrightsquigarrow$

backward: $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{b}} \\ \bar{\boldsymbol{H}}^{\mathrm{b}} \end{pmatrix}(x, y)\ \mathrm{e}^{+\mathrm{i}\beta z}$, $\quad \bar{\boldsymbol{E}}^{\mathrm{b}} = (E_x, E_y, -E_z)$, $\bar{\boldsymbol{H}}^{\mathrm{b}} = (-H_x, -H_y, H_z)$.

Modal power

(FD) $\sim \exp(\mathrm{i}\omega t)$

- E.m. power density: $\boldsymbol{S} = \frac{1}{2}\mathrm{Re}\,(\boldsymbol{E}^* \times \boldsymbol{H})$.

- $$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}\\ \bar{\boldsymbol{H}}\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z},$$

$\bar{\boldsymbol{E}} = a(\bar{E}_x, \bar{E}_y, \mathrm{i}\bar{E}_z{}'),$
$\bar{\boldsymbol{H}} = a(\bar{H}_x, \bar{H}_y, \mathrm{i}\bar{H}_z{}'),$
$a \in \mathbb{C},\ \bar{E}_x, \ldots, \bar{H}_z' \in \mathbb{R},$
a guided mode, $\beta \in \mathbb{R}$.

$$\hookrightarrow \quad \boldsymbol{S} = \frac{|a|^2}{2}\begin{pmatrix}0\\ 0\\ \bar{E}_x\bar{H}_y - \bar{E}_y\bar{H}_x\end{pmatrix},$$

or $S_x = 0,\ \ S_y = 0,\ \ S_z = \frac{1}{2}\mathrm{Re}\,\left(E_x^* H_y - E_y^* H_x\right).$ $\quad$ ($S_z(x,y)$)

- Power carried by the mode:

$$P = \iint S_z\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\iint \left(E_x^*H_y - E_y^*H_x + E_xH_y^* - E_yH_x^*\right)\,\mathrm{d}x\,\mathrm{d}y\,.$$

(backward mode, $E_x \to E_x,\ E_y \to E_y,\ H_x \to -H_x,\ H_y \to -H_y:\ \ P \to -P$)

Mode orthogonality

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z},$$

$\boldsymbol{\nabla} \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0\boldsymbol{H}_m,$
$\boldsymbol{\nabla} \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_m,$
$\beta_l \neq \beta_m$, if $l \neq m$.

- $$P_m = \frac{1}{4}\iint \left(E_{mx}^*H_{my} - E_{my}^*H_{mx} + E_{mx}H_{my}^* - E_{my}H_{mx}^*\right)\,\mathrm{d}x\,\mathrm{d}y\,.$$

- $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.

- $\boldsymbol{\nabla}\cdot(\boldsymbol{E}_l^* \times \boldsymbol{H}_m + \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = 0$ $\qquad$ for all l, m

$$\hookrightarrow \quad 0 = \mathrm{i}(\beta_l - \beta_m)\left\{\iint \left(\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m + \bar{\boldsymbol{E}}_m \times \bar{\boldsymbol{H}}_l^*\right)_z \mathrm{d}x\,\mathrm{d}y\right\}\ \mathrm{e}^{\mathrm{i}(\beta_l - \beta_m)z},$$

$$(\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2) := \frac{1}{4}\iint \left(E_{1x}^*H_{2y} - E_{1y}^*H_{2x} + H_{1y}^*E_{2x} - H_{1x}^*E_{2y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m) = \begin{cases} 0, & \text{if } l \neq m,\\ P_m, & \text{otherwise.}\end{cases}$$

(The modes are "power orthogonal".)
(Statements hold for propagating guided modes.)
$((.,.;.,.)$ is frequently used for mode normalization.)

Power transport by a mode superposition

$\beta \in \mathbb{R}$

- A set of guided modes of the same waveguide (ϵ):

$$\begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}, \qquad P_m = (\boldsymbol{E}_m, \boldsymbol{H}_m; \boldsymbol{E}_m, \boldsymbol{H}_m)\,.$$

- Superposition with amplitudes $a_m \in \mathbb{C}$:

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \sum_m a_m \begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z) = \sum_m a_m \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}.$$

Power flow along the waveguide:

$$\begin{aligned}\iint S_z\,\mathrm{d}x\,\mathrm{d}y &= (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H})\\ &= \sum_l \sum_m a_l^*\,a_m\,(\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m)\\ &= \sum_m |a_m|^2 P_m\,.\end{aligned}$$

(Forward / backward modes: $P \gtrless 0$.)

Mode interference

$\beta \in \mathbb{R}$

- Two modes $m = 1, 2$:

$$\begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}.$$

- Superposition with amplitudes a_1, a_2:

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = a_1\begin{pmatrix}\bar{\boldsymbol{E}}_1\\ \bar{\boldsymbol{H}}_1\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_1 z} + a_2\begin{pmatrix}\bar{\boldsymbol{E}}_2\\ \bar{\boldsymbol{H}}_2\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_2 z}.$$

- Fix a position x, y and component F: $\qquad$ Omit (x, y).

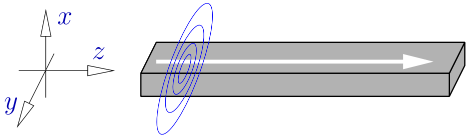
$F(z) = a_1\,\bar{F}_1\ \mathrm{e}^{-\mathrm{i}\beta_1 z} + a_2\,\bar{F}_2\ \mathrm{e}^{-\mathrm{i}\beta_2 z}, \qquad r\,\mathrm{e}^{-\mathrm{i}\phi} := a_1^* a_2 \bar{F}_1^* \bar{F}_2,$

$$\hookrightarrow \quad |F|^2(z) = |a_1|^2|\bar{F}_1|^2 + |a_2|^2|\bar{F}_2|^2 + 2r\cos((\beta_1 - \beta_2)z + \phi).$$

Periodic beating pattern with half-beat-length $\quad L_\mathrm{c} = \dfrac{\pi}{|\beta_1 - \beta_2|}$.

(Supermodes ▶) $\quad$ (Evanescent coupling ▶) $\qquad$ ("Coupling length" L_c.)

Polarization of a guided wave field



Unidirectional guided waves in a "long" dielectric channel that supports fundamental TE- and TM-like modes only:

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = a_{\mathrm{TE}}\begin{pmatrix}\bar{\boldsymbol{E}}_{\mathrm{TE}}\\ \bar{\boldsymbol{H}}_{\mathrm{TE}}\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{\mathrm{TE}}z} + a_{\mathrm{TM}}\begin{pmatrix}\bar{\boldsymbol{E}}_{\mathrm{TM}}\\ \bar{\boldsymbol{H}}_{\mathrm{TM}}\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{\mathrm{TM}}z},$$

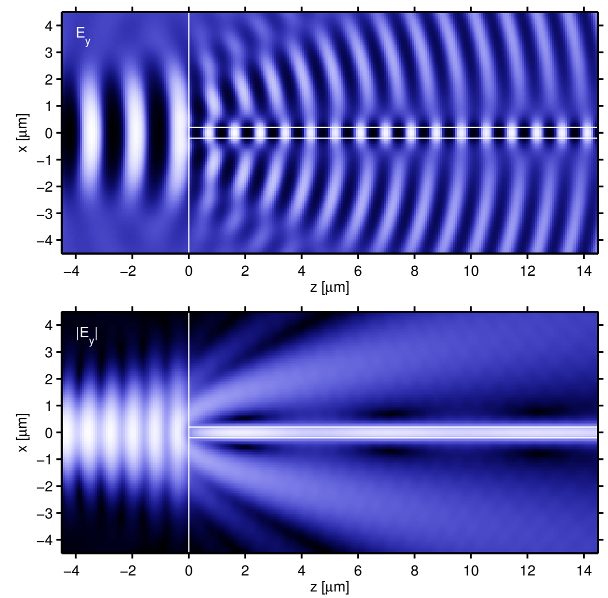
amplitudes $a_{\mathrm{TE}}, a_{\mathrm{TM}} \in \mathbb{C}$.

- $E_{\mathrm{TE}\,z} \neq 0,\ E_{\mathrm{TM}\,z} \neq 0.$
- $\bar{\boldsymbol{E}}_{\mathrm{TE}}(x,y) \not\parallel \bar{\boldsymbol{E}}_{\mathrm{TM}}(x,y).$
- At (x,y): adjust $\boldsymbol{E}/|\boldsymbol{E}|$ via $a_{\mathrm{TE}}, a_{\mathrm{TM}}$.
- $a_{\mathrm{TE}}, a_{\mathrm{TM}}$ fixed: $(\boldsymbol{E}/|\boldsymbol{E}|)(x,y)$ varies.

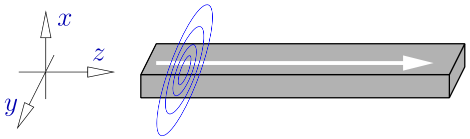
"Polarization" frequently indicates the presence of only one mode.

What about non-guided fields?

(2-D, TE)



Normal modes: real mode problems



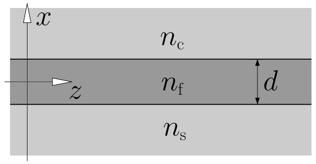
- lossless waveguide, $\epsilon \in \mathbb{R}$,
- "real" boundary conditions at x, y "far away" from the core,
- "real" vectorial mode equations:

$$\partial_x^2 H_x + \epsilon\partial_y\frac{1}{\epsilon}\partial_y H_x + \partial_{xy}H_y - \epsilon\partial_y\frac{1}{\epsilon}\partial_x H_y + (k^2\epsilon - \beta^2)H_x = 0\,,$$

$$\epsilon\partial_x\frac{1}{\epsilon}\partial_x H_y + \partial_y^2 H_y + \partial_{yx}H_x - \epsilon\partial_x\frac{1}{\epsilon}\partial_y H_x + (k^2\epsilon - \beta^2)H_y = 0\,,$$

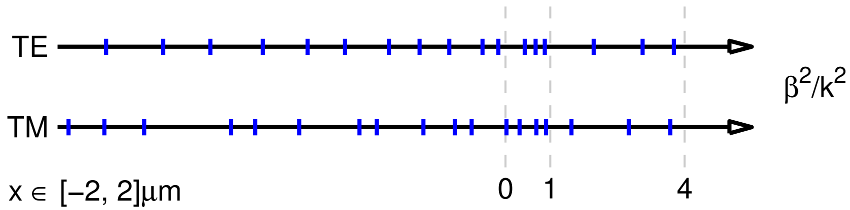
$\hookrightarrow$ real principal components $H_x(x,y), H_y(x,y)$, $\beta^2 \in \mathbb{R}$.

2-D slab waveguide, normal mode spectrum



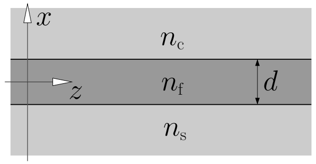
$n_{\mathrm{s}} = n_{\mathrm{c}} = 1.0,\ n_{\mathrm{f}} = 2.0,$
$d = 1.3\ \mu\mathrm{m},\ \lambda = 1.55\ \mu\mathrm{m},$

$E_y = 0, H_y = 0$ at $x = \pm 2\ \mu\mathrm{m}.$



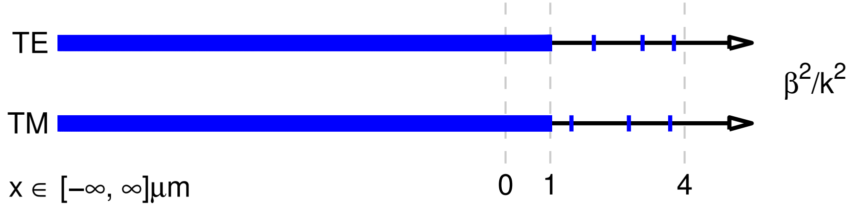
- $n_{\mathrm{f}}^2 < \beta^2/k^2$: no modal solutions.
- $n_{\mathrm{s}}^2 < \beta^2/k^2 < n_{\mathrm{f}}^2$: guided modes.
- $0 < \beta^2/k^2 < n_{\mathrm{s}}^2$: propagating radiation modes.
- $\beta^2/k^2 < 0$: evanescent radiation modes.

2-D slab waveguide, normal mode spectrum



$n_s = n_c = 1.0,\ n_f = 2.0,$
$d = 1.3\,\mu\text{m},\ \lambda = 1.55\,\mu\text{m},$

$E_y = 0, H_y = 0$ at $x = \pm\infty$.



- $n_f^2 < \beta^2/k^2$: no modal solutions.
- $n_s^2 < \beta^2/k^2 < n_f^2$: guided modes (discrete spectrum).
- $0 < \beta^2/k^2 < n_s^2$: propagating radiation modes (continuous spec.).
- $\beta^2/k^2 < 0$: evanescent radiation modes (continuous spec.).

Propagating & evanescent modes

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f,b}} \\ \bar{\boldsymbol{H}}^{\mathrm{f,b}} \end{pmatrix}(x,y)\ \mathrm{e}^{\mp \mathrm{i}\beta z}. \qquad \sim \exp(\mathrm{i}\omega t)\quad \text{(FD)}$$

- $\beta^2 > 0 \leftrightsquigarrow \beta = \sqrt{\beta^2},\ \beta \in \mathbb{R},\ \beta > 0,$
  $\sim \mathrm{e}^{\mp \mathrm{i}\beta z}$, a forward / backward propagating mode.

  (Physical relevance of individual modes.)

- $\beta^2 < 0 \leftrightsquigarrow \beta = -\mathrm{i}\sqrt{|\beta^2|} = -\mathrm{i}\alpha,\ \alpha = \sqrt{|\beta^2|} \in \mathbb{R},\ \alpha > 0,$
  $\sim \mathrm{e}^{\mp \alpha z}$, a forward / backward traveling evanescent mode.

  "forward": $\sim \mathrm{e}^{-\alpha z}$, field decays with z,
  "backward": $\sim \mathrm{e}^{+\alpha z}$, field grows with z.

  (Relevant for purposes of field expansions.)

- $\{$forward & backward, propagating & evanescent modes$\}$
  $=$ the set of normal modes.

Evanescent modes

$\beta = -\mathrm{i}\alpha,\ \alpha \in \mathbb{R}$ $\qquad \epsilon \in \mathbb{R}$

$$\begin{pmatrix} \partial_y E_z + \alpha E_y \\ -\alpha E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} = -\mathrm{i}\omega\mu_0 \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}, \quad \begin{pmatrix} \partial_y H_z + \alpha H_y \\ -\alpha H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} = \mathrm{i}\omega\epsilon_0\epsilon \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Plane mode profiles: real PDE for $E_x, E_y, E_z,\ \mathrm{i}H_x, \mathrm{i}H_y, \mathrm{i}H_z$;
  common phase with real E_x, E_y, E_z, imaginary H_x, H_y, H_z.

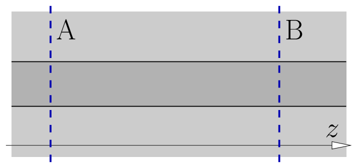
- Directional evanescent modes:
  $\{E_x, E_y, E_z, H_x, H_y, H_z;\ \alpha\}^{\mathrm{f}} \rightsquigarrow \{E_x, E_y, -E_z, -H_x, -H_y, H_z;\ -\alpha\}^{\mathrm{b}}$.

- Modal power:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\alpha z}, \qquad \begin{aligned} &\bar{\boldsymbol{E}} = a(E'_x, E'_y, E'_z), \\ &\bar{\boldsymbol{H}} = \mathrm{i}a(H'_x, H'_y, H'_z), \\ &E'_x, \ldots, H'_z \in \mathbb{R},\quad a \in \mathbb{C} \end{aligned}$$

$$\hookrightarrow \quad S_z = \frac{1}{2}\mathrm{Re}\,\left(E_x^* H_y - E_y^* H_x\right) = 0\,, \qquad \iint S_z\,\mathrm{d}x\,\mathrm{d}y = 0\,.$$

Completeness of normal modes



$\epsilon \in \mathbb{R},\ \sim \exp(\mathrm{i}\omega t)$ (FD)

A lossless, z-homogeneous waveguide configuration; general solution of the Maxwell equations between cross sectional planes A and B:

$$\begin{aligned} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) &= \sum_{m \in \mathcal{N}} F_m \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{f}}_m \\ \bar{\boldsymbol{H}}^{\mathrm{f}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z} \\ &+ \sum_{m \in \mathcal{N}} B_m \begin{pmatrix} \bar{\boldsymbol{E}}^{\mathrm{b}}_m \\ \bar{\boldsymbol{H}}^{\mathrm{b}}_m \end{pmatrix}(x,y)\ \mathrm{e}^{+\mathrm{i}\beta_m z}, \qquad \sum \to \not\!\!\int\!\!\!\!\sum \end{aligned}$$

$\mathcal{N}$: the set of forward normal modes supported by the waveguide.

("Solution": obvious; "general": without proof.)

Completeness of normal modes

Stronger statement:
"any" transverse 2-component field on a cross sectional plane can be expanded into alternatively

- the transverse electric components of forward normal modes,
- the transverse magnetic components of forward normal modes,
- the transverse electric components of backward normal modes,
- the transverse magnetic components of backward normal modes.

Orthogonality of normal modes

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}\\ \bar{\boldsymbol{H}}\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}. \qquad \sim \exp(\mathrm{i}\omega t) \quad \text{(FD)}$$

	$\bar{\boldsymbol{E}}$	$\bar{\boldsymbol{H}}$	β
[prop., f]	$(E'_x, E'_y, \mathrm{i}E'_z)$	$(H'_x, H'_y, \mathrm{i}H'_z)$	$\beta > 0$
[prop., b]	$(E'_x, E'_y, -\mathrm{i}E'_z)$	$(-H'_x, -H'_y, \mathrm{i}H'_z)$	$\beta < 0$
[evan., f]	(E'_x, E'_y, E'_z)	$(\mathrm{i}H'_x, \mathrm{i}H'_y, \mathrm{i}H'_z)$	$\beta = -\mathrm{i}\alpha,\ \alpha > 0$
[evan., b]	$(E'_x, E'_y, -E'_z)$	$(-\mathrm{i}H'_x, -\mathrm{i}H'_y, \mathrm{i}H'_z)$	$\beta = \mathrm{i}\alpha,\ \alpha > 0$

individual $E'_x, \ldots H'_z \in \mathbb{R}$.

$$(\boldsymbol{E}_a, \boldsymbol{H}_a; \boldsymbol{E}_b, \boldsymbol{H}_b) := \frac{1}{4}\iint \left(E^*_{ax}H_{by} - E^*_{ay}H_{bx} + H^*_{ay}E_{bx} - H^*_{ax}E_{by}\right) \mathrm{d}x\,\mathrm{d}y$$

$$\begin{pmatrix}\boldsymbol{E}_{1,2}\\ \boldsymbol{H}_{1,2}\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}_{1,2}\\ \bar{\boldsymbol{H}}_{1,2}\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{1,2} z}, \qquad \begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E}_{1,2} &= -\mathrm{i}\omega\mu_0\boldsymbol{H}_{1,2},\\ \boldsymbol{\nabla}\times\boldsymbol{H}_{1,2} &= \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_{1,2},\end{aligned}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{E}^*_1\times\boldsymbol{H}_2 + \boldsymbol{E}_2\times\boldsymbol{H}^*_1) = 0 \ \rightsquigarrow\ 0 = (\beta^*_1 - \beta_2)\,(\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2).$$

$\hookrightarrow$ …

Orthogonality of normal modes

Nondegenerate directional normal modes of the same waveguide (ϵ):

$$\begin{pmatrix}\boldsymbol{E}^{\mathrm{f,b}}_m\\ \boldsymbol{H}^{\mathrm{f,b}}_m\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}^{\mathrm{f,b}}_m\\ \bar{\boldsymbol{H}}^{\mathrm{f,b}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta^{\mathrm{f,b}}_m z}, \qquad \begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E}_m &= -\mathrm{i}\omega\mu_0\boldsymbol{H}_m,\\ \boldsymbol{\nabla}\times\boldsymbol{H}_m &= \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}_m,\end{aligned}$$

$$\beta_l \neq \beta_m, \text{ if } l \neq m.$$

- A propagating mode m:
  $(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m) =: P_m, \ (\bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m; \bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m) = -P_m, \qquad P_m \in \mathbb{R},$
  $(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m) = (\bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m; \bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m) = 0,$
  $(\bar{\boldsymbol{E}}^{\mathrm{d}}_m, \bar{\boldsymbol{H}}^{\mathrm{d}}_m; \bar{\boldsymbol{E}}^{\mathrm{r}}_l, \bar{\boldsymbol{H}}^{\mathrm{r}}_l) = (\bar{\boldsymbol{E}}^{\mathrm{r}}_l, \bar{\boldsymbol{H}}^{\mathrm{r}}_l; \bar{\boldsymbol{E}}^{\mathrm{d}}_m, \bar{\boldsymbol{H}}^{\mathrm{d}}_m) = 0 \ \text{ for all } \ l \neq m, \ \mathrm{d,r} = \mathrm{f,b}.$
- An evanescent mode m:
  $(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m) = (\bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m; \bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m) = 0,$
  $(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m) =: P_m, \ (\bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m; \bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m) = -P_m, \qquad P_m \notin \mathbb{R},$
  $(\bar{\boldsymbol{E}}^{\mathrm{d}}_m, \bar{\boldsymbol{H}}^{\mathrm{d}}_m; \bar{\boldsymbol{E}}^{\mathrm{r}}_l, \bar{\boldsymbol{H}}^{\mathrm{r}}_l) = (\bar{\boldsymbol{E}}^{\mathrm{r}}_l, \bar{\boldsymbol{H}}^{\mathrm{r}}_l; \bar{\boldsymbol{E}}^{\mathrm{d}}_m, \bar{\boldsymbol{H}}^{\mathrm{d}}_m) = 0 \ \text{ for all } \ l \neq m, \ \mathrm{d,r} = \mathrm{f,b}.$

(This implies orthogonality of propagating and evanescent modes.)
($1/\sqrt{|P_m|}$ is frequently used for mode normalization.)

Power flow associated with a normal mode expansion

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \sum_{m\in\mathcal{N}}\left\{F_m \begin{pmatrix}\bar{\boldsymbol{E}}^{\mathrm{f}}_m\\ \bar{\boldsymbol{H}}^{\mathrm{f}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z} + B_m \begin{pmatrix}\bar{\boldsymbol{E}}^{\mathrm{b}}_m\\ \bar{\boldsymbol{H}}^{\mathrm{b}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{+\mathrm{i}\beta_m z}\right\}$$

Power carried along z:

$$P = \iint S_z\,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H})$$

$$= \sum_{m \text{ propag.}} \left(|F_m|^2 - |B_m|^2\right) P_m \ + \sum_{m \text{ evanesc.}} (F^*_m B_m - B^*_m F_m)\, P_m\,.$$

- P is indepedent of z.
- Individual contributions from forward and backward propagating modes.
- Contributions from evanescent modes require forward and backward fields to be present.
- Unidirectional field (forward: $B_m = 0$ for all m): Only propagating modes carry power.

Projection onto normal modes

$\boldsymbol{E}, \boldsymbol{H}$: a solution of the Maxwell equations for the z-homogeneous waveguide between two cross sectional planes A and B.

Extract local mode amplitudes by projection onto normal modes:

- A propagating mode m, $\beta_m > 0$:
  $(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \boldsymbol{E}, \boldsymbol{H}) = F_m\, P_m\, \mathrm{e}^{-\mathrm{i}\beta z}$,
  $(\bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m; \boldsymbol{E}, \boldsymbol{H}) = -B_m\, P_m\, \mathrm{e}^{\mathrm{i}\beta z}$.

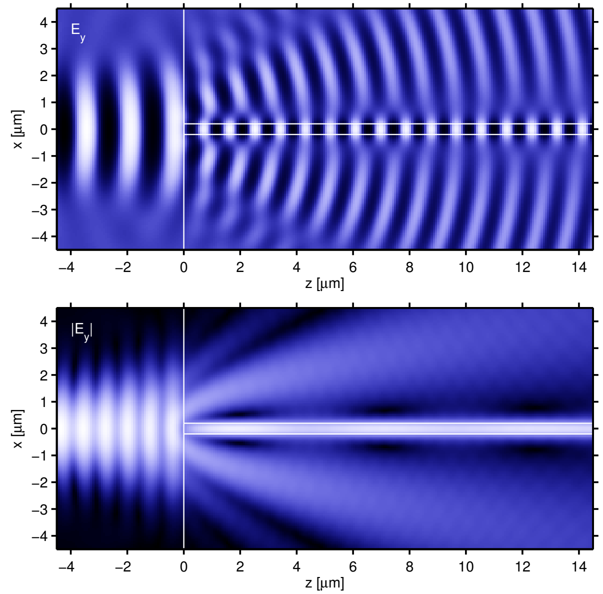
  $F_m\, \mathrm{e}^{-\mathrm{i}\beta z} = \dfrac{(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \boldsymbol{E}, \boldsymbol{H})}{(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m)}$

- An evanescent mode m, $\beta_m = -\mathrm{i}\alpha_m$, $\alpha_m > 0$:
  $(\bar{\boldsymbol{E}}^{\mathrm{f}}_m, \bar{\boldsymbol{H}}^{\mathrm{f}}_m; \boldsymbol{E}, \boldsymbol{H}) = B_m\, P_m\, \mathrm{e}^{\alpha z}$, $(\bar{\boldsymbol{E}}^{\mathrm{b}}_m, \bar{\boldsymbol{H}}^{\mathrm{b}}_m; \boldsymbol{E}, \boldsymbol{H}) = -F_m\, P_m\, \mathrm{e}^{-\alpha z}$.

$\leftrightsquigarrow$ Ports of a photonic integrated circuit.

