

# Optical Waveguide Theory (I)



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Paderborn University — Summer Semester 2018

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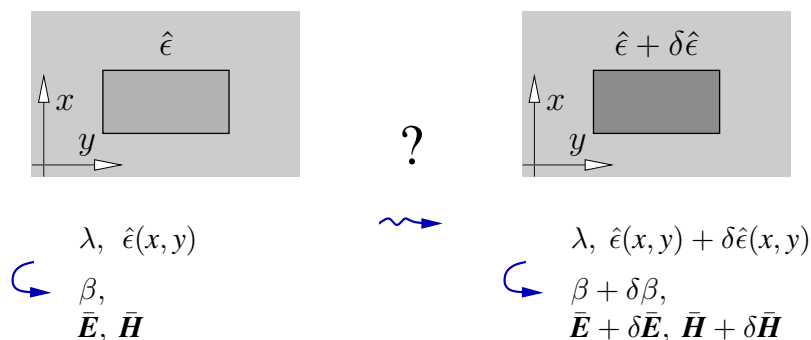
## Course overview

### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

### Perturbations of single modes

$\sim \exp(i\omega t)$  (FD)



### A functional for guided modes of 3-D dielectric waveguides

(→ Exercise.)

$$\bullet \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\mathbf{E}} \\ \bar{\mathbf{H}} \end{pmatrix}(x, y) e^{-i\beta z}, \quad \beta \in \mathbb{R},$$

$$\bar{\mathbf{E}}, \bar{\mathbf{H}} \rightarrow 0 \text{ for } x, y \rightarrow \pm\infty.$$

$$\bullet (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{E}} = -i\omega\mu_0\bar{\mathbf{H}}, \quad (\mathbf{C} + i\beta\mathbf{R})\bar{\mathbf{H}} = i\omega\epsilon_0\hat{\epsilon}\bar{\mathbf{E}},$$

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.$$

$$\bullet \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) := \frac{\omega\epsilon_0\langle \bar{\mathbf{E}}, \hat{\epsilon}\bar{\mathbf{E}} \rangle + \omega\mu_0\langle \bar{\mathbf{H}}, \bar{\mathbf{H}} \rangle + i\langle \bar{\mathbf{E}}, \mathbf{C}\bar{\mathbf{H}} \rangle - i\langle \bar{\mathbf{H}}, \mathbf{C}\bar{\mathbf{E}} \rangle}{\langle \bar{\mathbf{E}}, \mathbf{R}\bar{\mathbf{H}} \rangle - \langle \bar{\mathbf{H}}, \mathbf{R}\bar{\mathbf{E}} \rangle},$$

$$\langle \bar{\mathbf{F}}, \bar{\mathbf{G}} \rangle = \iint \bar{\mathbf{F}}^* \cdot \bar{\mathbf{G}} \, dx \, dy.$$

$$\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta \quad (*), \quad \left. \frac{d}{ds} \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}} + s\bar{\mathbf{F}}, \bar{\mathbf{H}} + s\bar{\mathbf{G}}) \right|_{s=0} = 0 \quad (**)$$

at valid mode fields  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ , for arbitrary  $\bar{\mathbf{F}}, \bar{\mathbf{G}}$ .

(\*): "arbitrary"  $\hat{\epsilon}$ .  
(\*\*): Hermitian  $\hat{\epsilon}$ .

## Perturbations of single modes

- Available: Mode  $\beta, \bar{\mathbf{E}}, \bar{\mathbf{H}}$  for parameters  $\lambda, \hat{\epsilon}$ ; ( $\hat{\epsilon} = \hat{\epsilon}^\dagger$ )  
 $\mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$ ,  $\mathcal{B}_{\hat{\epsilon}}$  stationary at  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ .

- Investigate parameters  $\lambda, \hat{\epsilon} + \delta\hat{\epsilon}$ , for a “small” change  $\delta\hat{\epsilon}$ :

$$\mathcal{B}_{\hat{\epsilon} + \delta\hat{\epsilon}}(\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}) = \beta + \delta\beta$$

$$\curvearrowright \dots \quad \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}} + \delta\bar{\mathbf{E}}, \bar{\mathbf{H}} + \delta\bar{\mathbf{H}}) \approx \mathcal{B}_{\hat{\epsilon}}(\bar{\mathbf{E}}, \bar{\mathbf{H}}) = \beta$$