Waveguide facet: Port definition

(2-D, TE)



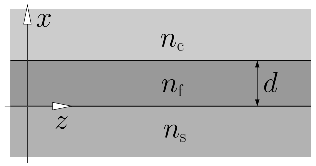
Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

2-D waveguide configurations

$\epsilon \in \mathbb{R},\ \mu = 1,\ \sim \exp(\mathrm{i}\omega t)$ (FD)



- 2-D waveguide, 1-D cross section.
- Permittivity $\epsilon = n^2$, refractive index $n(x)$. (1-D waveguide)

- $\partial_y \epsilon = 0$ $\leftrightsquigarrow$ $\partial_y \boldsymbol{E} = 0$, $\partial_y \boldsymbol{H} = 0$, 2-D TE / TM setting.
- $\partial_z \epsilon = 0$ $\leftrightsquigarrow$ Modal solutions that vary harmonically with z:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x)\ \mathrm{e}^{-\mathrm{i}\beta z},$$

mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
propagation constant β,
effective index $n_{\mathrm{eff}} = \beta/k$.

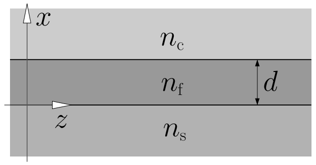
(TE): principal component $\bar{E}_y$, $\partial_x^2 \bar{E}_y + (k^2\epsilon - \beta^2)\bar{E}_y = 0$,

$$\bar{E}_x = 0,\quad \bar{E}_z = 0,\quad \bar{H}_x = \frac{-\beta}{\omega\mu_0}\bar{E}_y,\quad \bar{H}_y = 0,\quad \bar{H}_z = \frac{\mathrm{i}}{\omega\mu_0}\partial_x \bar{E}_y,$$

$\bar{E}_y$ & $\partial_x \bar{E}_y$ continuous at dielectric interfaces.

2-D waveguide configurations

$\epsilon \in \mathbb{R},\ \mu = 1,\ \sim \exp(i\omega t)$ (FD)



- 2-D waveguide, 1-D cross section.
- Permittivity $\epsilon = n^2$, refractive index $n(x)$. (1-D waveguide)

- $\partial_y \epsilon = 0 \longleftrightarrow \partial_y \boldsymbol{E} = 0,\ \partial_y \boldsymbol{H} = 0,$ 2-D TE / TM setting.
- $\partial_z \epsilon = 0 \longleftrightarrow$ Modal solutions that vary harmonically with z:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x)\ \mathrm{e}^{-\mathrm{i}\beta z},$$

mode profile $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$,
propagation constant β,
effective index $n_{\text{eff}} = \beta/k$.

(TM): principal component $\bar{H}_y$, $\quad \epsilon\partial_x \frac{1}{\epsilon}\partial_x \bar{H}_y + (k^2\epsilon - \beta^2)\bar{H}_y = 0,$

$$\bar{E}_x = \frac{\beta}{\omega\epsilon_0\epsilon}\bar{H}_y,\quad \bar{E}_y = 0,\quad \bar{E}_z = \frac{-\mathrm{i}}{\omega\epsilon_0\epsilon}\partial_x\bar{H}_y,\quad \bar{H}_x = 0,\quad \bar{H}_z = 0,$$

$\bar{H}_y$ & $\epsilon^{-1}\partial_x\bar{H}_y$ continuous at dielectric interfaces.

Guided 2-D TE / TM modes, orthogonality properties

($\rightarrow$ *Exercise.*)

- A set (index m) of guided modes of a 2-D waveguide (ϵ), $\psi_m^{\mathrm{p}} = (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m)$, $\ ^{\mathrm{p=TE,TM}}$ & β_m, $\beta_m \neq \beta_l$, if $l \neq m$.
- $(\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2) := \frac{1}{4}\int \left(E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}\right) \mathrm{d}x\,.$
- Power P_m per lateral (y) unit length carried by mode $\psi_m^{\mathrm{p}}, \beta_m$:

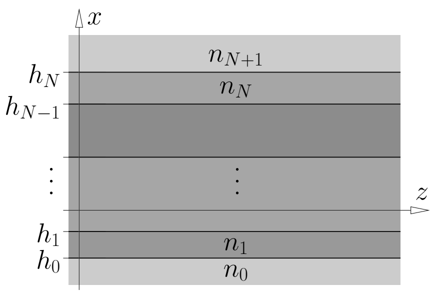
$$P_m := \int S_z\,\mathrm{d}x = (\psi_m^{\mathrm{p}}; \psi_m^{\mathrm{p}}) = \begin{cases} \dfrac{\beta_m}{2\omega\mu_0}\displaystyle\int |E_{m,y}|^2\,\mathrm{d}x, & \text{if p = TE}, \\ \dfrac{\beta_m}{2\omega\epsilon_0}\displaystyle\int \frac{1}{\epsilon}|H_{m,y}|^2\,\mathrm{d}x, & \text{if p = TM}. \end{cases}$$

$$(\psi_l^{\mathrm{TE}}; \psi_m^{\mathrm{TM}}) = 0\,, \qquad (\psi_l^{\mathrm{TE}}; \psi_m^{\mathrm{TE}}) = \frac{\beta_m}{2\omega\mu_0}\int E_{l,y}^*\, E_{m,y}\,\mathrm{d}x = \delta_{lm}P_m\,,$$

$$(\psi_l^{\mathrm{TM}}; \psi_m^{\mathrm{TE}}) = 0\,, \qquad (\psi_l^{\mathrm{TM}}; \psi_m^{\mathrm{TM}}) = \frac{\beta_m}{2\omega\epsilon_0}\int \frac{1}{\epsilon} H_{l,y}^*\, H_{m,y}\,\mathrm{d}x = \delta_{lm}P_m\,.$$

Dielectric multilayer slab waveguide

$\epsilon \in \mathbb{R},\ \mu = 1,\ \sim \exp(i\omega t)$ (2-D, FD)

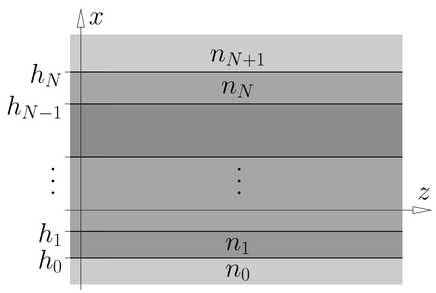


- N interior layers, piecewise constant $\epsilon = n^2$:

$$n(x) = \begin{cases} n_{N+1} & \text{if } h_N < x, \\ n_l & \text{if } h_{l-1} < x < h_l, \\ n_0 & \text{if } x < h_0. \end{cases}$$

- Principal component $\phi(x)$ $\qquad$ (TE: $\phi = \bar{E}_y$, TM: $\phi = \bar{H}_y$).
- $\partial_x^2\phi + (k^2 n_l^2 - \beta^2)\phi = 0, \qquad x \in$ layer l, $\quad l = 0, \dots, N+1$
(Half-infinte substrate ($l = 0$) and cover ($l = N+1$) layers.)
- ϕ & $\eta\partial_x\phi$ continuous at $x = h_l$, $\qquad$ (TE: $\eta = 1$, TM: $\eta = n^{-2}$).

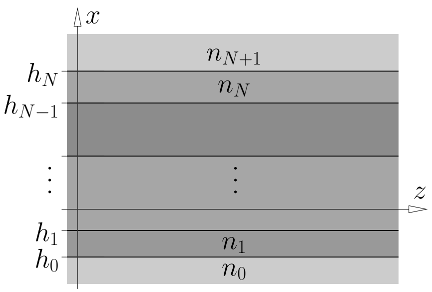
Dielectric multilayer slab waveguide



- Interior layer l, $h_{l-1} < x < h_l$, local refractive index n_l,
- $\partial_x^2\phi = (\beta^2 - k^2 n_l^2)\phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_l^2 \rightsquigarrow \partial_x^2\phi = -\kappa_l^2\,\phi, \quad \kappa_l := \sqrt{k^2 n_l^2 - \beta^2},$
$\phi(x) = A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x).$
- $\beta^2 > k^2 n_l^2 \rightsquigarrow \partial_x^2\phi = \kappa_l^2\,\phi, \quad \kappa_l := \sqrt{\beta^2 - k^2 n_l^2},$
$\phi(x) = A_l\,\mathrm{e}^{\kappa_l x} + B_l\,\mathrm{e}^{-\kappa_l x}.$
- Unknowns $A_l, B_l \in \mathbb{C}$. $\qquad$ (Local coordinate offsets required to cope with the exponentials.)

Dielectric multilayer slab waveguide, guided modes



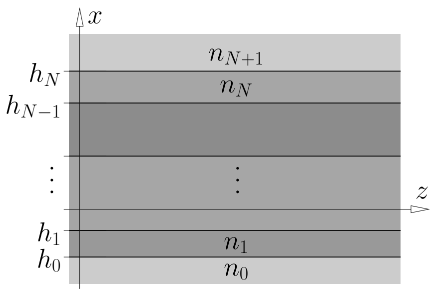
- Substrate region,
  $x < h_0$,
  local refractive index n_0,
- $\partial_x^2 \phi = (\beta^2 - k^2 n_0^2)\phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_0^2 \rightsquigarrow \quad \partial_x^2\phi = -\kappa_0^2\,\phi, \quad \kappa_0 := \sqrt{k^2 n_0^2 - \beta^2},$

  $\cancel{\phi(x) = A_0 \sin(\kappa_0 x) + B_0 \cos(-\kappa_0 x)}.$
- $\beta^2 > k^2 n_0^2 \rightsquigarrow \quad \partial_x^2\phi = \kappa_0^2\,\phi, \quad \kappa_0 := \sqrt{\beta^2 - k^2 n_0^2},$

  $\phi(x) = A_0\,\mathrm{e}^{\kappa_0 x} + \cancel{B_0\,\mathrm{e}^{-\kappa_0 x}}.$
- Unknown $A_0 \in \mathbb{C}$. Guided modes: $n_{\mathrm{eff}} = \beta/k > n_0$.

Dielectric multilayer slab waveguide, guided modes



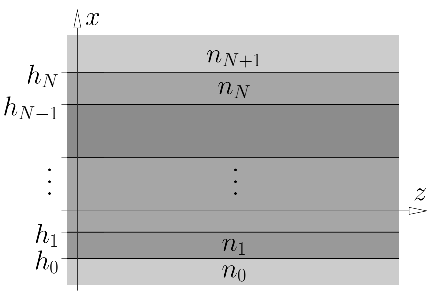
- Cover region,
  $h_N < x$,
  local refractive index n_{N+1},
- $\partial_x^2 \phi = (\beta^2 - k^2 n_{N+1}^2)\phi$.
- Consider a trial value $\beta^2 \in \mathbb{R}$.

- $\beta^2 < k^2 n_{N+1}^2 \rightsquigarrow \quad \partial_x^2\phi = -\kappa_{N+1}^2\,\phi, \quad \kappa_{N+1} := \sqrt{k^2 n_{N+1}^2 - \beta^2},$

  $\cancel{\phi(x) = A_{N+1} \sin(\kappa_{N+1} x) + B_{N+1} \cos(-\kappa_{N+1} x)}.$
- $\beta^2 > k^2 n_{N+1}^2 \rightsquigarrow \quad \partial_x^2\phi = \kappa_{N+1}^2\,\phi, \quad \kappa_{N+1} := \sqrt{\beta^2 - k^2 n_{N+1}^2},$

  $\phi(x) = \cancel{A_{N+1}\,\mathrm{e}^{\kappa_{N+1} x}} + B_{N+1}\,\mathrm{e}^{-\kappa_{N+1} x}.$
- Unknown $B_{N+1} \in \mathbb{C}$. Guided modes: $n_{\mathrm{eff}} = \beta/k > n_{N+1}$.

Dielectric multilayer slab waveguide

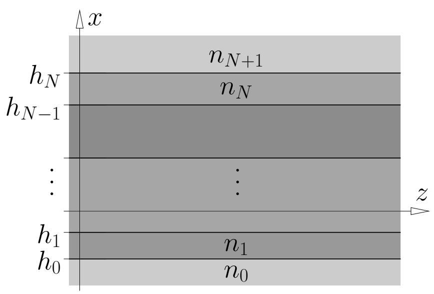


Trial value $\beta^2 \in \mathbb{R}$,
$\beta/k > n_0, n_{N+1}$,
$\rightsquigarrow \quad \kappa_l, \quad l = 0, \dots, N+1.$

$$\phi(x) = \begin{cases} B_{N+1}\,\mathrm{e}^{-\kappa_{N+1} x}, & \text{for } h_N < x, \\ \left\{ \begin{array}{ll} A_l \sin(\kappa_l x) + B_l \cos(\kappa_l x), & \text{if } \beta^2 < k^2 n_l^2, \\ A_l\,\mathrm{e}^{\kappa_l x} + B_l\,\mathrm{e}^{-\kappa_l x}, & \text{if } \beta^2 > k^2 n_l^2, \end{array} \right\} & \text{for } h_{l-1} < x < h_l, \\ A_0\,\mathrm{e}^{\kappa_0 x}, & \text{for } x < h_0. \end{cases}$$

- $2N+2$ unknowns $A_0, A_1, B_1, \dots, A_N, B_N, B_{N+1}$.
- Continuity of $\phi, \eta\partial_x\phi$ at $N+1$ interfaces $\rightsquigarrow$ $2N+2$ equations.

Dielectric multilayer slab waveguide



Trial value $\beta^2 \in \mathbb{R}$,
$\beta/k > n_0, n_{N+1}$.

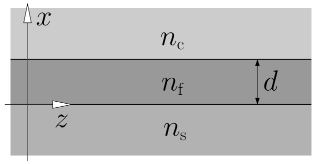
- $2N+2$ unknowns $A_0, A_1, B_1, \dots, A_N, B_N, B_{N+1}$.
- Continuity of $\phi, \eta\partial_x\phi$ at $N+1$ interfaces $\rightsquigarrow$ $2N+2$ equations.
- Arrange as linear system of equations $\mathsf{M}(\beta^2)\,(A_0, \dots, B_{N+1})^{\mathsf{T}} = 0$.
- Identify propagation constants where $\mathsf{M}(\beta^2)$ becomes singular.

(Equations relate to the series of interfaces $\leftrightarrow$ A transfer-matrix technique can be applied.)

- Choose e.g. $A_0 = 1$, fill $A_1, \dots, B_{N+1}$, normalize. $(.\,, .\,; .\,, .)$

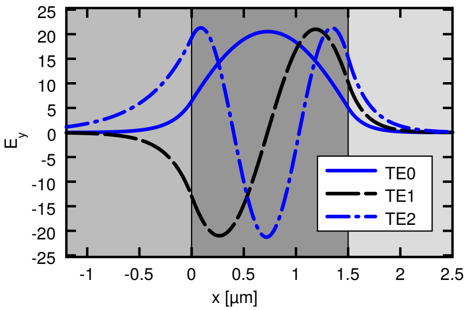
Guided modes $\{\beta_m, (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m)\}$.

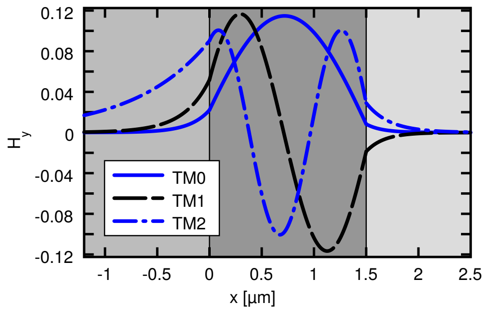
A nonsymmetric 3-layer slab waveguide



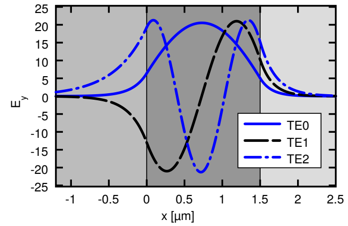
$n_s = 1.45$, $n_f = 1.99$, $n_c = 1.0$,
$d = 1.5\ \mu\text{m}$, $\lambda = 1.55\ \mu\text{m}$.

TE$_0$: $n_{\text{eff}} = 1.944$, TM$_0$: $n_{\text{eff}} = 1.933$,
TE$_1$: $n_{\text{eff}} = 1.804$, TM$_1$: $n_{\text{eff}} = 1.759$,
TE$_2$: $n_{\text{eff}} = 1.562$, TM$_2$: $n_{\text{eff}} = 1.490$.





Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE / TM.)

$$\partial_x(\partial_x\phi) = -(k^2\,n^2 - \beta^2)\phi.$$

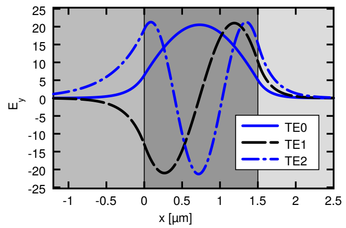
$k^2n^2 - \beta^2$ determines the rate of change of the slope of ϕ.

Imagine a numerical ODE algorithm of "shooting-type".

- Guided modes with a growing number of nodes (x with $\phi(x) = 0$) with decreasing effective indices
  ⟿ mode indices = number of nodes in ϕ.

  "Quantum numbers".
- A fundamental mode with zero nodes and highest effective index.
- Modes of the same polarization are non-degenerate.

Dielectric multilayer slab waveguide, nodal properties



(Fixed polarization, TE / TM.)

$$\partial_x(\partial_x\phi) = -(k^2\,n^2 - \beta^2)\phi.$$

$k^2n^2 - \beta^2$ determines the rate of change of the slope of ϕ.

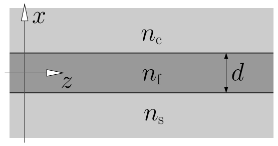
Imagine a numerical ODE algorithm of "shooting-type".

- A sign change of $\partial_x\phi$ is required to form a guided mode
  ⟿ There must be some region (layer) with $k^2n^2 - \beta^2 > 0$.

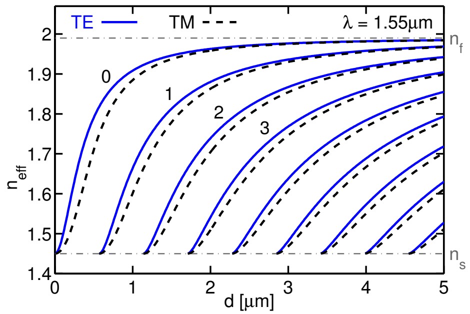
Interval for effective indices n_{eff} of guided modes:

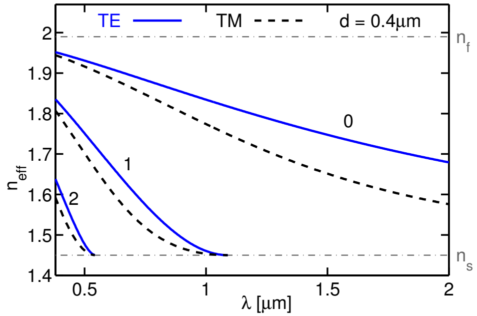
$$\max\{n_0, n_{N+1}\} < n_{\text{eff}} < \max_l\{n_l\}.$$

3-layer slab waveguide, dispersion curves



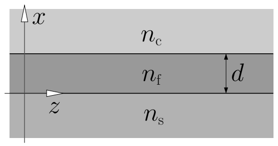
Symmetric waveguide,
moderate refractive index contrast,
$n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$.



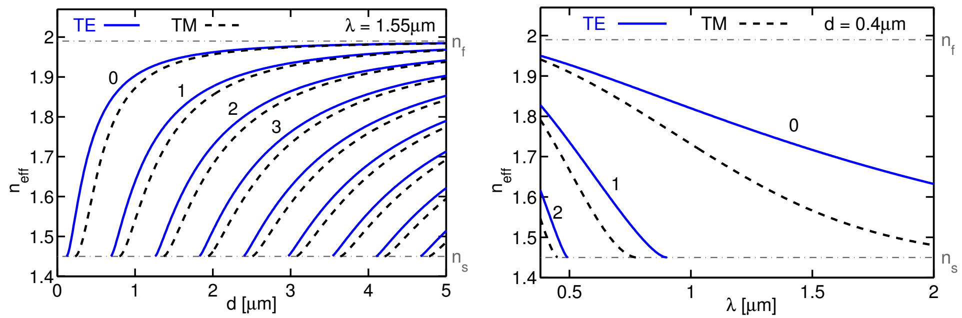


(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves

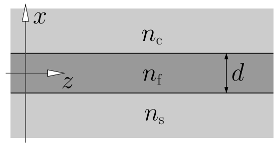


Nonsymmetric waveguide,
moderate refractive index contrast,
$n_{\rm s} = 1.45,\ n_{\rm f} = 1.99,\ n_{\rm c} = 1.0.$

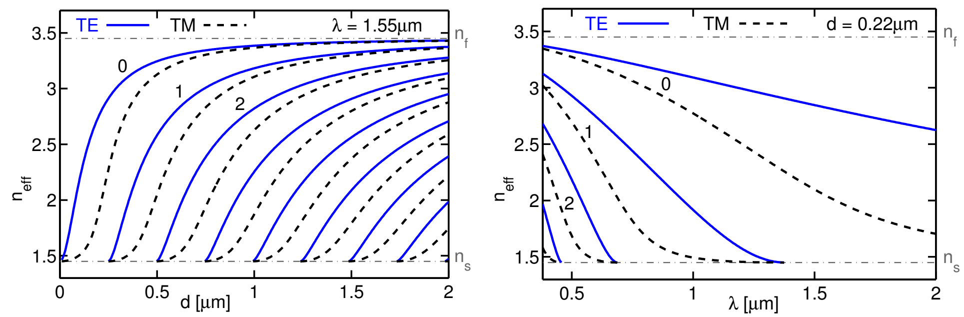


(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves

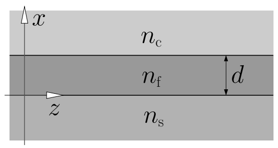


Symmetric waveguide,
high refractive index contrast,
$n_{\rm s} = 1.45,\ n_{\rm f} = 3.45,\ n_{\rm c} = 1.45.$

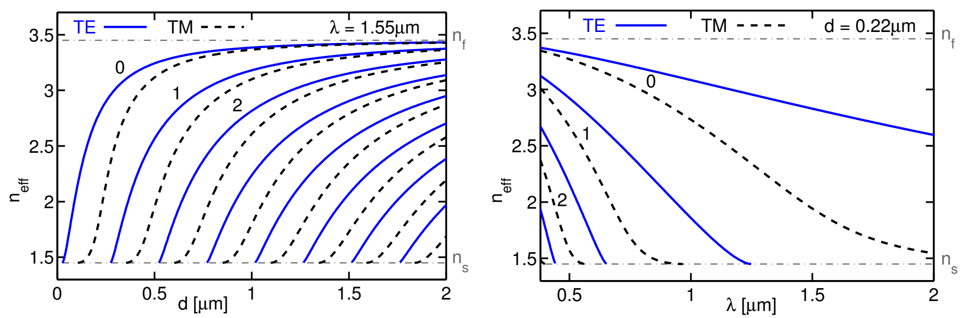


(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves



Nonsymmetric waveguide,
high refractive index contrast,
$n_{\rm s} = 1.45,\ n_{\rm f} = 3.45,\ n_{\rm c} = 1.0.$



(Caution: $\partial_\lambda \epsilon = 0$ assumed !)

3-layer slab waveguide, dispersion curves

Remarks / observations:

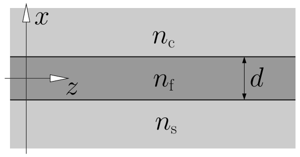
- At large core thicknesses, or short wavelengths, for all modes: $n_{\rm eff}$ approaches the level $n_{\rm f}$ of bulk waves in the core material.
- Modes of higher order at the same $n_{\rm eff}$ supported by waveguides with thickness increased by specific distances.

  Guided mode, layer l with $\kappa_l^2 = (k^2n^2 - \beta^2) > 0$, field $\phi(x) \sim \cos(\kappa_l x + \chi)$ for $x \in$ layer l;
  increase layer thickness by $\Delta x = \pi/\kappa_l$, such that $\kappa_l(x + \Delta x) = \kappa_l x + \pi$
  $\longrightarrow$ the thicker waveguide supports a mode of order $+1$ with the same propagation constant.
- Cutoff thicknesses at fixed wavelength.

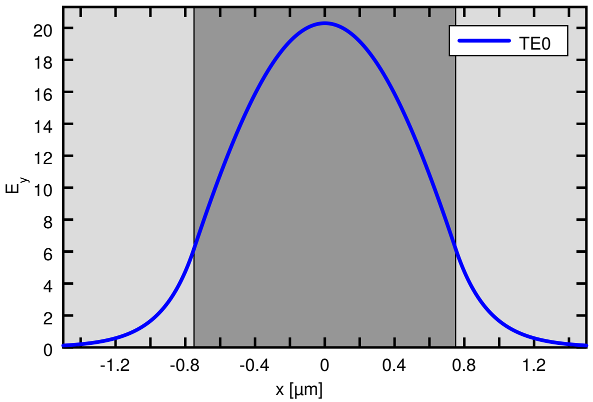
  Nonsymmetric 3-layer waveguide $n_{\rm s} \neq n_{\rm c}$: There exist cutoff thicknesses for all modes.
  Symmetric 3-layer waveguide $n_{\rm s} = n_{\rm c}$: Cutoff thicknesses exist for all modes of order ≥ 1,
  no cutoff thickness for the fundamental TE/TM modes.
- λ is the "length-defining" quantity; wavelength scaling, factor a :
  $n_{\rm eff}(\lambda, d) = n_{\rm eff}(a\lambda, ad), \quad \beta(\lambda, d) = a^{-1}\,\beta(a\lambda, ad).$
- Cutoff wavelengths for waveguides with fixed thickness.

  For all modes; exception: no cutoff wavelength for the fundamental TE/TM modes in a symmetric 3-layer waveguide.

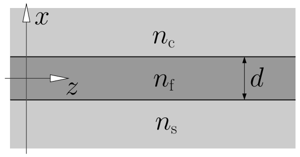
3-layer slab waveguide, mode confinement



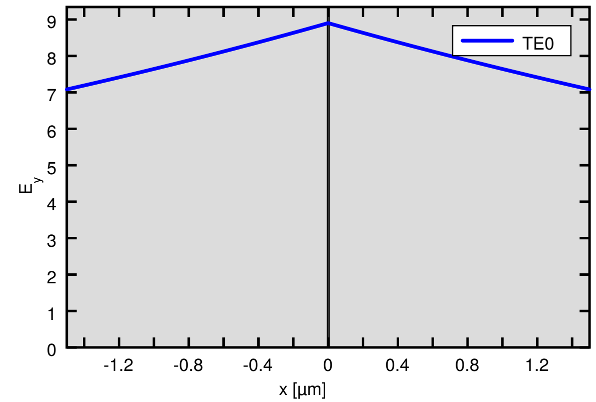
Symmetric waveguide,
moderate refractive index contrast,
$n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55$ µm,
$d = 1.50$ µm, TE_0: $n_{\text{eff}} = 1.946$.



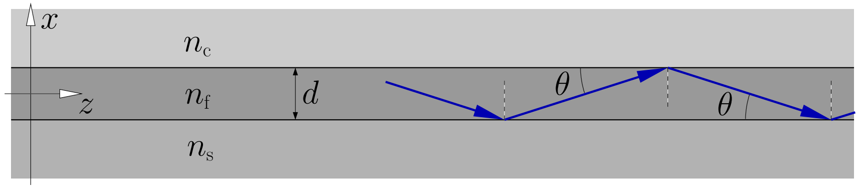
3-layer slab waveguide, mode confinement



Symmetric waveguide,
moderate refractive index contrast,
$n_s = 1.45$, $n_f = 1.99$, $n_c = 1.45$, $\lambda = 1.55$ µm,
$d = 0.01$ µm, TE_0: $n_{\text{eff}} = 1.450$.



3-layer slab waveguide, ray model



Field in the core:

$$\sim a_u \, e^{-i(\kappa x + \beta z)} + a_d \, e^{-i(-\kappa x + \beta z)}, \qquad k^2 n_f^2 = \beta^2 + \kappa^2$$

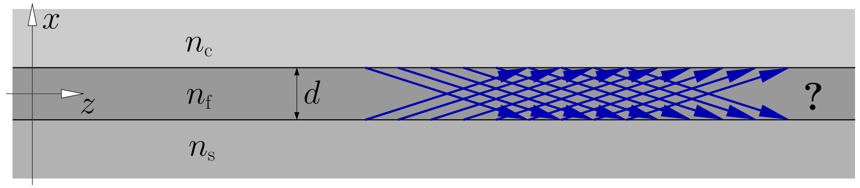
$\leftrightsquigarrow$ propagation angle θ with $\beta = k n_f \cos\theta$, $\kappa = k n_f \sin\theta$.

Guided mode formation:

- Repleated total internal reflection of waves in the core at upper and lower interfaces
- Calculate optical phase gain, including phase jumps for reflection at interfaces (polarization dependent).
- Phase gain of 2π for one "round trip", "transverse resonance condition" $\longleftrightarrow$ constructive interference of waves.

(A frequently encountered intuitive model ... of very limited applicability.)

3-layer slab waveguide, ray model



Field in the core:

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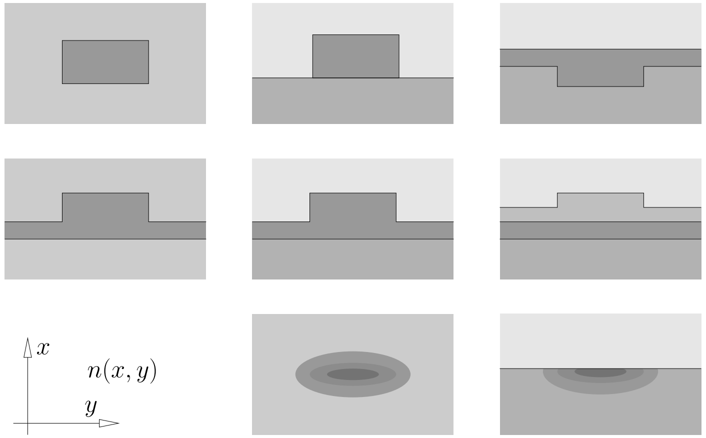
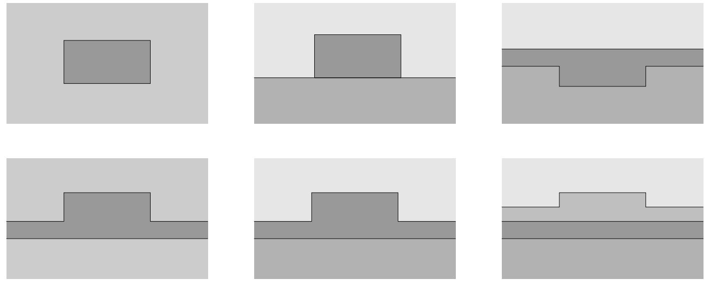
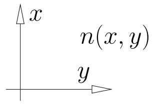
3-D waveguides



Cross sections (2-D) of typical integrated-optical waveguides.

3-D rectangular waveguides

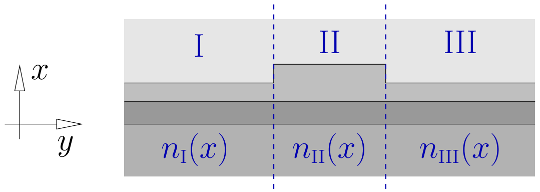




No analytical solutions :

- numerical mode solvers.
- approximations.

Effective index method



Outline: (!)

- Divide into slices $\rho = \mathrm{I}, \mathrm{II}, \mathrm{III}$: $n(x, y) = n_\rho(x)$, if $y \in$ slice ρ.
- Compute polarized modes $X_\rho(x), \beta_\rho$, $X_\rho'' + (k^2 n_\rho^2 - \beta_\rho^2) X_\rho = 0$, $N_\rho = \beta_\rho / k$.
- Consider a scalar mode equation for the principal component Ψ of the 3-D waveguide

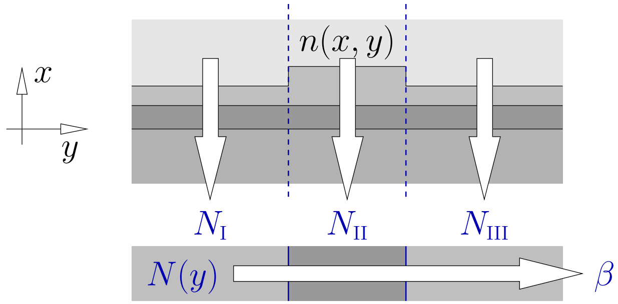
$$\partial_x^2 \Psi + \partial_y^2 \Psi + (k^2 n^2 - \beta^2)\Psi = 0, \qquad \Psi = E_y \text{ (TE)}, \ \Psi = H_y \text{ (TM)}.$$

- Ansatz: $\Psi(x, y) = X_\rho(x)\, Y(y)$, if $y \in$ slice ρ; require continuity of Y and Y'.
- Effective index profile: $N(y) := N_\rho$, if $y \in$ slice ρ.

$$\hookrightarrow \quad Y'' + (k^2 N^2 - \beta^2) Y = 0,$$

a 1-D mode equation for Y, β with the effective index profile N in place of the refractive indices.

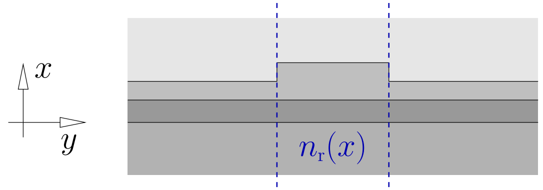
Effective index method, schematically



Remarks / issues:

- A popular, quite intuitive method.
- Frequently an (often informal) basis for discussion of waveguide properties.
- $\leftrightarrow$ Relevance of the slab waveguide model.
- Manifold variants / ways of improvements exist.
- What if a slice does not support a guided slab mode?
- What about higher order modes?
- How to evaluate modal fields? What about other than principal components?
- . . .

Variational effective index method



Outline: (!)

- Identify a reference slice, refractive index profile $n_{\mathrm{r}}(x)$.
- Compute polarized guided slab modes $(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}})_{\mathrm{r}}$, β_{r} for the reference slice.
- For each reference slab mode : ...
- Choose an ansatz: (VEIM)

$$\begin{pmatrix} E_x, E_y, E_z \\ H_x, H_y, H_z \end{pmatrix}(x,y,z) = \begin{pmatrix} 0, & \bar{E}_{\mathrm{r},y}(x)Y^{E_y}(y), & \bar{E}_{\mathrm{r},y}(x)Y^{E_z}(y) \\ \bar{H}_{\mathrm{r},x}(x)Y^{H_x}(y), & \bar{H}_{\mathrm{r},z}(x)Y^{H_y}(y), & \bar{H}_{\mathrm{r},z}(x)Y^{H_z}(y) \end{pmatrix} \qquad \text{(TE)}$$

$$\begin{pmatrix} E_x, E_y, E_z \\ H_x, H_y, H_z \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{E}_{\mathrm{r},x}(x)Y^{E_x}(y), & \bar{E}_{\mathrm{r},z}(x)Y^{E_y}(y), & \bar{E}_{\mathrm{r},z}(x)Y^{E_z}(y) \\ 0, & \bar{H}_{\mathrm{r},y}(x)Y^{H_y}(y), & \bar{H}_{\mathrm{r},y}(x)Y^{H_z}(y) \end{pmatrix} \qquad \text{(TM)}$$

↪ $Y^{\cdot}(y) = ?$

A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

- $$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}, \qquad \beta \in \mathbb{R},$$
  $$\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} \to 0 \ \text{ for } \ x, y \to \pm\infty.$$

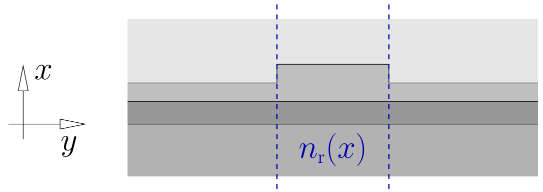
- $$(\mathsf{C} + \mathrm{i}\beta\mathsf{R})\bar{\boldsymbol{E}} = -\mathrm{i}\omega\mu_0\bar{\boldsymbol{H}}, \quad (\mathsf{C} + \mathrm{i}\beta\mathsf{R})\bar{\boldsymbol{H}} = \mathrm{i}\omega\epsilon_0\epsilon\bar{\boldsymbol{E}},$$
  $$\mathsf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

- $$\mathcal{B}(\boldsymbol{E}, \boldsymbol{H}) := \frac{\omega\epsilon_0\langle\boldsymbol{E}, \epsilon\boldsymbol{E}\rangle + \omega\mu_0\langle\boldsymbol{H}, \boldsymbol{H}\rangle + \mathrm{i}\langle\boldsymbol{E}, \mathsf{C}\boldsymbol{H}\rangle - \mathrm{i}\langle\boldsymbol{H}, \mathsf{C}\boldsymbol{E}\rangle}{\langle\boldsymbol{E}, \mathsf{R}\boldsymbol{H}\rangle - \langle\boldsymbol{H}, \mathsf{R}\boldsymbol{E}\rangle},$$
  $$\langle\boldsymbol{F}, \boldsymbol{G}\rangle = \iint \boldsymbol{F}^* \cdot \boldsymbol{G}\, \mathrm{d}x\, \mathrm{d}y.$$

$$\mathcal{B}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{B}(\bar{\boldsymbol{E}} + s\,\delta\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} + s\,\delta\bar{\boldsymbol{H}})\Big|_{s=0} = 0$$

at valid mode fields $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$, for arbitrary $\delta\bar{\boldsymbol{E}}$, $\delta\bar{\boldsymbol{H}}$.

Variational effective index method

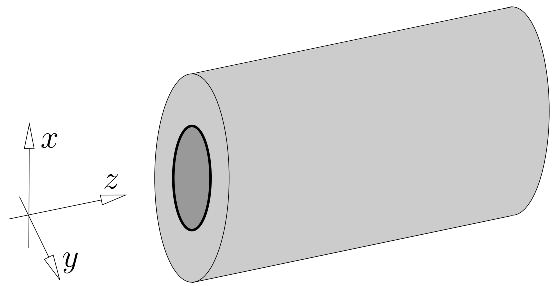


Outline, continued: (!)

- Restrict $\mathcal{B}$ to the VEIM ansatz, require stationarity with respect to the $\{Y^{\cdot}\}$.

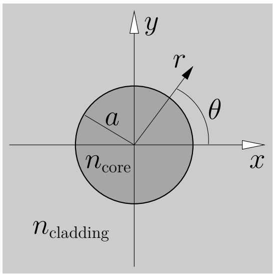
↪ 1-D mode (“-like”) equations for principal unknowns Y^{H_x} (TE) and Y^{E_x} (TM) with effective quantities in place of refractive indices, all other $Y^{\cdot}$ can be computed.

Optical fibers



[Optical Communication A-D]

Circular step index optical fibers



(FD)

Circular symmetry
$\rightsquigarrow$ cylindrical coordinates r, θ, z.

$$\epsilon = n^2, \quad n(r) = \begin{cases} n_{\text{core}}, & r \le a, \\ n_{\text{cladding}}, & r > a. \end{cases}$$

Circular and axial symmetry:

$$\hookrightarrow \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} (r, \theta, z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix} (r)\ \mathrm{e}^{-\mathrm{i} l\theta - \mathrm{i}\beta z}, \qquad l \in \mathbb{Z}, \beta \in \mathbb{R}.$$

$(E_r, E_\theta, E_z, H_r, H_\theta, H_z)$

Where $\partial\epsilon = 0$: $\quad \Delta\psi + k^2 n^2 \psi = 0, \quad \psi \in \{E_r, \dots H_z\}$.

$$\hookrightarrow \quad \partial_r^2 \phi + \frac{1}{r}\partial_r \phi + (k^2 n^2 - \beta^2 - \frac{l^2}{r^2})\phi = 0, \qquad \phi \in \{\bar{E}_r, \dots \bar{H}_z\}$$

(An ODE of Bessel type.)

& vectorial interface conditions at $r = a$. (Alternatively: Scalar theory, LP modes.)

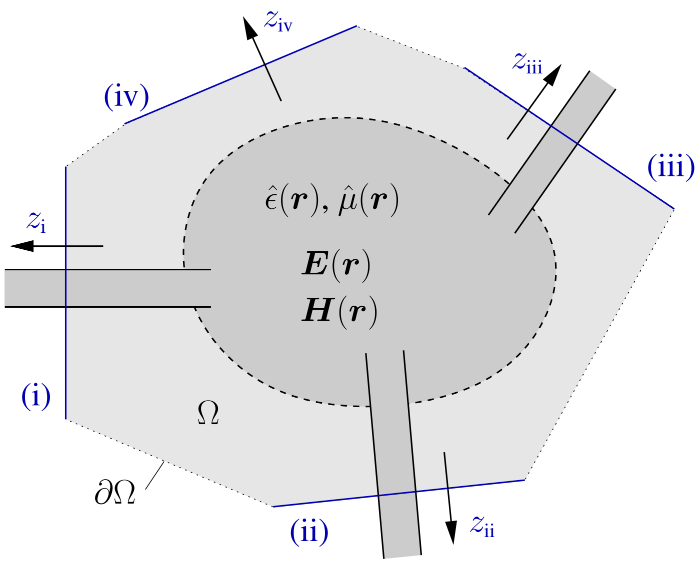
(. . .)

Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

PICs, OICs, scattering matrices



Scattering matrices, prerequisites

$\sim \exp(\mathrm{i}\omega t)$ (FD)

- Passive, linear circuit.
- (Computational) domain of interest Ω, its boundary $\partial\Omega$.
- Connecting channels: lossless waveguides (or “half-spaces”).
- Physical ports $p = \text{i}, \text{ii}, \dots$: waveguide cross-section planes, local coordinates x_p, y_p, z_p; local axis z_p oriented outwards of Ω.
- Establish sets $\mathcal{N}_p$ of *propagating* directional normal modes $\{\psi^d_{p,m} := (\boldsymbol{E}^d_{p,m}, \boldsymbol{H}^d_{p,m}),\ \beta_{p,m};\ d = \text{f,b}\}$ on each port p.

(Restriction to propagating fields: a condition on port positioning / a model assumption.)

- Ports & modes are such that all mode fields vanish on all “other” port planes, and on $\partial\Omega$ outside the ports.

$\hookrightarrow$ Field on port plane p and “outside”:

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} (x_p, y_p, z_p) = \sum_{m \in \mathcal{N}_p} F_{p,m} \psi^{\text{f}}_{p,m}(x_p, y_p)\ \mathrm{e}^{-\mathrm{i}\beta_{p,m} z_p} + B_{p,m} \psi^{\text{b}}_{p,m}(x_p, y_p)\ \mathrm{e}^{\mathrm{i}\beta_{p,m} z_p}.$$

Scattering matrices

$\sim \exp(\mathrm{i}\omega t)$ (FD)

- Merge all mode indices $\{m\}$ and port IDs $\{p\}$ into one set of mode identifiers $\{\nu\}$, $\mathcal{N} = \cup_p \mathcal{N}_p$.
- Assert that $\psi^{\cdot}_{p,\cdot}(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \in \partial\Omega$, $\boldsymbol{r} \notin$ port p.
- Field on $\partial\Omega$: $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = \sum_{\nu\in\mathcal{N}} \{F_\nu \boldsymbol{\psi}^{\mathrm{f}}_\nu + B_\nu \boldsymbol{\psi}^{\mathrm{b}}_\nu\}.$

(Position arguments omitted.)

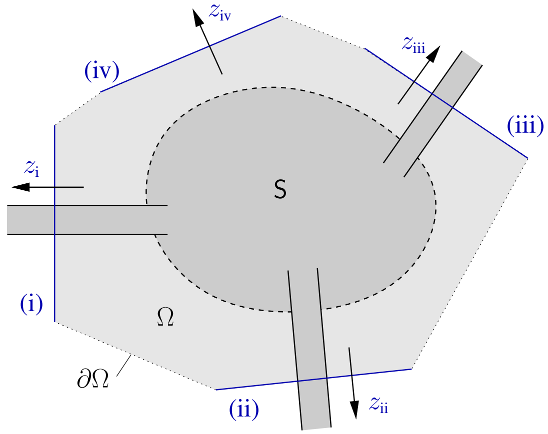
- B_ν : $\sim$ incident modes, traveling towards the interior of Ω.
  F_ν : $\sim$ outgoing modes, traveling towards the exterior of Ω.
  Combine into amplitude vectors $\boldsymbol{B}, \boldsymbol{F}$.

Linear circuit $\leftrightsquigarrow$ linear dependence of $\boldsymbol{F}$ on $\boldsymbol{B}$,
Scattering matrix S of the circuit: $\boldsymbol{F} = \mathsf{S}\,\boldsymbol{B}$, $\mathsf{S} = (S_{\nu\mu})$.

- $S_{\nu\nu}$: $\sim (\nu, \mathrm{b}) \to (\nu, \mathrm{f})$, reflection coefficient for mode ν.
- $S_{\nu\mu}$: $\sim (\mu, \mathrm{b}) \to (\nu, \mathrm{f})$, transmission coefficient for modes μ, ν.

PICs, OICs, scattering matrices, scenarios

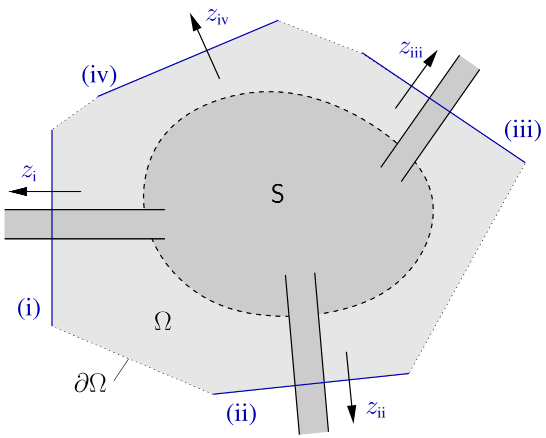
$\sim \exp(\mathrm{i}\omega t)$ (FD)



- Scenario: Full matrix S, including guided and radiation modes, large $\dim \mathsf{S}$ $\leftrightarrow$ theoretical results.

PICs, OICs, scattering matrices, scenarios

$\sim \exp(\mathrm{i}\omega t)$ (FD)

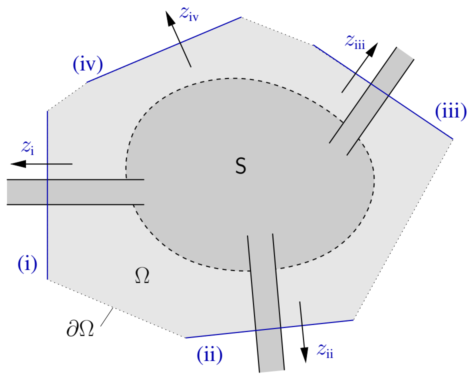


- Scenario: Restrict to a specific set of (guided) modes, or:
  Only a small set of guided modes are relevant:
  small $\dim \mathsf{S} = N \times N$ $\leftrightarrow$ an N-port circuit, a 2-N-pole.

(N: the total number of relevant modes, not the number of ports.)

Scattering matrices, port plane positions

$\sim \exp(\mathrm{i}\omega t)$ (FD)

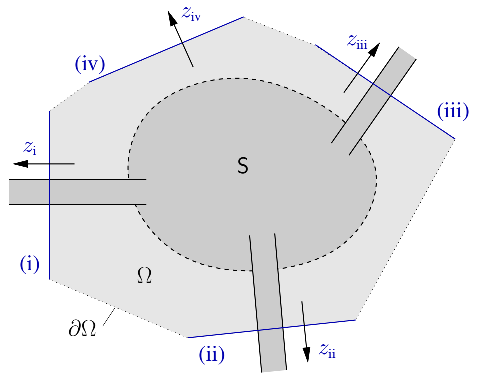


- Shift port plane of mode ν by Δz_ν: $F_\nu \to F'_\nu = F_\nu\, \mathrm{e}^{-\mathrm{i}\beta_\nu \Delta z_\nu}$,
  Shift port plane of mode μ by Δz_μ: $B_\mu \to B'_\mu = B_\mu\, \mathrm{e}^{\mathrm{i}\beta_\mu \Delta z_\mu}$,

$F'_\nu = S'_{\nu\mu} B'_\mu, \quad S'_{\nu\mu} = S_{\nu\mu}\, \mathrm{e}^{-\mathrm{i}(\beta_\nu \Delta z_\nu + \beta_\mu \Delta z_\mu)}.$

(Moving port planes $\leftrightarrow$ Phase change in reflection / transmission coefficients.)
(Moving port planes $\leftrightarrow$ No effect on reflectances / transmittances.)

Scattering matrices, port mode orthogonality



$\sim \exp(\mathrm{i}\omega t)$ (FD)

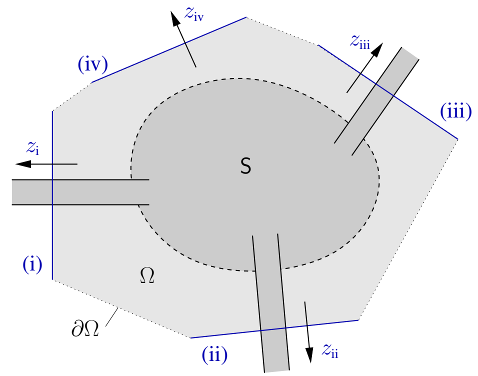
- Orthogonality relations on port plane p:

$$(\boldsymbol{E}_a, \boldsymbol{H}_a; \boldsymbol{E}_b, \boldsymbol{H}_b) = \frac{1}{4} \iint_p \left(E^*_{ax} H_{by} - E^*_{ay} H_{bx} + H^*_{ay} E_{bx} - H^*_{ax} E_{by} \right) \mathrm{d}x_p\, \mathrm{d}y_p$$

$$(\boldsymbol{\psi}^d_{p,l}; \boldsymbol{\psi}^r_{p,m}) = \pm \delta_{dr} \delta_{lm} P_{p,m}.$$

(Things restricted to propagating modes.)

Scattering matrices, port mode orthogonality



$\sim \exp(\mathrm{i}\omega t)$ (FD)

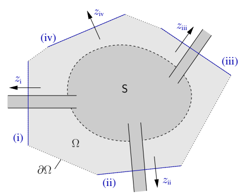
- Extend to the full boundary $\partial\Omega$:

$$(\boldsymbol{E}_a, \boldsymbol{H}_a; \boldsymbol{E}_b, \boldsymbol{H}_b) := \frac{1}{4} \int_{\partial\Omega} (\boldsymbol{E}^*_a \times \boldsymbol{H}_b + \boldsymbol{E}_b \times \boldsymbol{H}^*_a) \cdot \mathrm{d}\boldsymbol{a}$$

$$(\boldsymbol{\psi}^d_{p,l}; \boldsymbol{\psi}^r_{q,m}) = \pm \delta_{dr} \delta_{pq} \delta_{lm} P_{p,m} \quad \text{or} \quad (\boldsymbol{\psi}^d_{\nu}; \boldsymbol{\psi}^r_{\mu}) = \pm \delta_{dr} \delta_{\nu\mu} P_{\nu}.$$

(Modes belonging to different ports are mutually orthogonal.)

Scattering matrices, power balance



$\sim \exp(\mathrm{i}\omega t)$ (FD)

- Net power outflow across the border of the circuit:

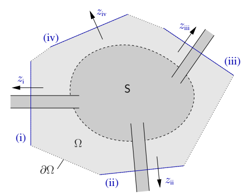
$$P = \int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H}) = \sum_p \sum_{m \in \mathcal{N}_p} \left(|F_{p,m}|^2 - |B_{p,m}|^2 \right) P_{p,m}$$
$$= \sum_{\nu \in \mathcal{N}} \left(|F_\nu|^2 - |B_\nu|^2 \right) P_\nu \,,$$

$|B_\mu|^2 P_\mu$: incident power carried by mode μ,
$|F_\nu|^2 P_\nu$: outgoing power carried by mode ν, $\qquad F_\nu = S_{\nu\mu} B_\mu$.

$$|S_{\nu\mu}|^2 \frac{P_\nu}{P_\mu} = \frac{|F_\nu|^2 P_\nu}{|B_\mu|^2 P_\mu}, \quad \begin{array}{ll} \mu \neq \nu: & \text{power transmittance } \mu \to \nu, \\ \mu = \nu: & \text{power reflectance for mode } \nu. \end{array}$$

(Uniform normalized modes, $P_\nu = P_\mu$: transmittances are directly given by elements of the scattering matrix).