$$\curvearrowright \dots \quad \delta(\cdot)\delta(\cdot)$$

$$\curvearrowright \delta\beta = \frac{\omega\epsilon_0 \langle \bar{\mathbf{E}}, \delta\hat{\epsilon} \bar{\mathbf{E}} \rangle}{\langle \bar{\mathbf{E}}, \mathbf{R}\bar{\mathbf{H}} \rangle - \langle \bar{\mathbf{H}}, \mathbf{R}\bar{\mathbf{E}} \rangle}, \quad \text{or} \quad \delta\beta = \frac{\omega\epsilon_0 \iint_{\square} \bar{\mathbf{E}}^* \cdot \delta\hat{\epsilon} \bar{\mathbf{E}} \, dx \, dy}{2 \operatorname{Re} \iint_{\square} (\bar{\mathbf{E}}_x^* \bar{\mathbf{H}}_y - \bar{\mathbf{E}}_y^* \bar{\mathbf{H}}_x) \, dx \, dy}.$$

(Valid for *small* perturbations: The original mode profiles are good approximations of the true fields in the modified structure.)

## Small uniform change in refractive index



- $n \rightarrow n + \delta n$  on  $\square$ ,  $n, \delta n$  constant on  $\square$

$$\curvearrowright \beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{\omega\epsilon_0 n \iint_{\square} |\bar{\mathbf{E}}|^2 \, dx \, dy}{\operatorname{Re} \iint_{\square} (\bar{\mathbf{E}}_x^* \bar{\mathbf{H}}_y - \bar{\mathbf{E}}_y^* \bar{\mathbf{H}}_x) \, dx \, dy} \delta n.$$

( $\delta\epsilon = 2n\delta n$ .)  
 (Plausible:  $\delta\beta \sim \delta n$ ,  $\delta\beta \sim |\bar{\mathbf{E}}|^2|_{\square}$ .)

## Small attenuation



- $n \rightarrow n - in''$  on  $\square$ ,  $n, n''$  constant on  $\square, n, n'' \in \mathbb{R}$

$$\curvearrowright \beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{-i\omega\epsilon_0 n \iint_{\square} |\bar{\mathbf{E}}|^2 \, dx \, dy}{\operatorname{Re} \iint_{\square} (\bar{\mathbf{E}}_x^* \bar{\mathbf{H}}_y - \bar{\mathbf{E}}_y^* \bar{\mathbf{H}}_x) \, dx \, dy} n''.$$

( $\delta\epsilon = -i2n''$ .)  
 (Different attenuation for each mode.)  
 (Damping, power, plane wave:  $\sim \exp(-2kn''z)$ , mode:  $\sim \exp(-2kn''z)$ .)

## Small anisotropy

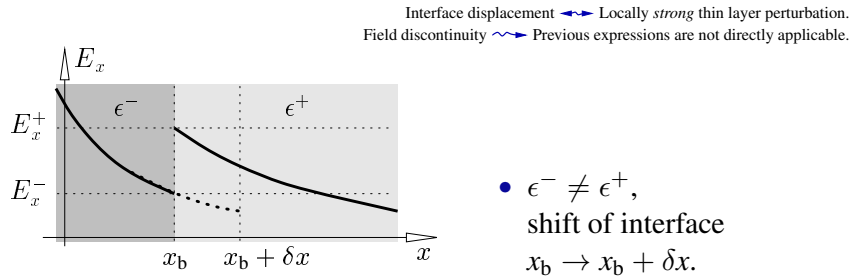


- $\hat{\epsilon} \rightarrow \hat{\epsilon} + \delta\hat{\epsilon}$  on  $\square$ ,  $\epsilon, \delta\hat{\epsilon}$  constant on  $\square$

$$\curvearrowright \beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{\omega\epsilon_0 \iint_{\square} \bar{\mathbf{E}}^* \cdot \delta\hat{\epsilon} \bar{\mathbf{E}} \, dx \, dy}{2 \operatorname{Re} \iint_{\square} (\bar{\mathbf{E}}_x^* \bar{\mathbf{H}}_y - \bar{\mathbf{E}}_y^* \bar{\mathbf{H}}_x) \, dx \, dy}.$$

(Phase shifts due to anisotropic contributions to the permittivity.)  
 (Polarization coupling might occur for modes with “close” propagation constants  $\rightarrow$  CMT.)