Scattering matrices, power balance



$\sim \exp(\mathrm{i}\omega t)$ (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = (\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{E}, \boldsymbol{H}) = P_0 \left(\boldsymbol{B}^* \cdot (\mathsf{S}^\dagger \mathsf{S} - 1) \boldsymbol{B} \right),$$

uniform normalization, $P_\nu = P_0$ for all ν.

- Lossless circuit $\longleftrightarrow$ $\int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} = 0$ $\longleftrightarrow$ $\mathsf{S}^\dagger \mathsf{S} = 1$,

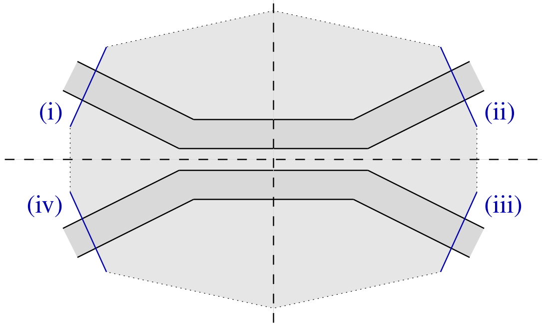
  the scattering matrix of a lossless circuit is unitary.

- Lossy circuit $\longleftrightarrow$ $\int_{\partial\Omega} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} \leq 0$ $\rightsquigarrow$ $\boldsymbol{B}^* \cdot \mathsf{S}^\dagger \mathsf{S} \boldsymbol{B} \leq \boldsymbol{B}^* \boldsymbol{B}$,

  $\sum_\nu |S_{\nu\mu}|^2 \leq 1$ for all μ. (The sum of transmittances mode μ to all other modes ν is less than one.)

(Interior lossy media, or radiative losses: outgoing propagating modes not taken into account.)

Scattering matrices, symmetry

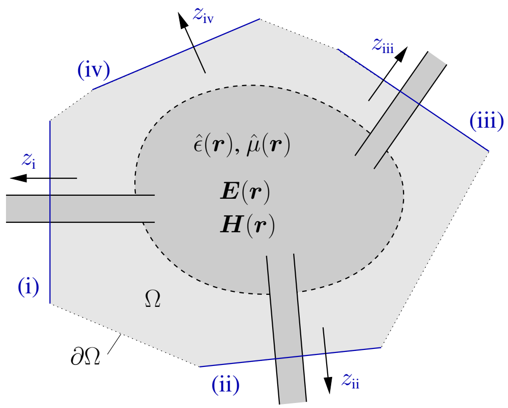


Circuit with specific spatial symmetry
& symmetrical setting of the port planes

respective symmetry in related coefficients of S,
symmetric power transmission properties.

Scattering matrices, reciprocity

$\sim \exp(\mathrm{i}\omega t)$ (FD)



Circuit properties
for reversed
wave propagation ?
$S_{\nu\mu} \leftrightarrow S_{\mu\nu}$?

- $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$ solve $\nabla\times\boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H}$, $\nabla\times\boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}$.

$\nabla\cdot(\boldsymbol{E}_1\times\boldsymbol{H}_2 + \boldsymbol{H}_1\times\boldsymbol{E}_2) = 0$, if $\hat{\epsilon}$ and $\hat{\mu}$ are symmetric.

(i.e. if $\hat{\epsilon}^{\mathsf{T}} = \hat{\epsilon}$, $\hat{\mu}^{\mathsf{T}} = \hat{\mu}$.)
(Note: order of factors, no complex conjugates.)

Scattering matrices, reciprocity

- $\boldsymbol{E}_1, \boldsymbol{H}_1$ and $\boldsymbol{E}_2, \boldsymbol{H}_2$ solve $\nabla\times\boldsymbol{E} = -\mathrm{i}\omega\mu_0\hat{\mu}\boldsymbol{H}$, $\nabla\times\boldsymbol{H} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\boldsymbol{E}$

$\nabla\cdot(\boldsymbol{E}_1\times\boldsymbol{H}_2 + \boldsymbol{H}_1\times\boldsymbol{E}_2) = 0$, if $\hat{\epsilon}$ and $\hat{\mu}$ are symmetric,

$$0 = \int_\Omega \nabla\cdot(\boldsymbol{E}_1\times\boldsymbol{H}_2 + \boldsymbol{H}_1\times\boldsymbol{E}_2)\,\mathrm{d}^3r = \int_{\partial\Omega}(\boldsymbol{E}_1\times\boldsymbol{H}_2 + \boldsymbol{H}_1\times\boldsymbol{E}_2)\cdot\mathrm{d}\boldsymbol{a}.$$

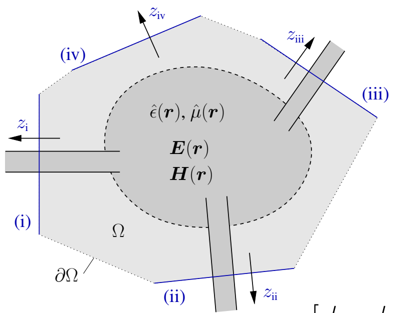
- Fields on $\partial\Omega$: $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}_j = \sum_{\nu\in\mathcal{N}}\{F_{j,\nu}\boldsymbol{\psi}^{\rm f}_\nu + B_{j,\nu}\boldsymbol{\psi}^{\rm b}_\nu\}$, $\quad j = 1, 2,$

$$[\boldsymbol{\psi}_a;\boldsymbol{\psi}_b] := \int_{\partial\Omega}(\boldsymbol{E}_a\times\boldsymbol{H}_b + \boldsymbol{H}_a\times\boldsymbol{E}_b)\cdot\mathrm{d}\boldsymbol{a},$$

$$0 = \sum_\nu\sum_\mu\big(\;F_{1,\nu}F_{2,\mu}[\boldsymbol{\psi}^{\rm f}_\nu;\boldsymbol{\psi}^{\rm f}_\mu] + F_{1,\nu}B_{2,\mu}[\boldsymbol{\psi}^{\rm f}_\nu;\boldsymbol{\psi}^{\rm b}_\mu] + B_{1,\nu}F_{2,\mu}[\boldsymbol{\psi}^{\rm b}_\nu;\boldsymbol{\psi}^{\rm f}_\mu] + B_{1,\nu}B_{2,\mu}[\boldsymbol{\psi}^{\rm b}_\nu;\boldsymbol{\psi}^{\rm b}_\mu]\big).$$

Scattering matrices, reciprocity

$\sim \exp(\mathrm{i}\omega t)$ (FD)



$$[\boldsymbol{\psi}_a;\boldsymbol{\psi}_b] := \int_{\partial\Omega}(\boldsymbol{E}_a\times\boldsymbol{H}_b + \boldsymbol{H}_a\times\boldsymbol{E}_b)\cdot\mathrm{d}\boldsymbol{a}.$$

- $[\boldsymbol{\psi}^{\cdot}_\nu;\boldsymbol{\psi}^{\cdot}_\mu] = 0$, if ν and μ relate to different ports.
- If ν and μ relate to the same port plane p:
$[\boldsymbol{\psi}^r_\nu;\boldsymbol{\psi}^d_\mu] = \iint_p \big(E^r_{\nu x}H^d_{\mu y} - E^r_{\nu y}H^d_{\mu x} - H^r_{\nu y}E^d_{\mu x} + H^r_{\nu x}E^d_{\mu y}\big)\,\mathrm{d}x_p\,\mathrm{d}y_p.$

Scattering matrices, reciprocity

- If ν and μ relate to the same port plane p:

$$[\psi^r_\nu; \psi^d_\mu] = \iint_p \left(E^r_{\nu x}H^d_{\mu y} - E^r_{\nu y}H^d_{\mu x} - H^r_{\nu y}E^d_{\mu x} + H^r_{\nu x}E^d_{\mu y}\right) \mathrm{d}x_p\, \mathrm{d}y_p.$$

- Compare with the modal orthogonality relations on port plane p, for propagating modes with real transverse components:

$$(\psi^r_\nu; \psi^d_\mu) = \frac{1}{4}\iint_p \left(E^r_{\nu x}H^d_{\mu y} - E^r_{\nu y}H^d_{\mu x} + H^r_{\nu y}E^d_{\mu x} - H^r_{\nu x}E^d_{\mu y}\right) \mathrm{d}x_p\, \mathrm{d}y_p,$$

$$(\psi^{\mathrm{f}}_\nu; \psi^{\mathrm{f}}_\mu) = \delta_{\nu\mu}P_\nu, \quad (\psi^{\mathrm{b}}_\nu; \psi^{\mathrm{b}}_\mu) = -\delta_{\nu\mu}P_\nu, \quad (\psi^{\mathrm{f}}_\nu; \psi^{\mathrm{b}}_\mu) = (\psi^{\mathrm{b}}_\nu; \psi^{\mathrm{f}}_\mu) = 0.$$

- $$\psi^{\mathrm{f}} = (E_x, E_y, \mathrm{i}E_z, H_x, H_y, \mathrm{i}H_z)^{\mathsf{T}}$$
  $$\leftrightarrow \quad \psi^{\mathrm{b}} = (E_x, E_y, -\mathrm{i}E_z, -H_x, -H_y, \mathrm{i}H_z)^{\mathsf{T}}.$$ (Real components).

$$[\psi^{\mathrm{f}}_\nu; \psi^{\mathrm{f}}_\mu] = [\psi^{\mathrm{b}}_\nu; \psi^{\mathrm{b}}_\mu] = 0, \quad [\psi^{\mathrm{f}}_\nu; \psi^{\mathrm{b}}_\mu] = -\delta_{\nu\mu}4P_\nu, \quad [\psi^{\mathrm{b}}_\nu; \psi^{\mathrm{f}}_\mu] = \delta_{\nu\mu}4P_\nu.$$

Scattering matrices, reciprocity

$$0 = \sum_\nu 4P_\nu \left(B_{1,\nu}F_{2,\nu} - F_{1,\nu}B_{2,\nu}\right),$$

uniform normalization $P_\nu = P_0$,

$$0 = \sum_\nu \left(B_{1,\nu}F_{2,\nu} - F_{1,\nu}B_{2,\nu}\right),$$

$$0 = \boldsymbol{B}_1 \cdot \boldsymbol{F}_2 - \boldsymbol{F}_1 \cdot \boldsymbol{B}_2,$$

$\boldsymbol{F}_j = \mathsf{S}\boldsymbol{B}_j$,

$$0 = \boldsymbol{B}_1 \cdot \mathsf{S}\boldsymbol{B}_2 - (\mathsf{S}\boldsymbol{B}_1) \cdot \boldsymbol{B}_2,$$

$$0 = \boldsymbol{B}_1 \cdot \mathsf{S}\boldsymbol{B}_2 - \boldsymbol{B}_1 \cdot \mathsf{S}^{\mathsf{T}}\boldsymbol{B}_2,$$

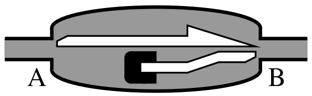
$$0 = \boldsymbol{B}_1 \cdot \left(\mathsf{S} - \mathsf{S}^{\mathsf{T}}\right)\boldsymbol{B}_2 \quad \text{for all } \boldsymbol{B}_1, \boldsymbol{B}_2.$$

$\mathsf{S} = \mathsf{S}^{\mathsf{T}}$, $\quad S_{\nu\mu} = S_{\mu\nu}$ for all ν, μ.

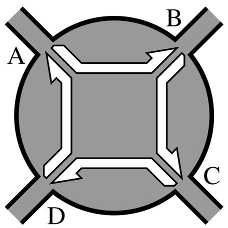
The scattering matrix of a reciprocal circuit is symmetric.

Reciprocal circuit: made of reciprocal media, with $\hat{\epsilon} = \hat{\epsilon}^{\mathsf{T}}$, $\hat{\mu} = \hat{\mu}^{\mathsf{T}}$.

Nonreciprocal devices



Isolator:
unidirectional transmission,
$S_{\mathrm{BA}} = 1, \quad S_{\mathrm{AB}} = 0.$



Circulator:
transmission cycle,
$S_{\mathrm{BA}} = 1, \quad S_{\mathrm{CB}} = 1, \quad S_{\mathrm{DC}} = 1, \quad S_{\mathrm{AD}} = 1,$
$S_{\cdot\cdot} = 0$ otherwise.

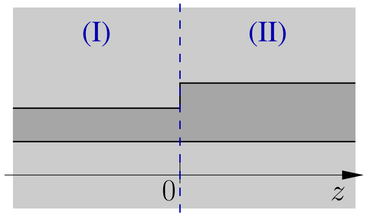
Required: nonreciprocal media with $\hat{\epsilon} \neq \hat{\epsilon}^{\mathsf{T}}$,
$\leftrightarrow$ magnetooptic media, Faraday effect.

Nonreciprocal devices

What about, for example,

- a long, "adiabatic" Y-junction ?
- a junction between a single mode core and a wider multimode waveguide ?

Waveguide discontinuities



Half-infinite waveguides (I), (II), discontinuity at $z = 0$.

- Expand into local normal modes $\{\psi^d_{s,m}, \beta_{s,m}\},\ m \in \mathcal{N}_s,\ s = \mathrm{I}, \mathrm{II}:$

  Transverse boundary conditions $\leftrightsquigarrow$ discrete sets.

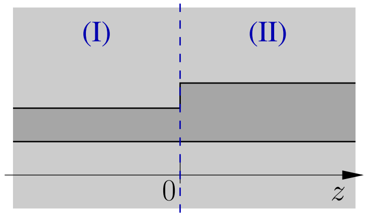
$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}_s(x,y,z) = \sum_{m\in\mathcal{N}_s}\left\{f_{s,m}\psi^{\mathrm{f}}_{s,m}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_{s,m}z} + b_{s,m}\psi^{\mathrm{b}}_{s,m}(x,y)\ \mathrm{e}^{+\mathrm{i}\beta_{s,m}z}\right\},$$

$z < 0$: $s = \mathrm{I}$, $f_{\mathrm{I},m}$ given influx, $b_{\mathrm{I},m}$ unknown,
$z > 0$: $s = \mathrm{II}$, $f_{\mathrm{II},m}$ unknown, $b_{\mathrm{II},m}$ given influx.

$\hookrightarrow$ $(\boldsymbol{E},\boldsymbol{H})_{\mathrm{I,II}}$ are solutions for $z < 0$ and $z > 0$.

- Continuity of the tangential components of $\boldsymbol{E},\boldsymbol{H}$ at the interface
  $\leftrightsquigarrow$ formally equate expressions for $(\boldsymbol{E},\boldsymbol{H})_{\mathrm{I,II}}$ at $z = 0$.
  (Only equality of E_x, E_y, H_x, H_y will be relevant.)
- Project on $\psi^d_{s,l}$ to extract coefficients ...

Waveguide discontinuities, scattering matrix



(Global coordinate $z \neq$ former local coordinate on port I.)
(One variant of a projection procedure.)

- $(\psi^{\mathrm{b}}_{\mathrm{I},l};\ \cdot = \cdot\ ),\ \ l \in \mathcal{N}_{\mathrm{I}}:$

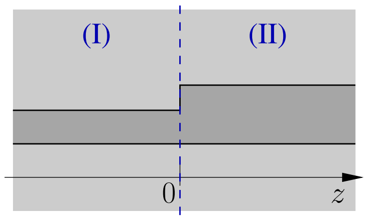
$$\sum_{m\in\mathcal{N}_{\mathrm{I}}}\left[f_{\mathrm{I},m}(\psi^{\mathrm{b}}_{\mathrm{I},l};\psi^{\mathrm{f}}_{\mathrm{I},m}) + b_{\mathrm{I},m}(\psi^{\mathrm{b}}_{\mathrm{I},l};\psi^{\mathrm{b}}_{\mathrm{I},m})\right] = \sum_{m\in\mathcal{N}_{\mathrm{II}}}\left[f_{\mathrm{II},m}(\psi^{\mathrm{b}}_{\mathrm{I},l};\psi^{\mathrm{f}}_{\mathrm{II},m}) + b_{\mathrm{II},m}(\psi^{\mathrm{b}}_{\mathrm{I},l};\psi^{\mathrm{b}}_{\mathrm{II},m})\right],$$

- $(\psi^{\mathrm{f}}_{\mathrm{II},l};\ \cdot = \cdot\ ),\ \ l \in \mathcal{N}_{\mathrm{II}}:$

$$\sum_{m\in\mathcal{N}_{\mathrm{I}}}\left[f_{\mathrm{I},m}(\psi^{\mathrm{f}}_{\mathrm{II},l};\psi^{\mathrm{f}}_{\mathrm{I},m}) + b_{\mathrm{I},m}(\psi^{\mathrm{f}}_{\mathrm{II},l};\psi^{\mathrm{b}}_{\mathrm{I},m})\right] = \sum_{m\in\mathcal{N}_{\mathrm{II}}}\left[f_{\mathrm{II},m}(\psi^{\mathrm{f}}_{\mathrm{II},l};\psi^{\mathrm{f}}_{\mathrm{II},m}) + b_{\mathrm{II},m}(\psi^{\mathrm{f}}_{\mathrm{II},l};\psi^{\mathrm{b}}_{\mathrm{II},m})\right],$$

$\hookrightarrow$ ... $\rightsquigarrow$ $\begin{pmatrix}\boldsymbol{b}_{\mathrm{I}}\\ \boldsymbol{f}_{\mathrm{II}}\end{pmatrix} = \mathsf{S}\begin{pmatrix}\boldsymbol{f}_{\mathrm{I}}\\ \boldsymbol{b}_{\mathrm{II}}\end{pmatrix} = \begin{pmatrix}\mathsf{S}_{\mathrm{I,I}} & \mathsf{S}_{\mathrm{I,II}}\\ \mathsf{S}_{\mathrm{II,I}} & \mathsf{S}_{\mathrm{II,II}}\end{pmatrix}\begin{pmatrix}\boldsymbol{f}_{\mathrm{I}}\\ \boldsymbol{b}_{\mathrm{II}}\end{pmatrix}.$

Waveguide discontinuities, overlap model



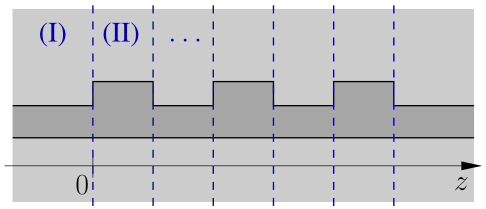
Most simplified variant:
Unidirectional overlap model.

- (I) : Incoming guided mode ψ_{I}, reflections & radiation neglected.
  (II) : Outgoing guided modes $\psi_{\mathrm{II},m}$, radiation neglected.
- $f_{\mathrm{I}}\,\psi_{\mathrm{I}} \approx \sum_m f_{\mathrm{II},m}\,\psi_{\mathrm{II},m}$ at $z = 0$.

$\hookrightarrow$ $f_{\mathrm{II},m} = \dfrac{(\psi_{\mathrm{II},m};\psi_{\mathrm{I}})}{(\psi_{\mathrm{II},m};\psi_{\mathrm{II},m})}\,f_{\mathrm{I}}$, or $f_{\mathrm{II},m} = \dfrac{1}{P_{\mathrm{II},m}}(\psi_{\mathrm{II},m};\psi_{\mathrm{I}})\,f_{\mathrm{I}}$.

(Transmission is given directly by the "overlaps" $\leftrightsquigarrow$ Relevance of the mode products $(\,\cdot\,;\,\cdot\,)$.)
(Cf. explicit expressions for overlaps of 2-D modes, involving only principal mode profile components.)

A sequence of waveguide discontinuities



- Divide into segments.
- Establish local normal mode expansions.
- Project on local modes.

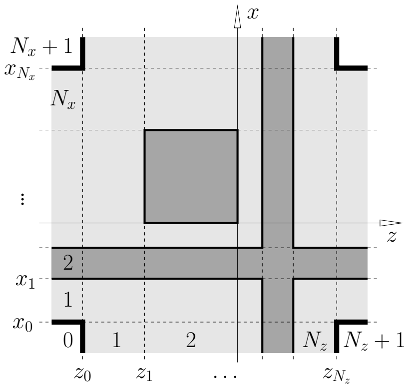
$\hookrightarrow$ Linear system of equations for all local mode amplitudes.

$\hookrightarrow$ Solve (...) $\rightsquigarrow$ $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z)$.

Bidirectional eigenmode propagation (BEP),
Eigenmode expansion method (EME),
... .

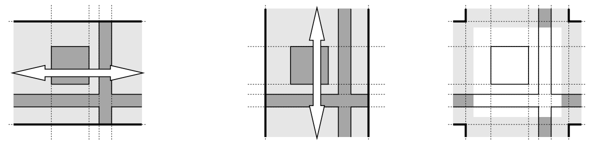
(Radiated outgoing fields: Open boundary conditions required (PMLs) $\leftrightsquigarrow$ Complex eigenmodes.)
(2-D: ok. 3-D: ?)

Rectangular 2-D circuits



Quadridirectional Eigenmode Propagation (QUEP)

- Divide into slices & layers.
- Establish local modes:
  Propagation along $\pm z$,
  & Propagation along $\pm x$,
  boundary conditions $\phi = 0$.
- Project at horizontal & vertical interfaces.



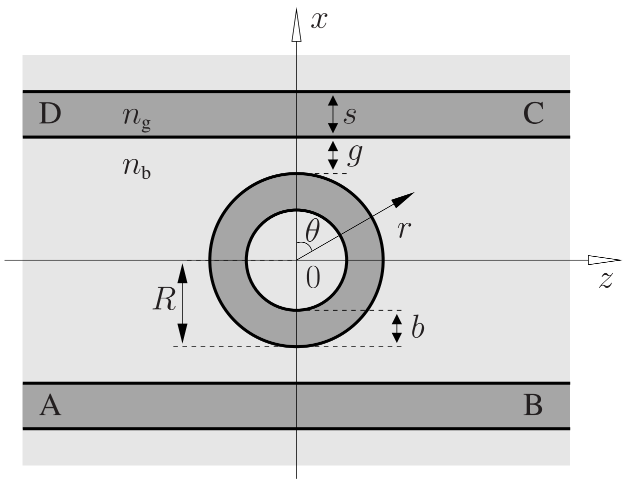
horizontal BEP, vertical BEP, continuity at x_0, x_N, z_0, z_N.

Course overview

Optical waveguide theory

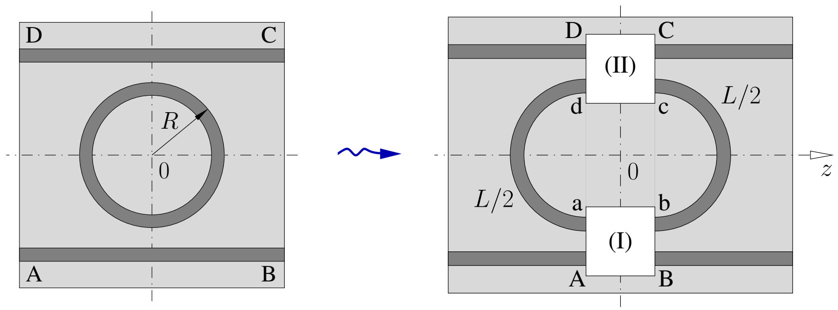
- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

Circular traveling wave resonators



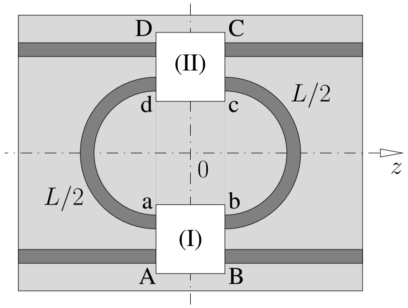
Integrated optical micro-ring or micro-disk resonators.

Ringresonator: Abstract model



- Ringresonator $\approx$ 2 couplers + 2 cavity segments
- CW description: $\boldsymbol{E}, \boldsymbol{H} \sim \mathrm{e}^{\mathrm{i}\omega t}$, $\omega = k\,\mathrm{c}$, $k = 2\pi/\lambda$.

Couplers: Scattering matrices



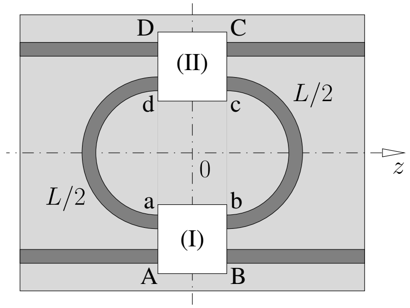
- Uniform polarization, single mode waveguides.
- Linear, nonmagnetic (attenuating) elements.
- Backreflections are negligible.
- Interaction restricted to the couplers ↔ “port” definition.

Symmetric coupler scattering matrices :

$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \chi & \tau \\ \rho & \chi & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

$A_\pm,\ B_\pm,\ a_\pm,\ b_\pm$: Amplitudes of waves traveling in $\pm z$-direction.

Coupler symmetries



Symmetry $z \to -z$:

$A_+ \to b_+ \quad \overset{!}{=} \quad B_- \to a_-$

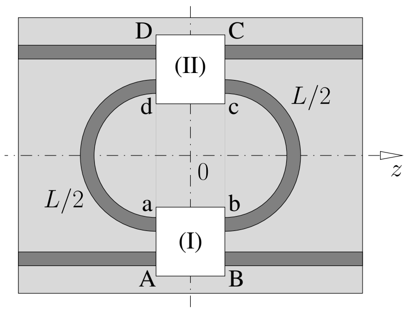
$$\begin{pmatrix} A_- \\ a_- \\ B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho & \kappa \\ 0 & 0 & \kappa & \tau \\ \rho & \kappa & 0 & 0 \\ \kappa & \tau & 0 & 0 \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \\ B_- \\ b_- \end{pmatrix}$$

$$\begin{pmatrix} A_- \\ a_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} B_- \\ b_- \end{pmatrix}, \qquad \begin{pmatrix} B_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} A_+ \\ a_+ \end{pmatrix}.$$

Symmetry $x \to -x$, (I) = (II) :

$$\begin{pmatrix} D_- \\ d_- \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} C_- \\ c_- \end{pmatrix}, \qquad \begin{pmatrix} C_+ \\ c_+ \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ \kappa & \tau \end{pmatrix} \begin{pmatrix} D_+ \\ d_+ \end{pmatrix}.$$

Cavity segments



Field evolution $\sim \mathrm{e}^{-\mathrm{i}\gamma s}$ along the cavity core, propagation distance s.

$\gamma = \beta - \mathrm{i}\alpha$,

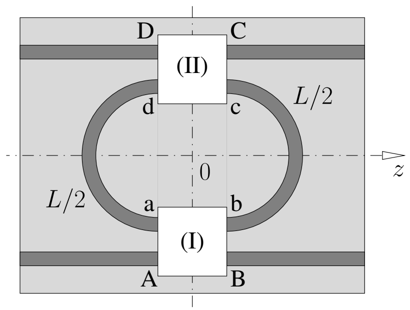
β : phase propagation constant,
α : attenuation constant.

(↔ bend modes, to come.)

Relations of amplitudes at the ends of the cavity segments :

$$c_- = b_+\,\mathrm{e}^{-\mathrm{i}\beta L/2}\ \mathrm{e}^{-\alpha L/2}, \qquad a_+ = d_-\,\mathrm{e}^{-\mathrm{i}\beta L/2}\ \mathrm{e}^{-\alpha L/2},$$
$$b_- = c_+\,\mathrm{e}^{-\mathrm{i}\beta L/2}\ \mathrm{e}^{-\alpha L/2}, \qquad d_+ = a_-\,\mathrm{e}^{-\mathrm{i}\beta L/2}\ \mathrm{e}^{-\alpha L/2}.$$

Output amplitudes



Coupler scattering matrices
+ Cavity field evolution
+ External input amplitudes

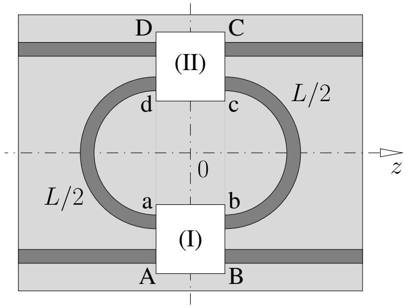
$A_+ = \sqrt{P_{\rm in}}$,
$B_- = C_- = D_+ = 0$

External output amplitudes :

$$A_- = 0, \quad C_+ = 0, \quad D_- = \frac{\kappa^2 p}{1-\tau^2 p^2} A_+, \quad B_+ = \left(\rho + \frac{\kappa^2 \tau p^2}{1-\tau^2 p^2}\right) A_+,$$

$p = \mathrm{e}^{-\mathrm{i}\beta L/2}\ \mathrm{e}^{-\alpha L/2}$.

Power transfer



Power drop : $P_{\mathrm{D}} = |D_-|^2$,

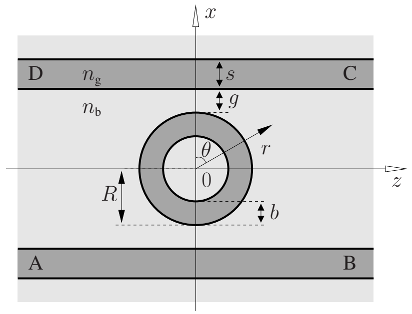
Transmission : $P_{\mathrm{T}} = |B_+|^2$.

$$P_{\mathrm{D}} = P_{\mathrm{in}} \frac{|\kappa|^4 \,\mathrm{e}^{-\alpha L}}{1 + |\tau|^4 \,\mathrm{e}^{-2\alpha L} - 2|\tau|^2 \,\mathrm{e}^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$P_{\mathrm{T}} = P_{\mathrm{in}} \frac{|\rho|^2 (1 + |\tau|^2 d^2 \,\mathrm{e}^{-2\alpha L} - 2|\tau| d \,\mathrm{e}^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 \,\mathrm{e}^{-2\alpha L} - 2|\tau|^2 \,\mathrm{e}^{-\alpha L} \cos(\beta L - 2\varphi)}$$

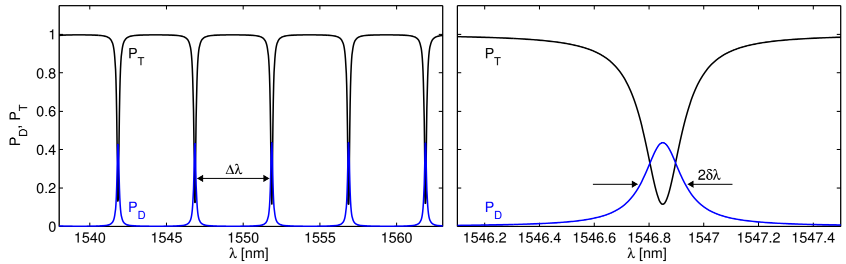
$\tau =: |\tau| \,\mathrm{e}^{\mathrm{i}\varphi}, \;\; d \,\mathrm{e}^{\mathrm{i}\psi} := \tau - \kappa^2/\rho, \;\; L \neq 2\pi R.$

Spectral response



$R = 50\,\mu\mathrm{m}, \; b = s = 1.0\,\mu\mathrm{m}, \; g = 0.9\,\mu\mathrm{m},$
$n_{\mathrm{b}} = 1.45, \; n_{\mathrm{g}} = 1.60; \;$ 2-D, TE.

$\Delta\lambda = 5.0\,\mathrm{nm}, \; 2\delta\lambda = 0.17\,\mathrm{nm},$
$F = 30, \; Q = 9400, \; P_{\mathrm{D,res}} = 0.44.$



Resonances

$$P_{\mathrm{D}} = P_{\mathrm{in}} \frac{|\kappa|^4 \,\mathrm{e}^{-\alpha L}}{1 + |\tau|^4 \,\mathrm{e}^{-2\alpha L} - 2|\tau|^2 \,\mathrm{e}^{-\alpha L} \cos(\beta L - 2\varphi)} (\lambda)$$

$$P_{\mathrm{T}} = P_{\mathrm{in}} \frac{|\rho|^2 (1 + |\tau|^2 d^2 \,\mathrm{e}^{-2\alpha L} - 2|\tau| d \,\mathrm{e}^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 \,\mathrm{e}^{-2\alpha L} - 2|\tau|^2 \,\mathrm{e}^{-\alpha L} \cos(\beta L - 2\varphi)} (\lambda)$$

Resonances

$$P_{\mathrm{D}} = P_{\mathrm{in}} \frac{|\kappa|^4 \,\mathrm{e}^{-\alpha L}}{1 + |\tau|^4 \,\mathrm{e}^{-2\alpha L} - 2|\tau|^2 \,\mathrm{e}^{-\alpha L} \cos(\beta L - 2\varphi)}$$

$$P_{\mathrm{T}} = P_{\mathrm{in}} \frac{|\rho|^2 (1 + |\tau|^2 d^2 \,\mathrm{e}^{-2\alpha L} - 2|\tau| d \,\mathrm{e}^{-\alpha L} \cos(\beta L - \varphi - \psi))}{1 + |\tau|^4 \,\mathrm{e}^{-2\alpha L} - 2|\tau|^2 \,\mathrm{e}^{-\alpha L} \cos(\beta L - 2\varphi)}$$

- Resonances :
  $\approx$ Singularities in the denominators of $P_{\mathrm{D}}, P_{\mathrm{T}}$, origin : $\beta(\lambda)$.
- Correction for finite coupler length l :
  $\beta L - 2\varphi = \beta L_{\mathrm{cav}} - \phi, \quad \phi = 2\beta l + 2\varphi, \quad L_{\mathrm{cav}} = 2\pi R, \quad \partial_\lambda \phi \approx 0.$
- Resonance condition : $\cos(\beta L_{\mathrm{cav}} - \phi) = 1$, or

$$\beta = \frac{2m\pi + \phi}{L_{\mathrm{cav}}} =: \beta_m \quad \text{integer } m; \qquad P_{\mathrm{D}}\big|_{\beta=\beta_m} = P_{\mathrm{in}} \frac{|\kappa|^4 \,\mathrm{e}^{-\alpha L}}{(1 - |\tau|^2 \,\mathrm{e}^{-\alpha L})^2}\,.$$

Free spectral range

- Resonance next to β_m :

$$\beta_{m-1} = \frac{2(m-1)\pi + \phi}{L_{\text{cav}}} = \beta_m - \frac{2\pi}{L_{\text{cav}}} \approx \beta_m + \left.\frac{\partial\beta}{\partial\lambda}\right|_m \Delta\lambda$$

- $\partial_\lambda \beta$ = ?
  q_j : waveguide parameters with dimension length,
  $\beta(a\lambda, aq_j) = \beta(\lambda, q_j)/a, \qquad \partial_a \,|_{a=1}$

$$\frac{\partial\beta}{\partial\lambda} = -\frac{1}{\lambda}\left(\beta + \sum_j q_j \frac{\partial\beta}{\partial q_j}\right) \approx -\frac{\beta}{\lambda}\,.$$

FSR : $\qquad \Delta\lambda = -\frac{2\pi}{L_{\text{cav}}}\left(\left.\frac{\partial\beta}{\partial\lambda}\right|_m\right)^{-1} \approx \left.\frac{\lambda^2}{n_{\text{eff}}\, L_{\text{cav}}}\right|_m, \qquad n_{\text{eff}} = \beta/k.$

(Free spectral range, the spectral distance (here: wavelength) between the drop peaks / the transmission dips).)

Spectral width of the resonances

- $P_{\text{D}} = P_{\text{in}} \dfrac{|\kappa|^4\, \mathrm{e}^{-\alpha L}}{1 + |\tau|^4\, \mathrm{e}^{-2\alpha L} - 2|\tau|^2\, \mathrm{e}^{-\alpha L} \cos(\beta L_{\text{cav}} - \phi)}\,,$
  $P_{\text{D}}|_{\beta_m} = P_{\text{D,res}}\,.$
- $P_{\text{D}}|_{\beta_m+\delta\beta} = P_{\text{D,res}}/2\,. \qquad \delta\beta$ = ?
- Expansion of cos-terms

$$\delta\beta = \pm\frac{1}{L_{\text{cav}}}\left(\frac{1}{|\tau|}\,\mathrm{e}^{\alpha L/2} - |\tau|\,\mathrm{e}^{-\alpha L/2}\right) \approx -\frac{\beta_m}{\lambda}\,\delta\lambda$$

FWHM : $\qquad 2\delta\lambda = \left.\frac{\lambda^2}{\pi L_{\text{cav}} n_{\text{eff}}}\right|_m \left(\frac{1}{|\tau|}\,\mathrm{e}^{\alpha L/2} - |\tau|\,\mathrm{e}^{-\alpha L/2}\right).$

(Full width at half maximum of the spectral drop peaks / the transmission dips (wavelength).)

Finesse & Q-factor

Finesse : $\qquad F = \dfrac{\Delta\lambda}{2\delta\lambda} = \pi\dfrac{|\tau|\,\mathrm{e}^{-\alpha L/2}}{1 - |\tau|^2\,\mathrm{e}^{-\alpha L}}\,.$

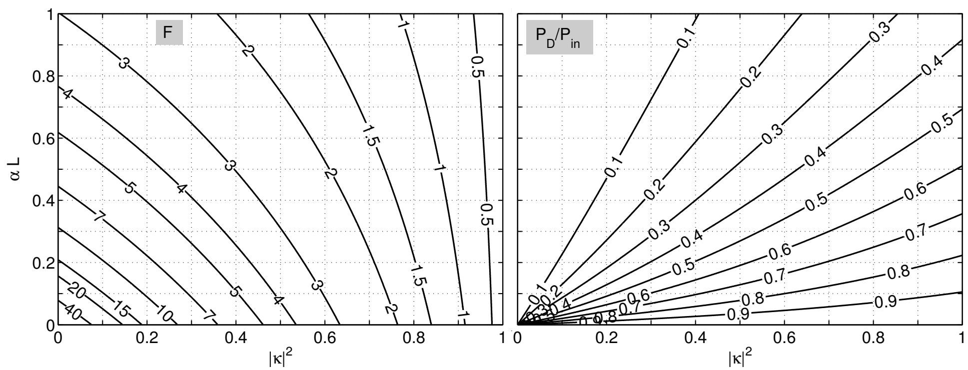
Q-factor : $\qquad Q = \dfrac{\lambda}{2\delta\lambda} = \pi\dfrac{n_{\text{eff}} L_{\text{cav}}}{\lambda}\,\dfrac{|\tau|\,\mathrm{e}^{-\alpha L/2}}{1 - |\tau|^2\,\mathrm{e}^{-\alpha L}} = \dfrac{n_{\text{eff}} L_{\text{cav}}}{\lambda} F\,.$

or $\qquad Q = kRn_{\text{eff}}F \qquad$ for $\qquad L_{\text{cav}} = 2\pi R\,.$

Performance versus coupling strength & losses

Assumption : Lossless coupler elements, $|\rho|^2 = |\tau|^2 = 1 - |\kappa|^2$.

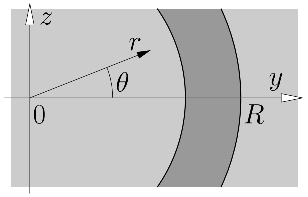
$$F = \pi\frac{(\sqrt{1-|\kappa|^2})\,\mathrm{e}^{-\alpha L/2}}{1 - (1-|\kappa|^2)\,\mathrm{e}^{-\alpha L}}\,, \qquad P_{\text{D}}|_{\text{res}} = P_{\text{in}}\frac{|\kappa|^4\,\mathrm{e}^{-\alpha L}}{(1 - (1-|\kappa|^2)\,\mathrm{e}^{-\alpha L})^2}\,.$$

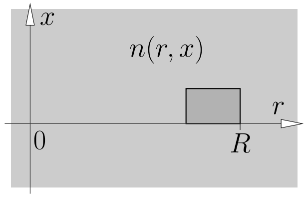


$\alpha, \kappa = ?$

Modes of bent waveguides

$\sim \exp(\mathrm{i}\omega t)$ (FD)





- Constant curvature $\leftrightsquigarrow$ cylindrical coordinates r, θ, x.
- Bend radius R, $\partial_\theta \epsilon = 0$, $\partial_\theta n = 0$

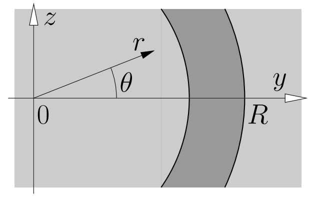
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(r,\theta,x) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(r,x)\,\mathrm{e}^{-\mathrm{i}\gamma R\theta}, \quad \text{bend modes,}$$

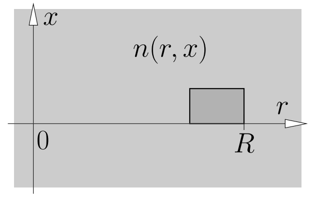
$\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$: bend mode profile, components $\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x$,
$\gamma = \beta - \mathrm{i}\alpha \in \mathbb{C}$: propagation constant,
$\beta \in \mathbb{R}$: phase constant,
$\alpha \in \mathbb{R}$: attenuation constant.

(Exponent $\mathrm{i}\gamma R\theta$: a convention, "propagation distance" $R\theta$.)

Modes of bent waveguides

$\sim \exp(\mathrm{i}\omega t)$ (FD)





- Piecewise constant $n(r,x)$, $\psi \in \{\bar{E}_r, \bar{E}_\theta, \bar{E}_x, \bar{H}_r, \bar{H}_\theta, \bar{H}_x\}$,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \left(k^2n^2 - \frac{\gamma^2R^2}{r^2}\right)\psi = 0, \qquad \text{where } \partial n = 0,$$

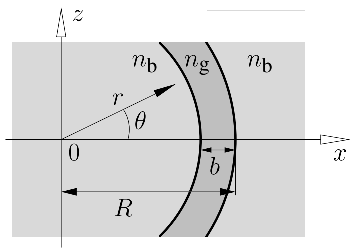
& continuity conditions at interfaces (cylindrical coordinates),
& boundary conditions:
regularity at $r = 0$, outgoing waves at $x = \pm\infty$, $r = \infty$.
(or: normalizability versus x.)

Vectorial 3-D bend mode eigenvalue problem.

(Practical setting: computational domain $r_\mathrm{i} < r < r_\mathrm{o}$, $x_\mathrm{b} < x < x_\mathrm{t}$, PML boundary conditions / $\psi = 0$ at $r = r_\mathrm{i}$.)

Modes of bent slab waveguides

$\sim \exp(\mathrm{i}\omega t)$ (FD)



2-D TE / TM, cylind. coord. r, θ, y,
$\partial_y n = \partial_\theta n = 0$

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(r,\theta) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(r)\,\mathrm{e}^{-\mathrm{i}\gamma R\theta},$$

bent slab mode $\{\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}, \gamma = \beta - \mathrm{i}\alpha\}$.

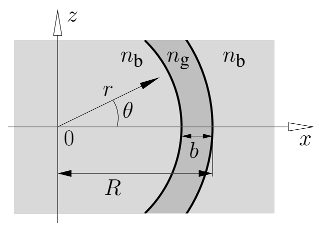
- Piecewise constant $n(r)$, $\phi = \bar{E}_y$ (TE), $\phi = \bar{H}_y$ (TM)

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \left(k^2n^2 - \frac{\gamma^2R^2}{r^2}\right)\phi = 0\,,$$

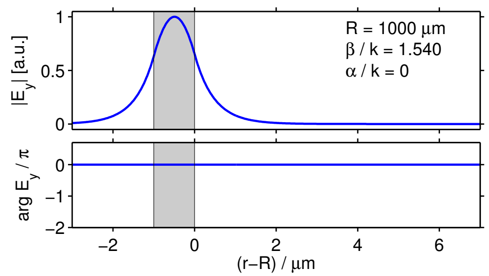
(Bessel differential equation with (complex) order γR.)

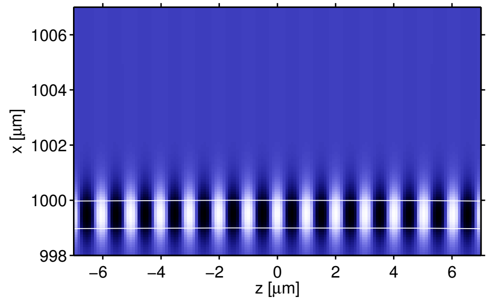
- Nonzero solutions,
- bounded at the origin, $\sim J_{\gamma R}(nkr)$ for $r < R - b$,
- outgoing exterior fields, $\sim H^{(2)}_{\gamma R}(nkr)$ for $r > R$, $(\sim \exp(\mathrm{i}\omega t))$,
- continuity at interfaces : ϕ, $\partial_r\phi$ (TE), ϕ, $(\partial_r\phi)/n^2$ (TM).

Bend modes, 2-D examples

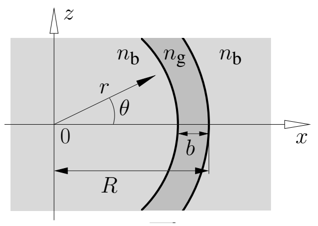


2-D, TE,
$n_\mathrm{b} = 1.45$, $n_\mathrm{g} = 1.60$, $b = 1.0\,\mu$m, $\lambda = 1.55\,\mu$m,
$R = 1000\,\mu$m.

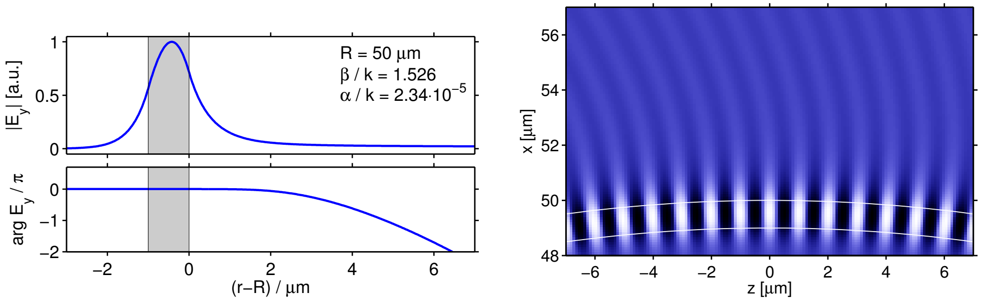




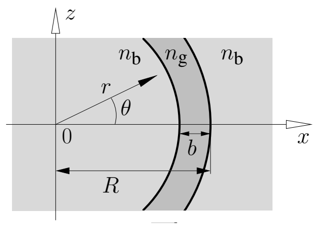
Bend modes, 2-D examples



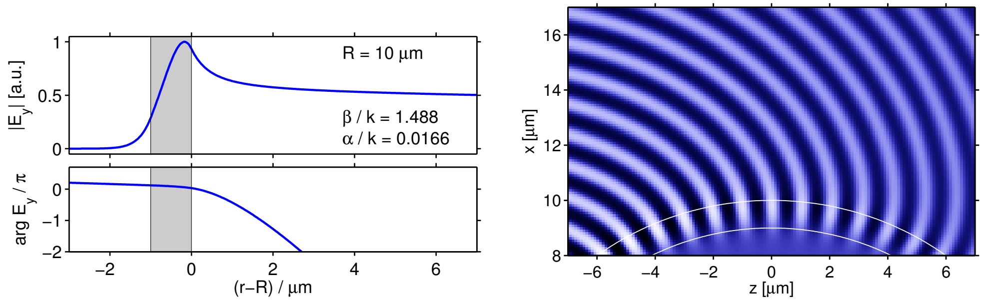
2-D, TE,
$n_b = 1.45$, $n_g = 1.60$, $b = 1.0\,\mu$m, $\lambda = 1.55\,\mu$m,
$R = 50\,\mu$m.



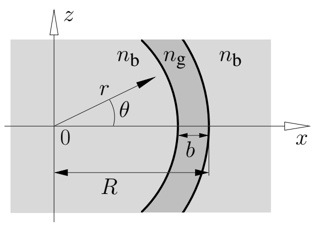
Bend modes, 2-D examples



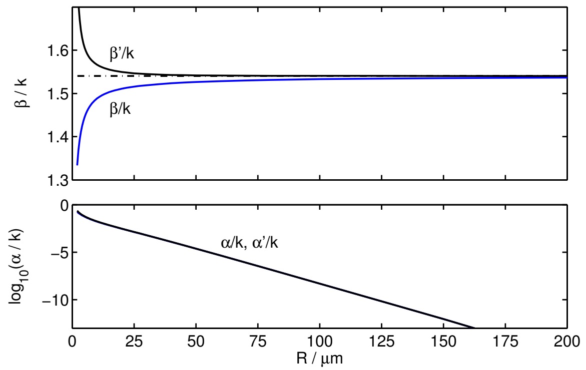
2-D, TE,
$n_b = 1.45$, $n_g = 1.60$, $b = 1.0\,\mu$m, $\lambda = 1.55\,\mu$m,
$R = 10\,\mu$m.



Propagation constant vs. bend radius



2-D, TE,
$n_b = 1.45$, $n_g = 1.60$, $b = 1.0\,\mu$m, $\lambda = 1.55\,\mu$m,
$R \in [2, 200]\,\mu$m.

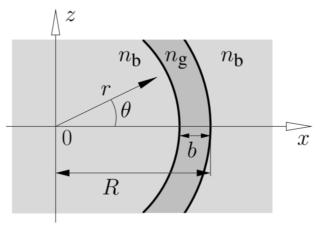


Alternative definition :
$R' = R - b/2.$

Identical physical fields

$$\gamma' R' = \gamma R, \qquad \gamma' = \gamma \, \frac{R}{R - b/2}\,.$$

Power & orthogonality



2-D TE / TM bend modes:

- Power flow: $S_r \neq 0$, $\quad S_r, S_\theta \sim \mathrm{e}^{-2\alpha R\theta}$, $\quad S_\theta \sim |\phi|^2/r$

  $\displaystyle\int_0^\infty S_\theta(r)\,\mathrm{d}r < \infty$ $\longleftrightarrow$ power normalization.

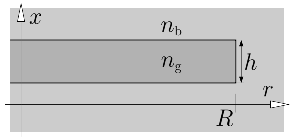
- Orthogonality of nondegenerate bend modes, product

  $$[\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2] = \int_0^\infty (\boldsymbol{E}_1 \times \boldsymbol{H}_2 + \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot \boldsymbol{e}_\theta \,\mathrm{d}r.$$

(Here [, ; ,] is complex valued.)

(Expressions $\sim \phi^2/r$ $\longleftrightarrow$ convergence of the integrals.)

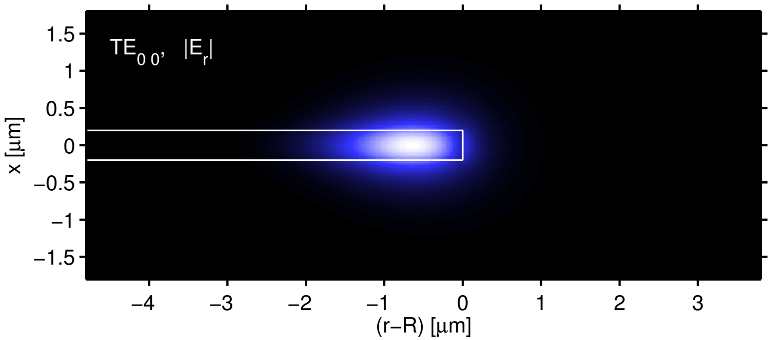
Bend modes supported by an angular disc segment



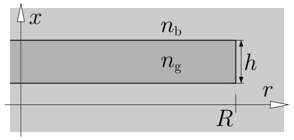
$x \in [-3, 3]$ µm, $(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm, $n_b = 1.45$, $n_g = 1.99$, $h = 0.4$ µm, $R = 20$ µm;

$TE_{0\,0}$
$\beta/k = 1.634$
$\alpha/k = 3.1 \cdot 10^{-8}$
[JCMwave].