## Small displacements of dielectric interfaces



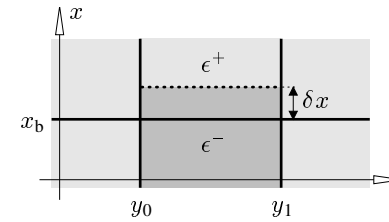
- Reposition discontinuity in field:  $E_x \rightarrow E_x + \delta E_x$ ,

$$\delta E_x(x, y) = \begin{cases} \frac{\epsilon^+ - \epsilon^-}{\epsilon^-} E_x(x, y), & \text{for } x_b < x < x_b + \delta x, \\ 0, & \text{otherwise.} \end{cases}$$

- Use functional with locally modified field

$\hookrightarrow$  ... (omitted) ...  $\rightsquigarrow$

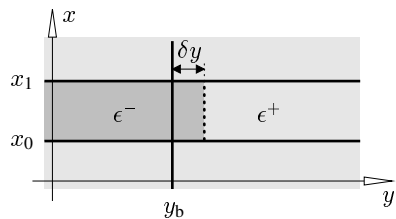
## Small displacements of dielectric interfaces



- Displacement of the interface at  $x_b$  between  $y_0$  and  $y_1$  by  $\delta x$ :

$$\delta \beta = \frac{\omega \epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{y_0}^{y_1} \left( \frac{1}{\epsilon^- \epsilon^+} |\epsilon \bar{E}_x|^2 + |\bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x_b, y) dy}{\text{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta x.$$

## Small displacements of dielectric interfaces

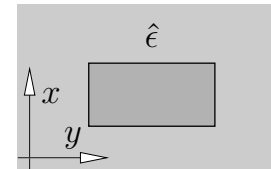


- Displacement of the interface at  $y_b$  between  $x_0$  and  $x_1$  by  $\delta y$ :

$$\hookrightarrow \beta \rightarrow \beta + \delta \beta,$$

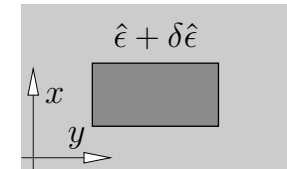
$$\delta \beta = \frac{\omega \epsilon_0}{2} \frac{(\epsilon^- - \epsilon^+) \int_{x_0}^{x_1} \left( |\bar{E}_x|^2 + \frac{1}{\epsilon^- \epsilon^+} |\epsilon \bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x, y_b) dx}{\text{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy} \delta y.$$

## Perturbations of single modes



$$\lambda, \hat{\epsilon}(x, y)$$

$$\hookrightarrow \beta, \bar{\mathbf{E}}, \bar{\mathbf{H}}$$



$$\lambda, \hat{\epsilon}(x, y) + \delta \hat{\epsilon}(x, y)$$

$$\hookrightarrow \beta + \delta \beta, \approx \bar{\mathbf{E}}, \approx \bar{\mathbf{H}}$$

- View  $\frac{\delta \beta}{\delta p}$  as  $\frac{\partial \beta}{\partial p}$ : slope of the dispersion curves  $\beta$  vs.  $p$ .
- Depending on the parametrization, change of a parameter value might require several perturbations.
- First order theory: In case of multiple perturbations, add the effects of the individual expressions.
- Estimation of fabrication tolerances: The phase shifts  $\delta \beta$  enter into respective scattering matrix models.
- Wavelength shifts ... ?

## Small shift of frequency or vacuum wavelength

(\*) : Explicit frequency dependence of  $\mathcal{B}$  & dependence through  $\hat{\epsilon}$ .  
 (\*\*) : Frequency dependence of  $\bar{\mathbf{E}}, \bar{\mathbf{H}}$ .

$$\beta(\omega) = \mathcal{B}_{\hat{\epsilon}}(\omega; \bar{\mathbf{E}}(\omega), \bar{\mathbf{H}}(\omega))$$

$$\begin{aligned} \hookrightarrow \frac{\partial \beta}{\partial \omega} &= \frac{\partial \mathcal{B}_{\hat{\epsilon}}}{\partial \omega} (*) + \frac{\partial}{\partial s} \mathcal{B}_{\hat{\epsilon}} \left( \omega; \bar{\mathbf{E}} + s \frac{\partial \bar{\mathbf{E}}}{\partial \omega}, \bar{\mathbf{H}} \right) \Big|_{s=0} (**) \\ &\quad + \frac{\partial}{\partial s} \mathcal{B}_{\hat{\epsilon}} \left( \omega; \bar{\mathbf{E}}, \bar{\mathbf{H}} + s \frac{\partial \bar{\mathbf{H}}}{\partial \omega} \right) \Big|_{s=0} (**) \\ &= \frac{\partial \mathcal{B}_{\hat{\epsilon}}}{\partial \omega}, \end{aligned} \quad \text{(Stationarity of } \mathcal{B} \text{ at } \bar{\mathbf{E}}, \bar{\mathbf{H}}.)$$