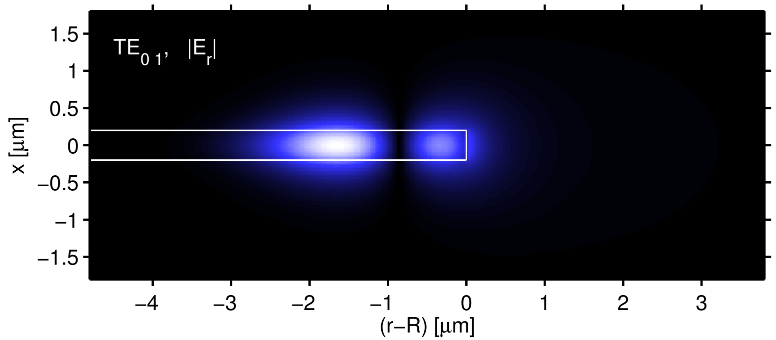
Bend modes supported by an angular disc segment



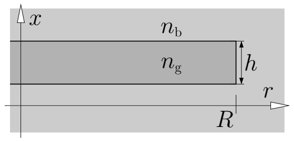
$x \in [-3, 3]$ µm, $(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm, $n_b = 1.45$, $n_g = 1.99$, $h = 0.4$ µm, $R = 20$ µm;

$TE_{0\,1}$
$\beta/k = 1.548$
$\alpha/k = 1.5 \cdot 10^{-5}$
[JCMwave].



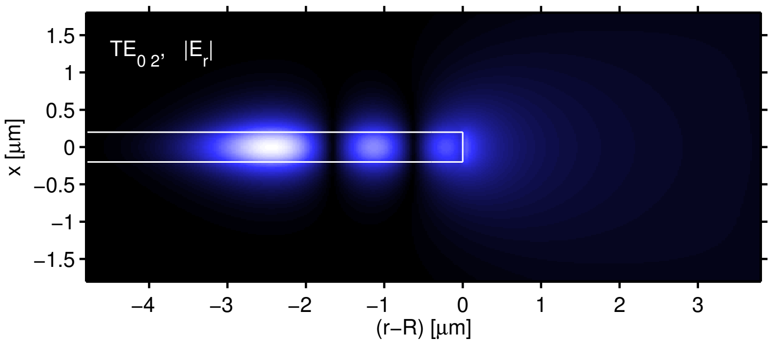
Bend modes supported by an angular disc segment



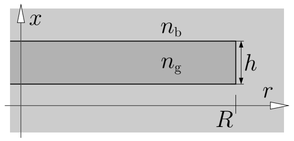
$x \in [-3, 3]$ µm, $(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm, $n_b = 1.45$, $n_g = 1.99$, $h = 0.4$ µm, $R = 20$ µm;

$TE_{0\,2}$
$\beta/k = 1.480$
$\alpha/k = 4.0 \cdot 10^{-4}$
[JCMwave].



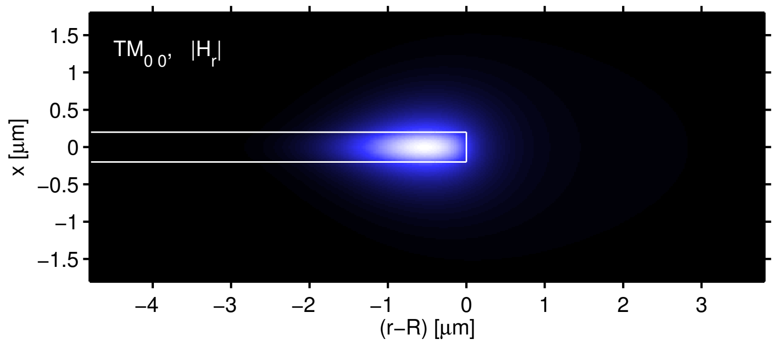
Bend modes supported by an angular disc segment



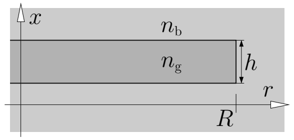
$x \in [-3, 3]$ µm, $(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm, $n_b = 1.45$, $n_g = 1.99$, $h = 0.4$ µm, $R = 20$ µm;

$TM_{0\,0}$
$\beta/k = 1.551$
$\alpha/k = 2.4 \cdot 10^{-5}$
[JCMwave].



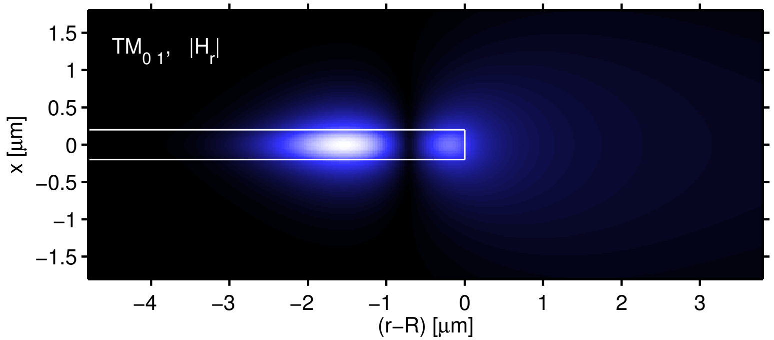
Bend modes supported by an angular disc segment



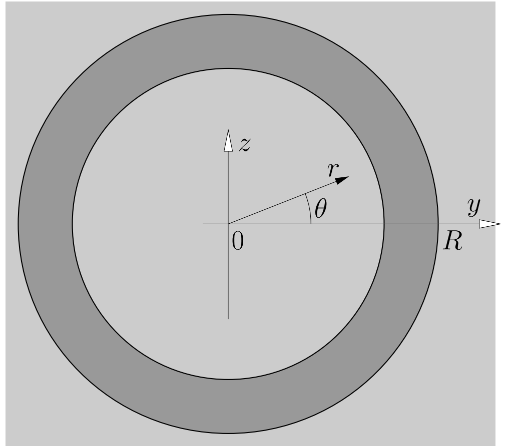
$\lambda = 1.55\,\mu\mathrm{m}$,
$n_\mathrm{b} = 1.45$,
$n_\mathrm{g} = 1.99$,
$h = 0.4\,\mu\mathrm{m}$,
$R = 20\,\mu\mathrm{m}$;

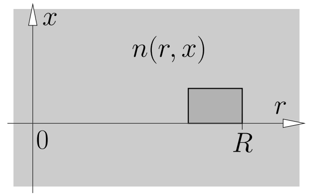
$\mathrm{TM}_{0\,1}$
$\beta/k = 1.468$
$\alpha/k = 7.6 \cdot 10^{-4}$
[JCMwave].

$x \in [-3, 3]\,\mu\mathrm{m}$,
$(r - R) \in [-8, 4]\,\mu\mathrm{m}$;



Circular microcavity



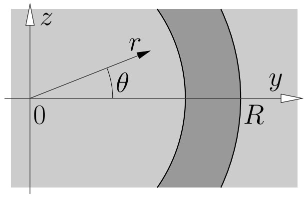


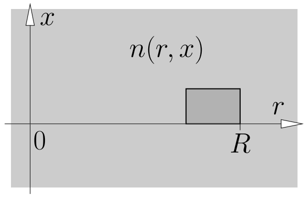
Bend modes $\leftrightsquigarrow$ Whispering gallery resonances.

(Terms not always clearly distinguished.)

Whispering gallery resonances

~~(FD)~~





- Full cavity, $\theta \in [0, 2\pi]$:
  Look for resonances in the form of whispering gallery modes

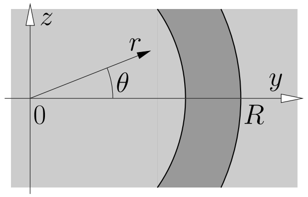
$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(r, \theta, x, t) = \begin{pmatrix} \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{H}} \end{pmatrix}(r, x)\, \mathrm{e}^{\mathrm{i}\omega_\mathrm{c} t \,-\, \mathrm{i} m\theta}, \qquad +\mathrm{c.c.}$$

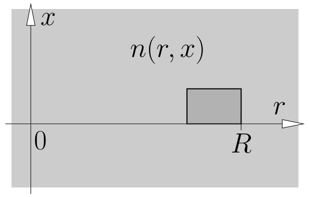
$\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$: WGM profile, components $\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x$,
$m \in \mathbb{Z}$: angular order,
$\omega_\mathrm{c} = \omega'_\mathrm{c} + \mathrm{i}\omega''_\mathrm{c} \in \mathbb{C}$: eigenfrequency, $\omega'_\mathrm{c}, \omega''_\mathrm{c} \in \mathbb{R}$.

Q-factor $Q = \omega'_\mathrm{c}/(2\omega''_\mathrm{c})$, resonance wavelength $\lambda_\mathrm{r} = 2\pi\mathrm{c}/\omega'_\mathrm{c}$, outgoing radiation, FWHM: $2\delta\lambda = \lambda_\mathrm{r}/Q$.

Whispering gallery resonances

~~(FD)~~





- Piecewise constant $n(r, x)$, $\psi \in \{\tilde{E}_r, \tilde{E}_\theta, \tilde{E}_x, \tilde{H}_r, \tilde{H}_\theta, \tilde{H}_x\}$,

(Dispersion ?)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \left(\frac{\omega_\mathrm{c}^2}{\mathrm{c}^2}n^2 - \frac{m^2}{r^2}\right)\psi = 0, \qquad \text{where } \partial n = 0,$$

& continuity conditions at interfaces (cylindrical coordinates),
& boundary conditions:
regularity at $r = 0$, outgoing waves at $x = \pm\infty$, $r = \infty$.

(or: normalizability versus x.)

Vectorial eigenproblem for whispering gallery resonances.

(Practical setting: computational domain $r_\mathrm{i} < r < r_\mathrm{o}$, $x_\mathrm{b} < x < x_\mathrm{t}$, PML boundary conditions / $\psi = 0$ at $r = r_\mathrm{i}$.)

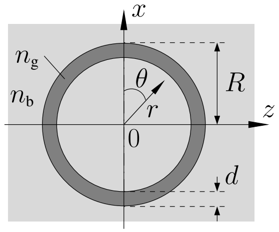
2-D whispering gallery resonances

. . . as discussed for the 2-D TE / TM bend modes.

(WGMs: Bessel differential equation of integer order.)

(Notation: WGM(ρ, m) — mode of radial order ρ and angular order m.)

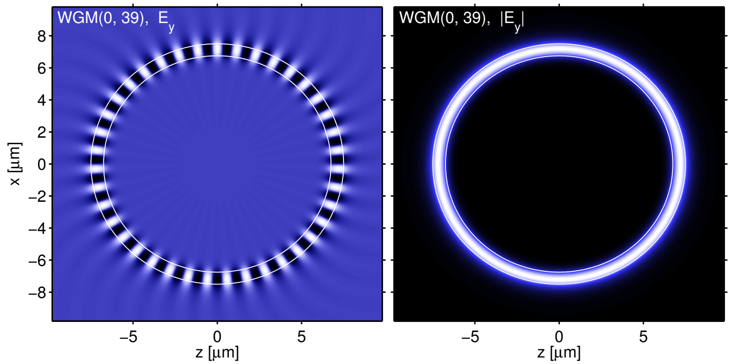
2-D whispering gallery resonances



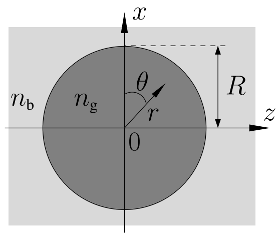
TE, $R = 7.5$ µm, $d = 0.75$ µm, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.5637$ µm, $Q = 1.1 \cdot 10^5$, $2\delta\lambda = 1.4 \cdot 10^{-5}$ µm.



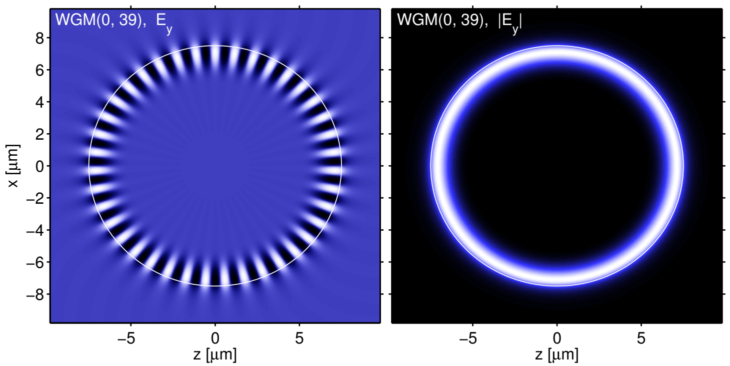
2-D whispering gallery resonances



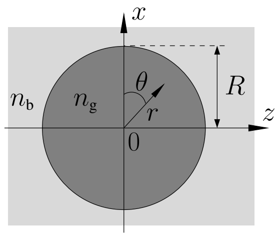
TE, $R = 7.5$ µm, $n_g = 1.5$, $n_b = 1.0$.

WGM(0, 39):

$\lambda_r = 1.6025$ µm, $Q = 5.7 \cdot 10^5$, $2\delta\lambda = 2.8 \cdot 10^{-6}$ µm.



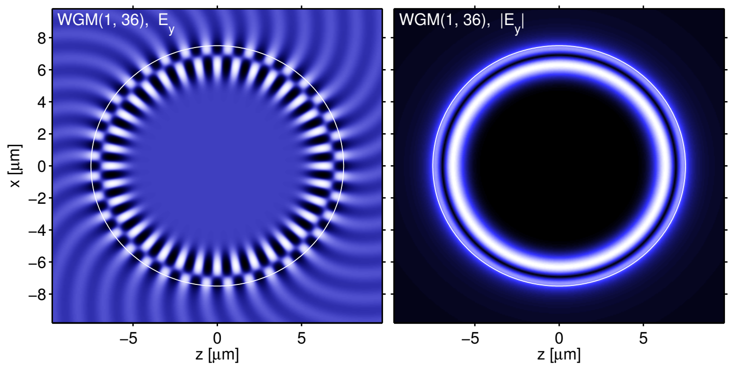
2-D whispering gallery resonances



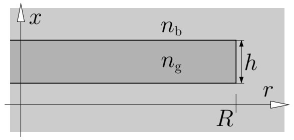
TE, $R = 7.5$ µm, $n_g = 1.5$, $n_b = 1.0$.

WGM(1, 36):

$\lambda_r = 1.5367$ µm, $Q = 2.2 \cdot 10^3$, $2\delta\lambda = 7.0 \cdot 10^{-4}$ µm.



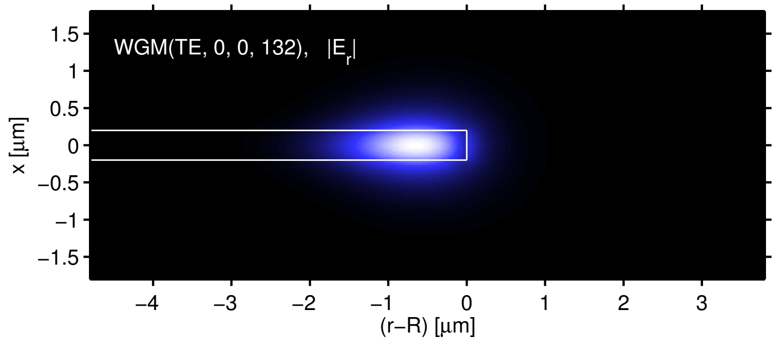
WGMs supported by a circular slab disc



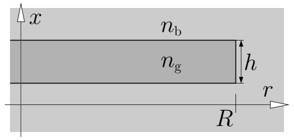
$x \in [-3, 3]$ µm,
$(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm,
$n_b = 1.45$,
$n_g = 1.99$,
$h = 0.4$ µm,
$R = 20$ µm;

WGM(TE, 0, 0, 132)
$\lambda_r = 1.555$ µm
$Q = 6.9 \cdot 10^6$
[JCMwave].



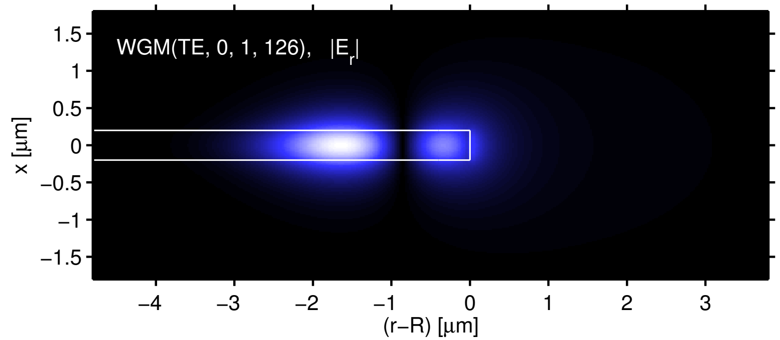
WGMs supported by a circular slab disc



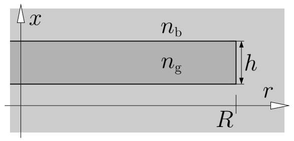
$x \in [-3, 3]$ µm,
$(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm,
$n_b = 1.45$,
$n_g = 1.99$,
$h = 0.4$ µm,
$R = 20$ µm;

WGM(TE, 0, 1, 126)
$\lambda_r = 1.545$ µm
$Q = 1.7 \cdot 10^4$
[JCMwave].



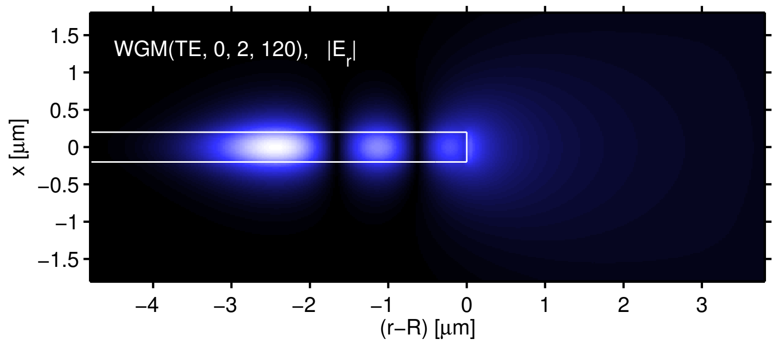
WGMs supported by a circular slab disc



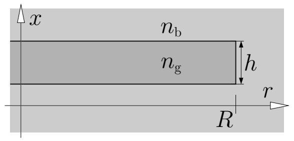
$x \in [-3, 3]$ µm,
$(r - R) \in [-8, 4]$ µm;

$\lambda = 1.55$ µm,
$n_b = 1.45$,
$n_g = 1.99$,
$h = 0.4$ µm,
$R = 20$ µm;

WGM(TE, 0, 2, 120)
$\lambda_r = 1.550$ µm
$Q = 5.7 \cdot 10^2$
[JCMwave].



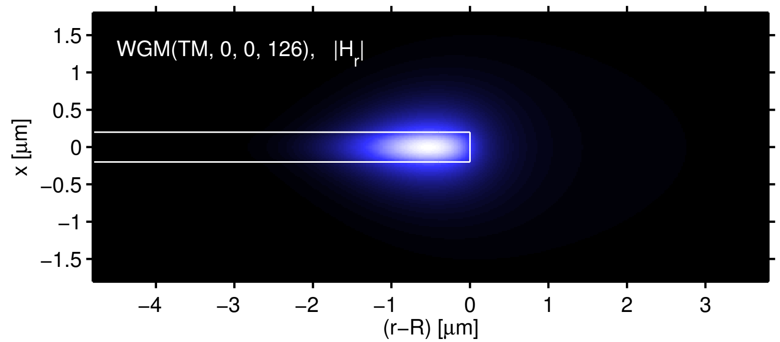
WGMs supported by a circular slab disc



$x \in [-3, 3]$ µm,
$(r - R) \in [-8, 4]$ µm;

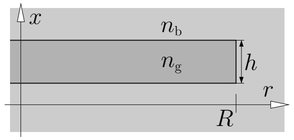
$\lambda = 1.55$ µm,
$n_b = 1.45$,
$n_g = 1.99$,
$h = 0.4$ µm,
$R = 20$ µm;

WGM(TM, 0, 0, 126)
$\lambda_r = 1.547$ µm
$Q = 1.0 \cdot 10^4$
[JCMwave].



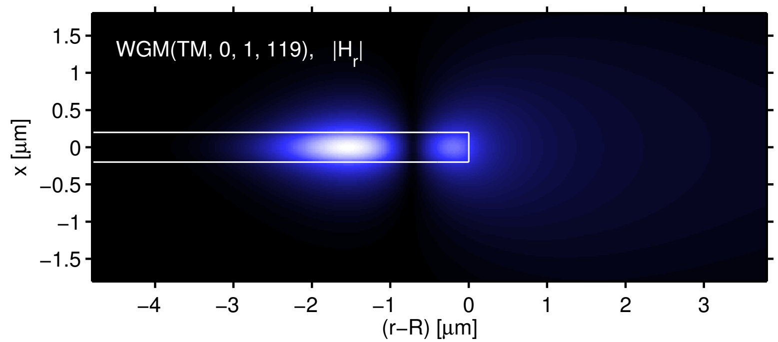
WGMs supported by a circular slab disc

$x \in [-3, 3]$ µm,
$(r - R) \in [-8, 4]$ µm;



$\lambda = 1.55$ µm,
$n_b = 1.45$,
$n_g = 1.99$,
$h = 0.4$ µm,
$R = 20$ µm;

WGM(TM, 0, 1, 119)
$\lambda_r = 1.550$ µm
$Q = 3.0 \cdot 10^2$
[JCMwave].



Bend modes versus whispering gallery resonances

(Field supported by a full circular cavity.)
(Incompatible models, in principle.)

[BWG] $\omega \in \mathbb{R}$ given, $\gamma = \beta - \mathrm{i}\alpha \in \mathbb{C}$ eigenvalue,

$$\Phi(r, \theta, t) = \phi(r)\ \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\beta R\theta}\ \mathrm{e}^{-\alpha R\theta}.$$

[WGM] $\omega_c = \omega_c + \mathrm{i}\omega_c'' \in \mathbb{C}$ eigenvalue, $m \in \mathbb{Z}$ given,

$$\Psi(r, \theta, t) = \psi(r)\ \mathrm{e}^{\mathrm{i}\omega_c' t - \mathrm{i}m\theta}\ \mathrm{e}^{-\omega_c'' t}.$$

Look at a resonant low-loss configuration:

- Translate $\omega \approx \omega_c'$, $m \approx \beta R$.
- Equate the power loss during one time period $T = 2\pi/\omega \approx 2\pi/\omega_c'$

$\rightsquigarrow$ $\beta/\alpha \approx \omega_c'/\omega_c'' = 2\,Q$.

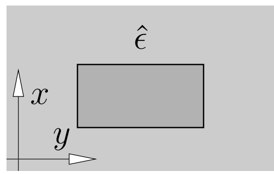
Course overview

Optical waveguide theory

A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
B Brush up on mathematical tools.
C Maxwell equations, different formulations, interfaces, energy and power flow.
D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
E Normal modes of dielectric optical waveguides, mode interference.
F Examples for dielectric optical waveguides.
G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
H Bent optical waveguides; whispering gallery resonances; circular microresonators.
I Coupled mode theory, perturbation theory.
• Hybrid analytical / numerical coupled mode theory.
J A touch of photonic crystals; a touch of plasmonics.
• Oblique semi-guided waves: 2-D integrated optics.
• Summary, concluding remarks.

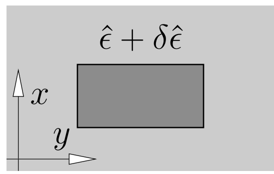
Perturbations of single modes

$\sim \exp(\mathrm{i}\omega t)$ (FD)



?

$\rightsquigarrow$



λ, $\hat\epsilon(x, y)$
β,
$\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{H}}$

λ, $\hat\epsilon(x, y) + \delta\hat\epsilon(x, y)$
$\beta + \delta\beta$,
$\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}$

A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

- $$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,y,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta z}, \qquad \beta \in \mathbb{R},$$
  $$\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} \to 0 \ \text{ for } \ x, y \to \pm\infty.$$

- $(\mathsf{C} + \mathrm{i}\beta\mathsf{R})\bar{\boldsymbol{E}} = -\mathrm{i}\omega\mu_0\bar{\boldsymbol{H}}, \quad (\mathsf{C} + \mathrm{i}\beta\mathsf{R})\bar{\boldsymbol{H}} = \mathrm{i}\omega\epsilon_0\hat{\epsilon}\bar{\boldsymbol{E}},$

  $$\mathsf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

- $$\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) := \frac{\omega\epsilon_0\langle\bar{\boldsymbol{E}}, \hat{\epsilon}\bar{\boldsymbol{E}}\rangle + \omega\mu_0\langle\bar{\boldsymbol{H}}, \bar{\boldsymbol{H}}\rangle + \mathrm{i}\langle\bar{\boldsymbol{E}}, \mathsf{C}\bar{\boldsymbol{H}}\rangle - \mathrm{i}\langle\bar{\boldsymbol{H}}, \mathsf{C}\bar{\boldsymbol{E}}\rangle}{\langle\bar{\boldsymbol{E}}, \mathsf{R}\bar{\boldsymbol{H}}\rangle - \langle\bar{\boldsymbol{H}}, \mathsf{R}\bar{\boldsymbol{E}}\rangle},$$
  $$\langle\bar{\boldsymbol{F}}, \bar{\boldsymbol{G}}\rangle = \iint \bar{\boldsymbol{F}}^* \cdot \bar{\boldsymbol{G}}\,\mathrm{d}x\,\mathrm{d}y.$$

$$\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta \ {\scriptstyle(*)}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}} + s\bar{\boldsymbol{F}}, \bar{\boldsymbol{H}} + s\bar{\boldsymbol{G}})\Big|_{s=0} = 0 \ {\scriptstyle(**)}$$

at valid mode fields $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$, for arbitrary $\bar{\boldsymbol{F}}, \bar{\boldsymbol{G}}$.

$(*)$: "arbitrary" $\hat{\epsilon}$.
$(**)$: Hermitian $\hat{\epsilon}$.

Perturbations of single modes

- Available: Mode $\beta, \bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$ for parameters λ, $\hat{\epsilon}$; $(\hat{\epsilon} = \hat{\epsilon}^\dagger)$
  $\mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta$, $\mathcal{B}_{\hat{\epsilon}}$ stationary at $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$.

- Investigate parameters λ, $\hat{\epsilon} + \delta\hat{\epsilon}$, for a "small" change $\delta\hat{\epsilon}$:

  $$\mathcal{B}_{\hat{\epsilon}+\delta\hat{\epsilon}}(\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}) = \beta + \delta\beta$$

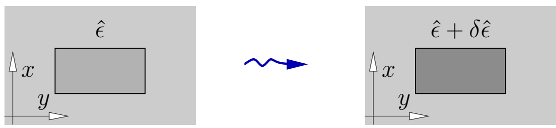
  ... $\quad \mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}} + \delta\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} + \delta\bar{\boldsymbol{H}}) \approx \mathcal{B}_{\hat{\epsilon}}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta$

  ... $\quad \cancel{\delta(\cdot)\,\delta(\cdot)}$

$$\delta\beta = \frac{\omega\epsilon_0\langle\bar{\boldsymbol{E}}, \delta\hat{\epsilon}\,\bar{\boldsymbol{E}}\rangle}{\langle\bar{\boldsymbol{E}}, \mathsf{R}\bar{\boldsymbol{H}}\rangle - \langle\bar{\boldsymbol{H}}, \mathsf{R}\bar{\boldsymbol{E}}\rangle}, \quad \text{or} \quad \delta\beta = \frac{\omega\epsilon_0 \iint \bar{\boldsymbol{E}}^* \cdot \delta\hat{\epsilon}\,\bar{\boldsymbol{E}}\,\mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re}\iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}.$$

(Valid for *small* perturbations: The original mode profiles are good approximations of the true fields in the modified structure.)

Small uniform change in refractive index

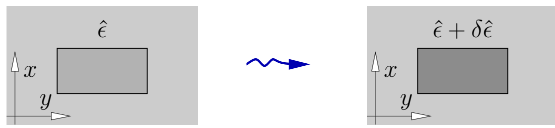


- $n \longrightarrow n + \delta n$ on $\square$, $\quad n, \delta n$ constant on $\square$

$$\beta \longrightarrow \beta + \delta\beta, \quad \delta\beta = \frac{\omega\epsilon_0\, n \iint_\square |\bar{\boldsymbol{E}}|^2\,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Re}\iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}\,\delta n\,.$$

$(\delta\epsilon = 2n\delta n.)$
(Plausible: $\delta\beta \sim \delta n$, $\delta\beta \sim |\bar{\boldsymbol{E}}|^2|_\square$.)

Small attenuation

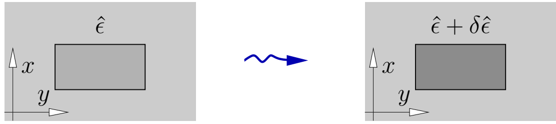


- $n \longrightarrow n - \mathrm{i}n''$ on $\square$, $\quad n, n''$ constant on $\square$, $n, n'' \in \mathbb{R}$

$$\beta \longrightarrow \beta + \delta\beta, \quad \delta\beta = \frac{-\mathrm{i}\omega\epsilon_0\, n \iint_\square |\bar{\boldsymbol{E}}|^2\,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Re}\iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}\,n''\,.$$

$(\delta\epsilon = -\mathrm{i}2nn''.)$
(Different attenuation for each mode.)
(Damping, power, plane wave: $\sim \exp(-2kn''z)$, mode: $\not\sim \exp(-2kn''z)$.)

Small anisotropy



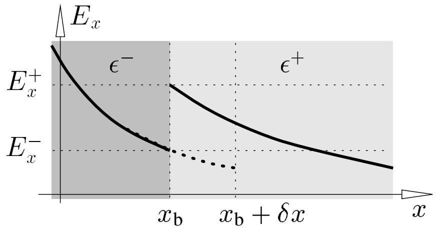
- $\epsilon \hat{1} \longrightarrow \epsilon\hat{1} + \delta\hat{\epsilon}$ on $\square$, $\epsilon, \delta\hat{\epsilon}$ constant on $\square$

$$\hookrightarrow \quad \beta \longrightarrow \beta + \delta\beta, \qquad \delta\beta = \frac{\omega\epsilon_0 \iint_{\square} \bar{\boldsymbol{E}}^* \cdot \delta\hat{\epsilon}\, \bar{\boldsymbol{E}} \,\mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re} \iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x\right) \,\mathrm{d}x\,\mathrm{d}y}.$$

(Phase shifts due to anisotropic contributions to the permittivity.)
(Polarization coupling might occur for modes with "close" propagation constants $\leftrightarrow$ CMT.)

Small displacements of dielectric interfaces

Interface displacement $\leftrightarrow$ Locally *strong* thin layer perturbation.
Field discontinuity $\leadsto$ Previous expressions are not directly applicable.



- $\epsilon^- \neq \epsilon^+$, shift of interface $x_\mathrm{b} \to x_\mathrm{b} + \delta x$.

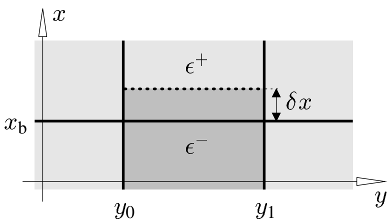
- Reposition discontinuity in field: $E_x \to E_x + \delta E_x$,

$$\delta E_x(x,y) = \begin{cases} \dfrac{\epsilon^+ - \epsilon^-}{\epsilon^-} E_x(x,y), & \text{for } x_\mathrm{b} < x < x_\mathrm{b} + \delta x, \\ 0, & \text{otherwise.} \end{cases}$$

- Use functional with locally modified field

$\hookrightarrow$ . . . (omitted) . . . $\leadsto$

Small displacements of dielectric interfaces

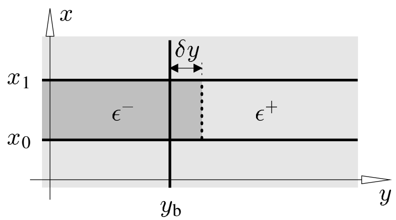


- Displacement of the interface at x_b between y_0 and y_1 by δx:

$\hookrightarrow \quad \beta \longrightarrow \beta + \delta\beta,$

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \displaystyle\int_{y_0}^{y_1} \left(\frac{1}{\epsilon^-\epsilon^+} |\epsilon \bar{E}_x|^2 + |\bar{E}_y|^2 + |\bar{E}_z|^2\right)(x_\mathrm{b}, y)\,\mathrm{d}y}{\mathrm{Re} \displaystyle\iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x\right) \mathrm{d}x\,\mathrm{d}y} \,\delta x\,.$$

Small displacements of dielectric interfaces

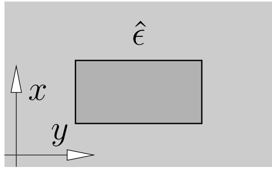


- Displacement of the interface at y_b between x_0 and x_1 by δy:

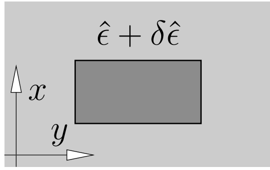
$\hookrightarrow \quad \beta \longrightarrow \beta + \delta\beta,$

$$\delta\beta = \frac{\omega\epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \displaystyle\int_{x_0}^{x_1} \left(|\bar{E}_x|^2 + \frac{1}{\epsilon^-\epsilon^+} |\epsilon \bar{E}_y|^2 + |\bar{E}_z|^2\right)(x, y_\mathrm{b})\,\mathrm{d}x}{\mathrm{Re} \displaystyle\iint \left(\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x\right) \mathrm{d}x\,\mathrm{d}y} \,\delta y\,.$$

Perturbations of single modes



$\rightsquigarrow$



$\lambda,\ \hat{\epsilon}(x, y)$ $\qquad\rightsquigarrow\qquad$ $\lambda,\ \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y)$

$\beta,$ $\quad\bar{\boldsymbol{E}},\ \bar{\boldsymbol{H}}$ $\qquad\qquad$ $\beta + \delta\beta,$ $\quad\approx\bar{\boldsymbol{E}},\ \approx\bar{\boldsymbol{H}}$

- View $\frac{\delta\beta}{\delta p}$ as $\frac{\partial\beta}{\partial p}$: slope of the dispersion curves β vs. p.
- Depending on the parametrization, change of a parameter value might require several perturbations.
- First order theory: In case of multiple pertubations, add the effects of the individual expressions.
- Estimation of fabrication tolerances: The phase shifts $\delta\beta$ enter into respective scattering matrix models.
- Wavelength shifts . . . ?

Small shift of frequency or vacuum wavelength

$(*)$: Explicit frequency dependence of $\mathcal{B}$ & dependence through $\hat{\epsilon}$.
$(**)$: Frequency dependence of $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$.

$$\beta(\omega) = \mathcal{B}_{\hat{\epsilon}}(\omega;\ \bar{\boldsymbol{E}}(\omega),\ \bar{\boldsymbol{H}}(\omega))$$

$$\frac{\partial\beta}{\partial\omega} = \frac{\partial\mathcal{B}_{\hat{\epsilon}}}{\partial\omega}_{\ (*)} + \frac{\partial}{\partial s}\mathcal{B}_{\hat{\epsilon}}\Big(\omega;\ \bar{\boldsymbol{E}} + s\frac{\partial\bar{\boldsymbol{E}}}{\partial\omega},\ \bar{\boldsymbol{H}}\Big)\Big|_{s=0}\ (**) + \frac{\partial}{\partial s}\mathcal{B}_{\hat{\epsilon}}\Big(\omega;\ \bar{\boldsymbol{E}},\ \bar{\boldsymbol{H}} + s\frac{\partial\bar{\boldsymbol{H}}}{\partial\omega}\Big)\Big|_{s=0}\ (**)$$

$$= \frac{\partial\mathcal{B}_{\hat{\epsilon}}}{\partial\omega},$$

(Stationarity of $\mathcal{B}$ at $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$.)

$$\frac{\partial\beta}{\partial\omega} = \frac{\iint \left(\epsilon_0\,\bar{\boldsymbol{E}}^* \cdot \frac{\partial(\omega\hat{\epsilon})}{\partial\omega}\,\bar{\boldsymbol{E}} + \mu_0\,|\bar{\boldsymbol{H}}|^2\right) \mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re}\iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}.$$

Small shift of frequency or vacuum wavelength

If dispersion can be neglected, $\partial_\omega\hat{\epsilon} = 0$:

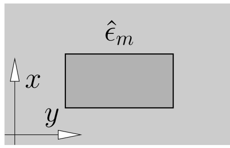
$$\frac{\partial\beta}{\partial\omega} = \frac{\iint \left(\epsilon_0\,\bar{\boldsymbol{E}}^* \cdot \hat{\epsilon}\,\bar{\boldsymbol{E}} + \mu_0\,|\bar{\boldsymbol{H}}|^2\right) \mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re}\iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y},$$

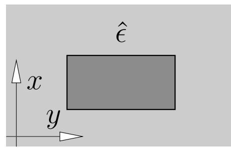
$$\frac{\partial\beta}{\partial\lambda} = -\frac{\pi\mathrm{c}}{\lambda^2}\,\frac{\iint \left(\epsilon_0\,\bar{\boldsymbol{E}}^* \cdot \hat{\epsilon}\,\bar{\boldsymbol{E}} + \mu_0\,|\bar{\boldsymbol{H}}|^2\right) \mathrm{d}x\,\mathrm{d}y}{\mathrm{Re}\iint \left(\bar{E}_x^*\bar{H}_y - \bar{E}_y^*\bar{H}_x\right)\,\mathrm{d}x\,\mathrm{d}y}.$$

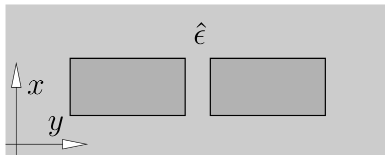
$(\omega = 2\pi\mathrm{c}/\lambda \leftrightsquigarrow \partial_\lambda\omega = -2\pi\mathrm{c}/\lambda^2)$
(Compare with expression based on homogeneity, H, 12.)

Coupled mode theory (CMT)

$\sim \exp(\mathrm{i}\omega t)$ (FD)







?

$$\left\{\hat{\epsilon}_m;\ \beta_m, (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m)\right\} \rightsquigarrow \begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x, y, z)$$

(Next: One of many variants of approaches to CMT.)
(Propagation & interaction of basis fields along a common propagation coordinate.)
[D.G. Hall, B.J. Thompson, *Selected papers on Coupled-Mode Theory in Guided-Wave Optics*, SPIE Milestone series MS 84 (1993)]
(Codirectional coupling (here), versus contradirectional coupling, coupling to radiation modes, nonlinear coupling.)
(Hybrid variant (HCMT): separate lecture.)

Coupled mode theory (CMT)

- Investigate a permittivity $\hat\epsilon$, look for fields $\boldsymbol{E}, \boldsymbol{H}$ with
  $\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0 \hat\epsilon \boldsymbol{E}.$
  ($\hat\epsilon(x, y, z)$, in general.)

- Available: A set of fields $\{\boldsymbol{E}_m, \boldsymbol{H}_m\}$ for permittivities $\hat\epsilon_m = \hat\epsilon_m^\dagger$;
  $\nabla \times \boldsymbol{E}_m = -\mathrm{i}\omega\mu_0 \boldsymbol{H}_m, \quad \nabla \times \boldsymbol{H}_m = \mathrm{i}\omega\epsilon_0 \hat\epsilon_m \boldsymbol{E}_m.$
  (Not necessarily "modes".)

- Assume that $(\boldsymbol{E}, \boldsymbol{H})$ can be well approximated by

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) \approx \sum_m C_m(z) \begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z),$$

  C_m: unknown amplitudes, common propagation coordinate z.
  (Choose $\hat\epsilon_m$ as close as possible to $\hat\epsilon$.)

Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\nabla \cdot (\boldsymbol{H} \times \boldsymbol{E}_l^* - \boldsymbol{E} \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat\epsilon - \hat\epsilon_l)\boldsymbol{E}.$$

(Insert CMT ansatz for $\boldsymbol{E}, \boldsymbol{H}$.)

...

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

...

(Apply identity $\nabla \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat\epsilon_m - \hat\epsilon_l)\boldsymbol{E}$.)

...

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

...

(Manipulate, arrange terms, tidy up.)

$$\sum_m o_{lm}\, \partial_z C_m = -\mathrm{i} \sum_m k_{lm}\, C_m \quad \forall\, l, \qquad \text{coupled mode equations.}$$

$$o_{lm} = \frac{1}{4} \iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \boldsymbol{E}_l \cdot (\hat\epsilon - \hat\epsilon_m)\boldsymbol{E}_m \,\mathrm{d}x\,\mathrm{d}y.$$

Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\nabla \cdot (\boldsymbol{H} \times \boldsymbol{E}_l^* - \boldsymbol{E} \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat\epsilon - \hat\epsilon_l)\boldsymbol{E}.$$

(Insert CMT ansatz for $\boldsymbol{E}, \boldsymbol{H}$.)

...

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

...

(Apply identity $\nabla \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat\epsilon_m - \hat\epsilon_l)\boldsymbol{E}$.)

...

($\iint \mathrm{d}x\,\mathrm{d}y$, assume $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

...

(Manipulate, arrange terms, tidy up.)

$$\mathsf{O}\, \partial_z \boldsymbol{C} = -\mathrm{i}\mathsf{K}\boldsymbol{C}, \qquad \text{coupled mode equations.}$$

$\boldsymbol{C} = (C_m),\ \mathsf{O} = (o_{lm}),\ \mathsf{K} = (k_{lm}).$

$$o_{lm} = \frac{1}{4} \iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \boldsymbol{E}_l \cdot (\hat\epsilon - \hat\epsilon_m)\boldsymbol{E}_m \,\mathrm{d}x\,\mathrm{d}y.$$

Coupled mode theory (CMT)

(Variational derivation of CMT equations.)

$$\mathcal{F}(\boldsymbol{E}, \boldsymbol{H}) = \iiint \Big\{ \boldsymbol{H}^* \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E}^* \cdot (\nabla \times \boldsymbol{H}) + \mathrm{i}\omega\mu_0 \boldsymbol{H}^* \cdot \boldsymbol{H} + \mathrm{i}\omega\epsilon_0 \boldsymbol{E}^* \cdot \hat\epsilon \boldsymbol{E} \Big\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$\delta\mathcal{F} = 0 \ \ \forall\, \delta\boldsymbol{E}, \delta\boldsymbol{H} \quad \longleftrightarrow \quad \nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0 \boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0 \hat\epsilon \boldsymbol{E}.$$

(Restrict $\mathcal{F}$ to the CMT ansatz for $\boldsymbol{E}, \boldsymbol{H} \rightsquigarrow \mathcal{F}_\mathrm{c}(\boldsymbol{C})$, require $\delta\mathcal{F}_\mathrm{c} = 0\ \forall \delta\boldsymbol{C}$.)

...

($\nabla \cdot (\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat\epsilon_m - \hat\epsilon_l)\boldsymbol{E}$, $\iint \mathrm{d}x\,\mathrm{d}y$, $\boldsymbol{E}_m, \boldsymbol{H}_m \to 0$ for $x, y \to \pm\infty$.)

...

(Manipulate, arrange terms, tidy up.)

$$\mathsf{O}\, \partial_z \boldsymbol{C} = -\mathrm{i}\mathsf{K}\boldsymbol{C}, \qquad \text{coupled mode equations.}$$

$\boldsymbol{C} = (C_m),\ \mathsf{O} = (o_{lm}),\ \mathsf{K} = (k_{lm}).$

$$o_{lm} = \frac{1}{4} \iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \boldsymbol{E}_l \cdot (\hat\epsilon - \hat\epsilon_m)\boldsymbol{E}_m \,\mathrm{d}x\,\mathrm{d}y.$$

Coupled mode equations

. . .

$$\mathsf{O}\,\partial_z \boldsymbol{C} = -\mathrm{i}\,\mathsf{K}\boldsymbol{C}, \qquad \boldsymbol{C} = (C_m),\ \ \mathsf{O} = (o_{lm}),\ \ \mathsf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4}\iint (\boldsymbol{E}_l^* \times \boldsymbol{H}_m - \boldsymbol{H}_l^* \times \boldsymbol{E}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\boldsymbol{E}_l, \boldsymbol{H}_l; \boldsymbol{E}_m, \boldsymbol{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4}\iint \boldsymbol{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)\boldsymbol{E}_m \,\mathrm{d}x\,\mathrm{d}y.$$

- A set of coupled *ordinary* linear differential equations, of first order. (Here.)
- o_{lm} : power coupling coefficients (field overlaps). (No reason to assume $o_{lm} = \delta_{lm}$, in general.)
- k_{lm} : coupling coefficients.
- z-dependence of $\hat{\epsilon}, \hat{\epsilon}_m, \boldsymbol{E}_m, \boldsymbol{H}_m$ $\rightsquigarrow$ $o_{lm}(z),\ k_{lm}(z),\ \mathsf{O}(z),\ \mathsf{K}(z)$.
(Compare the bend-straight couplers, Lecture H.)

. . . to be solved by numerical procedures. (In general.)

CMT for longitudinally homogeneous structures

$\partial_z \hat{\epsilon} = 0,\ \partial_z \hat{\epsilon}_m = 0,$

basis: guided modes $\begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z) = \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\ \mathrm{e}^{-\mathrm{i}\beta_m z}$,

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \sum_m C_m(z)\begin{pmatrix}\boldsymbol{E}_m\\ \boldsymbol{H}_m\end{pmatrix}(x,y,z) = \sum_m c_m(z)\begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y).$$

($c_m(z) = C_m(z)\exp(-\mathrm{i}\beta_m z)$, rewrite CMT equations for $c_m(z)$.)

. . .

($\nabla\cdot(\boldsymbol{H}_m \times \boldsymbol{E}_l^* - \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = \mathrm{i}\omega\epsilon_0 \boldsymbol{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)\boldsymbol{E}$, integrate, rewrite for $\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m$.)

. . .

(Symmetrize coefficients.)

$$\sum_m \sigma_{lm}\,\partial_z c_m = -\mathrm{i}\sum_m (b_{lm} + \kappa_{lm})\ c_m \quad \forall\, l,$$

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y, \qquad b_{lm} = \sigma_{lm}\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

Longitudinally constant structures, coupled mode equations

. . . ($\partial_z\hat{\epsilon} = \partial_z\hat{\epsilon}_m = 0$)

$$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}(\mathsf{B}+\mathsf{Q})\boldsymbol{c},\ \ \boldsymbol{c} = (c_m),\ \mathsf{S} = (\sigma_{lm}),\ \mathsf{B} = (b_{lm}),\ \mathsf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y, \qquad b_{lm} = \sigma_{lm}\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- A set of coupled *ordinary* linear differential equations, of first order (Here.)
- σ_{lm} : power coupling coefficients (field overlaps). (No reason to assume $\sigma_{lm} = \delta_{lm}$, in general.)
- κ_{lm} : coupling coefficients.
- $\partial_z\hat{\epsilon} = \partial_z\hat{\epsilon}_m = 0$ $\rightsquigarrow$ $\partial_z\sigma_{lm} = \partial_z b_{lm} = \partial_z\kappa_{lm} = 0$.
(ODEs with constant coefficents.)

. . . quasi-analytical solutions.

Longitudinally constant structures, coupled mode equations

. . . ($\partial_z\hat{\epsilon} = \partial_z\hat{\epsilon}_m = 0$)

$$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}(\mathsf{B}+\mathsf{Q})\boldsymbol{c},\ \ \boldsymbol{c} = (c_m),\ \mathsf{S} = (\sigma_{lm}),\ \mathsf{B} = (b_{lm}),\ \mathsf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4}\iint (\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m - \bar{\boldsymbol{H}}_l^* \times \bar{\boldsymbol{E}}_m)_z \,\mathrm{d}x\,\mathrm{d}y = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8}\iint \bar{\boldsymbol{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m)\bar{\boldsymbol{E}}_m \,\mathrm{d}x\,\mathrm{d}y, \qquad b_{lm} = \sigma_{lm}\frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- $\sigma_{ml}^* = \sigma_{lm},\quad b_{ml}^* = b_{lm};\quad \kappa_{ml}^* = \kappa_{lm}$, if $\hat{\epsilon}^\dagger = \hat{\epsilon},\ \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m$,
$\mathsf{S}^\dagger = \mathsf{S},\quad \mathsf{B}^\dagger = \mathsf{B};\quad \mathsf{Q}^\dagger = \mathsf{Q}$, if $\hat{\epsilon}^\dagger = \hat{\epsilon},\ \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m$.
- Power: $P = (\boldsymbol{E},\boldsymbol{H};\boldsymbol{E},\boldsymbol{H}) = \sum_{l,m} c_l^*(\bar{\boldsymbol{E}}_l,\bar{\boldsymbol{H}}_l;\bar{\boldsymbol{E}}_m,\bar{\boldsymbol{H}}_m)c_m = \boldsymbol{c}^*\cdot\mathsf{S}\boldsymbol{c}$

$$\partial_z P = \mathrm{i}\boldsymbol{c}^*\cdot\big((\mathsf{B}+\mathsf{Q})^\dagger - (\mathsf{B}+\mathsf{Q})\big)\boldsymbol{c}, \quad \partial_z P = 0 \ \text{ for } \mathsf{B}^\dagger = \mathsf{B},\ \mathsf{Q}^\dagger = \mathsf{Q}.$$

(For lossless waveguides the scheme is power conservative.)

Longitudinally constant structures, formal solution

$$\mathsf{S}\,\partial_z \boldsymbol{c} = -\mathrm{i}(\mathsf{B}+\mathsf{Q})\boldsymbol{c}, \qquad \partial_z\mathsf{S} = \partial_z\mathsf{B} = \partial_z\mathsf{Q} = 0.$$

Ansatz: $\boldsymbol{c}(z) = \boldsymbol{a}\,\mathrm{e}^{-\mathrm{i}bz}$, $\quad \boldsymbol{a}, b$ constants.

$\hookrightarrow$ $(\mathsf{B}+\mathsf{Q})\boldsymbol{a} = b\,\mathsf{S}\boldsymbol{a}$, a generalized eigenvalue problem.

(Dimension: number of basis modes included.)

Solutions: $\{\boldsymbol{a}, b\}$,

$\rightsquigarrow$ "supermodes" $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = \left(\sum_m a_m \begin{pmatrix}\bar{\boldsymbol{E}}_m\\ \bar{\boldsymbol{H}}_m\end{pmatrix}(x,y)\right)\mathrm{e}^{-\mathrm{i}bz}.$

(Superpositions of the original mode profiles with constant coefficients.)
(As many supermodes as there are basis modes.)
(Formalism can be continued: power / orthogonality of supermodes . . .)

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)
(Or: non-orthogonality neglected as a further approximation.)

$\sigma_{lm} = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) = \delta_{lm}P_0.$ (Orthogonal modes, uniform normalization $P_m = P_0$.)
(Or: apply inverse of S to CM equations, continue with redefined expressions for β_m, κ_{lm}.)

$\hookrightarrow$ $\begin{pmatrix}\partial_z c_1\\ \partial_z c_2\end{pmatrix} = -\mathrm{i}\begin{pmatrix}\beta'_1 & \kappa\\ \kappa^* & \beta'_2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$, $\quad \beta'_l = \beta_l + \kappa_{ll}/P_0$, $\kappa = \kappa_{12}/P_0$.

$\hookrightarrow$ . . .

$\hookrightarrow$ . . .

$\hookrightarrow$

$$\begin{pmatrix}c_1\\ c_2\end{pmatrix}(z) = \mathrm{e}^{-\mathrm{i}\frac{(\beta'_1+\beta'_2)}{2}z}\begin{pmatrix}\cos\rho z - \mathrm{i}\dfrac{\Delta\beta'}{2\rho}\sin\rho z & -\mathrm{i}\dfrac{\kappa}{\rho}\sin\rho z\\ -\mathrm{i}\dfrac{\kappa^*}{\rho}\sin\rho z & \cos\rho z + \mathrm{i}\dfrac{\Delta\beta'}{2\rho}\sin\rho z\end{pmatrix}\begin{pmatrix}c_{10}\\ c_{20}\end{pmatrix},$$

$$\Delta\beta' = \beta'_1 - \beta'_2, \quad \rho = \sqrt{\left(\frac{\Delta\beta'}{2}\right)^2 + |\kappa|^2}.$$

Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes $(\boldsymbol{E}_1, \boldsymbol{H}_1)$, $(\boldsymbol{E}_2, \boldsymbol{H}_2)$:

(Example: two modes supported by the same isotropic waveguide ($\hat{\epsilon}_1 = \hat{\epsilon}_2$); interaction due to small anisotropy ($\hat{\epsilon}$).)
(Or: non-orthogonality neglected as a further approximation.)

$\sigma_{lm} = (\bar{\boldsymbol{E}}_l, \bar{\boldsymbol{H}}_l; \bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m) = \delta_{lm}P_0.$ (Orthogonal modes, uniform normalization $P_m = P_0$.)
(Or: apply inverse of S to CM equations, continue with redefined expressions for β_m, κ_{lm}.)

$\hookrightarrow$ $\begin{pmatrix}\partial_z c_1\\ \partial_z c_2\end{pmatrix} = -\mathrm{i}\begin{pmatrix}\beta'_1 & \kappa\\ \kappa^* & \beta'_2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$, $\quad \beta'_l = \beta_l + \kappa_{ll}/P_0$, $\kappa = \kappa_{12}/P_0$.

- $c_{20} = 0$ $\rightsquigarrow$ $\left|\dfrac{c_2(z)}{c_1(0)}\right|^2 = \eta_{\max}\,\sin^2(\rho z)$, $\quad \eta_{\max} = \dfrac{|\kappa|^2}{|\kappa|^2 + (\Delta\beta'/2)^2}.$
- Maximum conversion $\eta_{\max}$ at $z = L_\mathrm{c}$ with $\rho L_\mathrm{c} = \pi/2$,
  coupling length $L_\mathrm{c} = \dfrac{\pi}{\sqrt{(\Delta\beta')^2 + 4|\kappa|^2}}$, (Conversion length, half-beat length.)
- In case of phase matching $\Delta\beta' = \beta'_1 - \beta'_2 = 0$: $\eta_{\max} = 1$, $L_\mathrm{c} = \dfrac{\pi}{2|\kappa|}$.

(Here the *phase-shifted* propagation constants are relevant.)
(Small interaction (small maximum conversion) for out-of-phase modes, i.e. for $|\Delta\beta'|^2 \gg |\kappa|^2$).)

Longitudinally constant structures, one "coupled" mode

CMT with one basis mode: $\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(x,y,z) = c_1(z)\begin{pmatrix}\bar{\boldsymbol{E}}_1\\ \bar{\boldsymbol{H}}_1\end{pmatrix}(x,y)$

$\hookrightarrow$ $\partial_z c_1 = -\mathrm{i}\dfrac{b_{11}+\kappa_{11}}{\sigma_{11}}c_1,$

$$\frac{b_{11}}{\sigma_{11}} = \beta_1, \quad \frac{\kappa_{11}}{\sigma_{11}} = \frac{\omega\epsilon_0 \iint \bar{\boldsymbol{E}}_1^* \cdot (\hat{\epsilon} - \hat{\epsilon}_1)\bar{\boldsymbol{E}}_1\,\mathrm{d}x\,\mathrm{d}y}{2\,\mathrm{Re}\iint \left(\bar{E}_{1x}^*\bar{H}_{1y} - \bar{E}_{1y}^*\bar{H}_{1x}\right)\mathrm{d}x\,\mathrm{d}y} =: \delta\beta_1,$$

$\hookrightarrow$ $\partial_z c_1 = -\mathrm{i}(\beta_1 + \delta\beta_1)c_1,$

$\hookrightarrow$ $c_1(z) = c_1(0)\,\mathrm{e}^{-\mathrm{i}(\beta_1+\delta\beta_1)z}.$

$\leftrightsquigarrow$ Theory of single mode perturbations.

Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- • Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- • Oblique semi-guided waves: 2-D integrated optics.
- • Summary, concluding remarks.

A touch of photonic crystals

"Photonic crystals": ?

Keywords:
- A branch of photonics.
- Optics involving strcutures with (1-D, 2-D, 3-D) spatial periodicity.
- 1-D periodicity: Multilayer stacks / coatings, gratings, corrugated waveguides.
- 2-D periodicity: Corrugated dielectric slabs, membranes, gratings.
- 3-D periodicity: Bulk photonic crystals.
- "Molding the flow of light" $\leftrightsquigarrow$ tunability, degrees of freedom in design.
- Defect cavities & defect waveguides in photonic crsytals.
- Phenomena & fundamental research.
- Photonic crystal fibers.

Context of this lecture:
- Problems of general classical electromagnetics & methods as discussed; different emphasis.
- Periodicity: Restrict computations to unit cells.

Structures with spatial periodicity

$\sim \exp(\mathrm{i}\omega t)$ (FD)

Infinite system with periodic permittivity:

$$\epsilon(\boldsymbol{r}+\boldsymbol{g}) = \epsilon(\boldsymbol{r}) \quad \text{for all lattice vectors } \boldsymbol{g}.$$

$\hookrightarrow$ Consider Floquet-Bloch waves

(Floquet: 1-D, context of mechanics;
Bloch: context of solid state physics.)

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(\boldsymbol{r}) = \boldsymbol{U}_{\boldsymbol{k}}(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}},$$

$\boldsymbol{k}$: wavevector of the FB wave,
$\boldsymbol{U}_{\boldsymbol{k}}$: a periodic function, $\boldsymbol{U}_{\boldsymbol{k}}(\boldsymbol{r}+\boldsymbol{g}) = \boldsymbol{U}_{\boldsymbol{k}}(\boldsymbol{r})$.

(A plane wave, modulated by a periodic function.)

{FB waves}: A complete basis for the periodic system.

(Bloch theorem: any solution can be written as a superposition of FB waves.)
(Background: Hilbert space theory, self-adjoint operators; familiar from Quantum theory.)
(Hermitian Hamiltonian and translation operators commute; Bloch waves are a simultaneous eigenbasis of these operators.)
(Required: Hermitian "Hamiltonian" $\leftrightsquigarrow$ Hermitian $\hat{\epsilon}$.)

($\boldsymbol{U}_k$ =?, but $\boldsymbol{U}_k$ satisfies different equations than $\boldsymbol{E}, \boldsymbol{H}$ . . .)

Structures with spatial periodicity

$\sim \exp(\mathrm{i}\omega t)$ (FD)

$\boldsymbol{g}$: a lattice vector, such that $\epsilon(\boldsymbol{r}+\boldsymbol{g}) = \epsilon(\boldsymbol{r})$

$$\hookrightarrow \begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(\boldsymbol{r}+\boldsymbol{g}) = \begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{g}}. \qquad \text{(QPBC)}$$

(. . . if $\boldsymbol{g}$ connects the boundaries of a unit cell.)

FB-wave eigenproblem:
Given a wavevector $\boldsymbol{k}$, look for frequencies $\omega \in \mathbb{R}$, such that there exist nonzero solutions $(\boldsymbol{E}, \boldsymbol{H})$ on a unit cell domain, with quasi-periodic boundary conditions (QPBC).

- Outcome:
  $\exists\,\omega$ with $(\boldsymbol{E}, \boldsymbol{H}) \neq 0$: $(\boldsymbol{k}, \omega) \in$ a frequency band, or
  $\nexists\,\omega$ with $(\boldsymbol{E}, \boldsymbol{H}) \neq 0$: $\omega \in$ a bandgap region.
  $\leftrightsquigarrow$ "Bandstructure" calculations.

- QPBC for $\boldsymbol{k}$ are the same as for $\boldsymbol{k}+\boldsymbol{K}$, if $\boldsymbol{K}\cdot\boldsymbol{g} = m\,2\pi$, $m \in \mathbb{Z}$.
  $\rightsquigarrow$ Restrict $\boldsymbol{k}$ to the first Brillouin zone.

(Exclude $\boldsymbol{k}+\boldsymbol{K}$ $\forall \boldsymbol{g}, m$.)
($\boldsymbol{K}$: A vector of the reciprocal lattice.)

Structures with spatial periodicity

$\sim \exp(\mathrm{i}\omega t)$ (FD)

$\boldsymbol{g}$: a lattice vector, such that $\epsilon(\boldsymbol{r}+\boldsymbol{g}) = \epsilon(\boldsymbol{r})$

$$\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(\boldsymbol{r}+\boldsymbol{g}) = \begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{g}}. \qquad \text{(QPBC)}$$

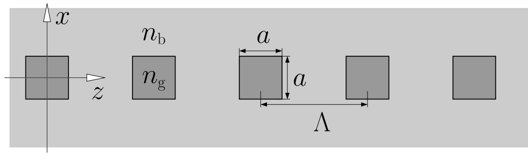
(. . . if $\boldsymbol{g}$ connects the boundaries of a unit cell.)

FB-wave eigenproblem:
Given a wavevector $\boldsymbol{k}$, look for frequencies $\omega \in \mathbb{R}$, such that there exist nonzero solutions $(\boldsymbol{E}, \boldsymbol{H})$ on a unit cell domain, with quasi-periodic boundary conditions (QPBC).

(Include this in the list of computational problems of lecture D.)
(Bandstructure calculations: Information on inifinite periodic strctures.)
(Calculations on a (small) unit cell domain, typically computationally cheap.)
(Finite structures, (most) defects, external excitation, etc.: scattering solvers (FD, TD) or resonance solvers required, on the full system domain.)

A sequence of dielectric rods

$\sim \exp(\mathrm{i}\omega t)$ (FD)



$a = 0.4\,\mu\mathrm{m}$, $\Lambda = 1\,\mu\mathrm{m}$,
$n_{\mathrm{b}} = 1.0$, $n_{\mathrm{g}} = \sqrt{12}$.

[Joannopoulos, Johnson, Winn, Meade, *Photonic Crystals: Molding the Flow of Light*, 2nd edition, Princeton, 2008.]

- 1-D periodicity, $\epsilon(x,z) = \epsilon(x, z+\Lambda)$.
- 2-D TE setting, $E_y(x,z) = ?$, $(\partial_x^2 + \partial_z^2 + k^2\epsilon)E_y = 0.$ $(*)$
- Look for FB waves $E_y(x,z) = u(x,z)\,\mathrm{e}^{-\mathrm{i}\beta z}$.
  (β: the FB wavenumber, $u(x,z) = u(x, z+\Lambda)\ \forall z$.)
- $E_y(x, z+\Lambda) = u(x, z+\Lambda)\,\mathrm{e}^{-\mathrm{i}\beta(z+\Lambda)} = E_y(x,z)\,\mathrm{e}^{-\mathrm{i}\beta\Lambda}$
  Restrict $(*)$ to $z \in [0, \Lambda]$ with boundary conditions
  $E_y(x,\Lambda) = \mathrm{e}^{-\mathrm{i}\beta\Lambda}E_y(x,0)$, $\partial_z E_y(x,\Lambda) = \mathrm{e}^{-\mathrm{i}\beta\Lambda}\partial_z E_y(x,0)$.
- Brillouin zone: $K\Lambda = \pm m\,2\pi \rightsquigarrow \beta \in [-\pi/\Lambda, \pi/\Lambda]$.

(BEP simulations (Lecture G.24), ω given, β determined from an eigenvalue problem.)
(Shaded region: above the "light line", $\omega^2 n_{\mathrm{b}}^2/c^2 > k_z^2$, potentially leaky solutions.)

Defect waveguides

(At a frequency in the bandgap of a photonic crystal: ∃ "forbidden" regions $\rightsquigarrow$ The waves travel elsewhere . . .)

Line defects in a square lattice of dielectric rods,
excitation through conventional waveguides, 2-D QUEP simulations.

- A straight defect waveguide.
- 90° corner in a defect waveguide.

A touch of plasmonics

"Plasmonics": ?

Keywords:
- A branch of photonics.
- Optics involving metals and metal surfaces.
- Interaction between the electromagnetic field and free electrons in the metal / at the surface.
- Strong field confinement, "beyond the diffraction limit".
- "Strong" local fields, near field enhancement (nonlinearity).
- "Small" structures: Nano . . . .
- Applications: Sensing, focusing ("antennas", microscopy), communication (short-range), chemistry, art.

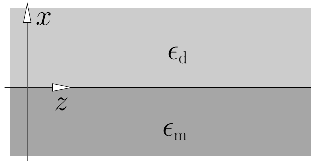
Context of this lecture:
- Problems of general classical electromagnetics & methods as discussed; different emphasis.
- Presence of metals: complex (negative) permittivity, strong dispersion, losses; some concepts do not apply.
- Among the phenomena not encountered so far: Surface plasmon polaritons (SPPs).

Surface plasmon polaritons

(Surface waves,
"plasmon": oscillations of the free electron plasma,
"polariton": strong interaction of the optical e.m. field with polarizable matter; here discussed merely as . . .)

Optical waves confined at a metal / dielectric interface.

(. . . accepting the permittivities as given, disregarding any processes in the metal or dielectric that lead to this permittivity.)

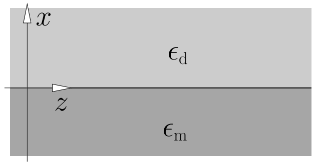


$x > 0$: dielectric, $\epsilon_{\rm d} = n_{\rm d}^2 \in \mathbb{R}$.
$x < 0$: metal, $\epsilon_{\rm m} \in \mathbb{C}$.

(Coordinates in line with the previous discussion in this lecture, but different from literature "standard".)

Surface plasmon polaritons

$\sim \exp(\mathrm{i}\omega t)$ (FD)



$x > 0$: dielectric, $\epsilon_{\rm d} = n_{\rm d}^2 \in \mathbb{R}$.
$x < 0$: metal, $\epsilon_{\rm m} \in \mathbb{C}$.

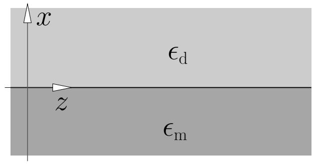
2-D TE / TM waves.

- Look for fields $\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x,z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x)\, \mathrm{e}^{-\mathrm{i}\gamma z}$,
  $\gamma = \beta - \mathrm{i}\alpha \in \mathbb{C}, \ \ \beta, \alpha \geq 0.$
- Principal component $\phi = \bar{E}_y$ (TE) and $\phi = \bar{H}_y$ (TM),
  continuity of ϕ, $\eta\partial_x\phi$ at the interface, $\eta = 1$ (TE), $\eta = 1/\epsilon$ (TM),
  $\partial_x^2\phi + (k^2\epsilon - \gamma^2)\phi = 0$ for $x < 0$ and $x > 0$.
- Ansatz:

$$\phi(x) = \begin{cases} \phi_0\, \mathrm{e}^{-\mathrm{i}k_{\rm d}x}, & x > 0, \qquad k_{\rm d} = \chi_{\rm d} - \mathrm{i}\kappa_{\rm d}, \quad \kappa_{\rm d} > 0, \\ \phi_0\, \mathrm{e}^{\,\mathrm{i}k_{\rm m}x}, & x < 0, \qquad k_{\rm m} = \chi_{\rm m} - \mathrm{i}\kappa_{\rm m}, \quad \kappa_{\rm m} > 0. \end{cases}$$

Surface plasmon polaritons

$\sim \exp(\mathrm{i}\omega t)$ (FD)



$x > 0$: dielectric, $\epsilon_{\rm d} = n_{\rm d}^2 \in \mathbb{R}$.
$x < 0$: metal, $\epsilon_{\rm m} \in \mathbb{C}$.

- $x > 0$: $k^2\epsilon_{\rm d} - k_{\rm d}^2 - \gamma^2 = 0$,
  $x < 0$: $k^2\epsilon_{\rm m} - k_{\rm m}^2 - \gamma^2 = 0$.
- $x = 0$: Continuity of ϕ. (Ansatz.)
  $x = 0$: Continuity of $\eta\partial_x\phi$ $\rightsquigarrow$ $-k_{\rm d}\eta_{\rm d} = k_{\rm m}\eta_{\rm m}$.

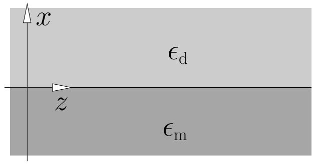
  (TE): $-k_{\rm d} = k_{\rm m}$ $\rightsquigarrow$ No TE solution. (Required: $\kappa_{\rm d} > 0$ & $\kappa_{\rm m} > 0$.)

  (TM): $-\dfrac{k_{\rm d}}{\epsilon_{\rm d}} = \dfrac{k_{\rm m}}{\epsilon_{\rm m}}$. (OK, if $\mathrm{Re}\,\epsilon_{\rm m} < 0$.)
  (No solution for an interface between pure dielectrics.)

$\hookrightarrow$ $\gamma = \dfrac{\omega}{\mathrm{c}}\sqrt{\dfrac{\epsilon_{\rm d}\epsilon_{\rm m}}{\epsilon_{\rm d} + \epsilon_{\rm m}}}$, the dispersion equation for SPPs.
(Note that, in general, $\epsilon_{\rm m}(\omega)$.)

Surface plasmon polaritons

$\sim \exp(\mathrm{i}\omega t)$ (FD)



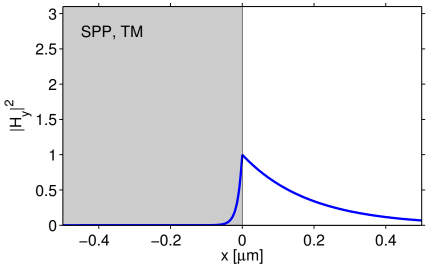
$x > 0$: dielectric, $\epsilon_{\rm d} = n_{\rm d}^2 \in \mathbb{R}$.
$x < 0$: metal, $\epsilon_{\rm m} \in \mathbb{C}$.

Characteristic lengths:

- $x > 0$: $|\phi(x)|^2 \sim \mathrm{e}^{-2\kappa_{\rm d}x}$ $\rightsquigarrow$ $d_{\rm d} = \dfrac{1}{2\kappa_{\rm d}}$. (Penetration depth, dielectric.)
- $x < 0$: $|\phi(x)|^2 \sim \mathrm{e}^{2\kappa_{\rm m}x}$ $\rightsquigarrow$ $d_{\rm m} = \dfrac{1}{2\kappa_{\rm m}}$. (Penetration depth, metal.)
- $|E.|^2 \sim \mathrm{e}^{-2\alpha z}$ $\rightsquigarrow$ $L_{\rm p} = \dfrac{1}{2\alpha}$, the SPP propagation length.

Field profiles

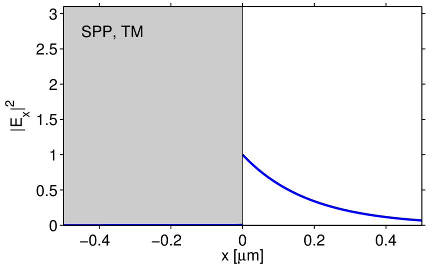
SPP, Ag / air, $\lambda = 0.633\ \mu\text{m}$,
$\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16\ \mu\text{m}$,
$\beta/k = 1.036$,
$d_d = 190\ \text{nm}$,
$d_m = 12\ \text{nm}$.

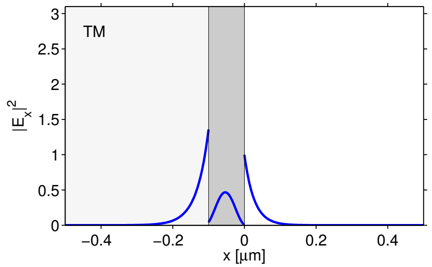
Field profiles

SPP, Ag / air, $\lambda = 0.633\ \mu\text{m}$,
$\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16\ \mu\text{m}$,
$\beta/k = 1.036$,
$d_d = 190\ \text{nm}$,
$d_m = 12\ \text{nm}$.

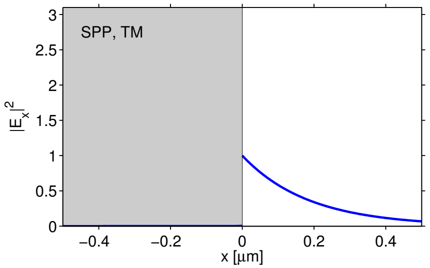
SiO_2 / Si(100 nm) / air, $\lambda = 0.633\ \mu\text{m}$,
$\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$,
$n_{\text{eff}} = 2.106$,
$d_{\text{air}} = 27\ \text{nm}$.

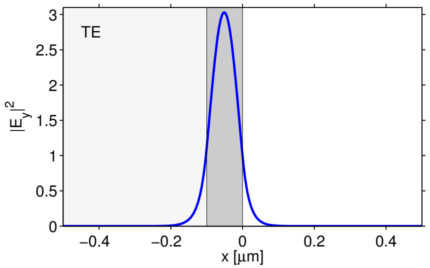
Field profiles

SPP, Ag / air, $\lambda = 0.633\ \mu\text{m}$,
$\epsilon_m = -14.5 - 1.2i$, $\epsilon_d = 1.0$



$L_p = 16\ \mu\text{m}$,
$\beta/k = 1.036$,
$d_d = 190\ \text{nm}$,
$d_m = 12\ \text{nm}$.

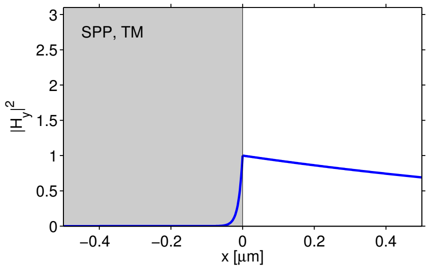
SiO_2 / Si(100 nm) / air, $\lambda = 0.633\ \mu\text{m}$,
$\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_p = \infty$,
$n_{\text{eff}} = 2.883$,
$d_{\text{air}} = 19\ \text{nm}$.

Field profiles

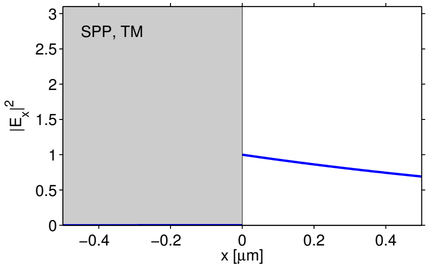
SPP, Ag / air, $\lambda = 1.550\ \mu\text{m}$,
$\epsilon_m = -121 - 4.4i$, $\epsilon_d = 1.0$



$L_p = 812\ \mu\text{m}$,
$\beta/k = 1.0042$,
$d_d = 1350\ \text{nm}$,
$d_m = 11\ \text{nm}$.

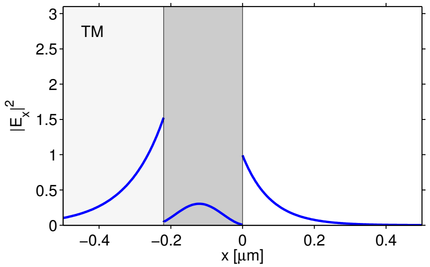
Field profiles

SPP, Ag / air, $\lambda = 1.550\,\mu\text{m}$,
$\epsilon_{\text{m}} = -121 - 4.4\text{i}$, $\epsilon_{\text{d}} = 1.0$



$L_{\text{p}} = 812\,\mu\text{m}$,
$\beta/k = 1.0042$,
$d_{\text{d}} = 1350\,\text{nm}$,
$d_{\text{m}} = 11\,\text{nm}$.

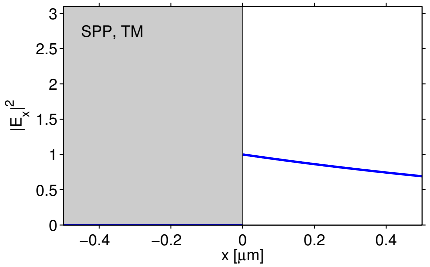
SiO_2 / Si(220 nm) / air, $\lambda = 1.550\,\mu\text{m}$,
$\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_{\text{p}} = \infty$,
$n_{\text{eff}} = 1.874$,
$d_{\text{air}} = 78\,\text{nm}$.

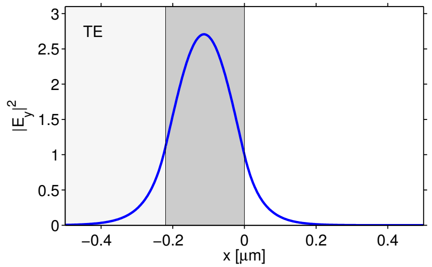
Field profiles

SPP, Ag / air, $\lambda = 1.550\,\mu\text{m}$,
$\epsilon_{\text{m}} = -121 - 4.4\text{i}$, $\epsilon_{\text{d}} = 1.0$



$L_{\text{p}} = 812\,\mu\text{m}$,
$\beta/k = 1.0042$,
$d_{\text{d}} = 1350\,\text{nm}$,
$d_{\text{m}} = 11\,\text{nm}$.

SiO_2 / Si(220 nm) / air, $\lambda = 1.550\,\mu\text{m}$,
$\epsilon = 1.45^2 : 3.45^2 : 1.0$



$L_{\text{p}} = \infty$,
$n_{\text{eff}} = 2.805$,
$d_{\text{air}} = 47\,\text{nm}$.

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Upcoming

Next lectures:

- Oblique semi-guided waves: 2-D integrated optics.
- Summary, concluding remarks.