$$\hookrightarrow \frac{\partial \beta}{\partial \omega} = \frac{\iint (\epsilon_0 \bar{\mathbf{E}}^* \cdot \frac{\partial(\omega \hat{\epsilon})}{\partial \omega} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2) dx dy}{2 \operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy}.$$

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## Small shift of frequency or vacuum wavelength

If dispersion can be neglected,  $\partial_\omega \hat{\epsilon} = 0$ :

$$\hookrightarrow \frac{\partial \beta}{\partial \omega} = \frac{\iint (\epsilon_0 \bar{\mathbf{E}}^* \cdot \hat{\epsilon} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2) dx dy}{2 \operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy},$$

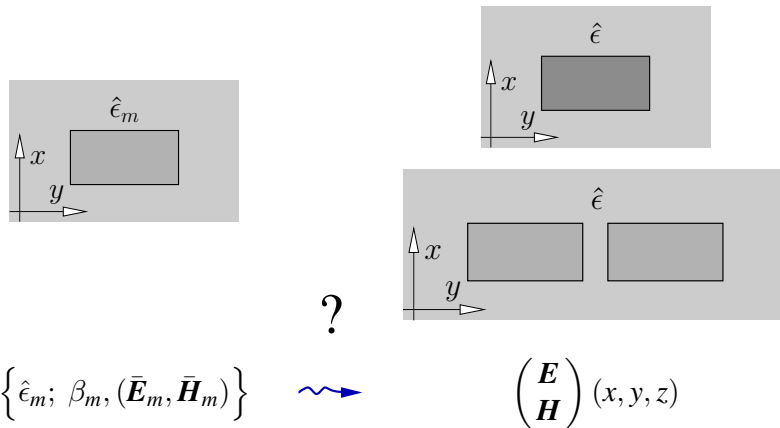
$$\hookrightarrow \frac{\partial \beta}{\partial \lambda} = -\frac{\pi c}{\lambda^2} \frac{\iint (\epsilon_0 \bar{\mathbf{E}}^* \cdot \hat{\epsilon} \bar{\mathbf{E}} + \mu_0 |\bar{\mathbf{H}}|^2) dx dy}{\operatorname{Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) dx dy}.$$

( $\omega = 2\pi c / \lambda \rightarrow \partial_\lambda \omega = -2\pi c / \lambda^2$ )  
 (Compare with expression based on homogeneity, H, 12.)

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## Coupled mode theory (CMT)

$\sim \exp(i\omega t)$  (FD)



(Next: One of many variants of approaches to CMT.)

(Propagation & interaction of basis fields along a common propagation coordinate.)

[D.G. Hall, B.J. Thompson, *Selected papers on Coupled-Mode Theory in Guided-Wave Optics*, SPIE Milestone series MS 84 (1993)]

(Codirectional coupling (here), versus contradirectional coupling, coupling to radiation modes, nonlinear coupling.)

(Hybrid variant (HCMT): separate lecture.)

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## Coupled mode theory (CMT)

- Investigate a permittivity  $\hat{\epsilon}$ , look for fields  $\mathbf{E}, \mathbf{H}$  with

$$\nabla \times \mathbf{E} = -i\omega \mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega \epsilon_0 \hat{\epsilon} \mathbf{E}.$$

( $\hat{\epsilon}(x, y, z)$ , in general.)

- Available: A set of fields  $\{\mathbf{E}_m, \mathbf{H}_m\}$  for permittivities  $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$ ;

$$\nabla \times \mathbf{E}_m = -i\omega \mu_0 \mathbf{H}_m, \quad \nabla \times \mathbf{H}_m = i\omega \epsilon_0 \hat{\epsilon}_m \mathbf{E}_m.$$

(Not necessarily "modes".)

- Assume that  $(\mathbf{E}, \mathbf{H})$  can be well approximated by

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) \approx \sum_m C_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z),$$

$C_m$ : unknown amplitudes, common propagation coordinate  $z$ .

(Choose  $\hat{\epsilon}_m$  as close as possible to  $\hat{\epsilon}$ .)


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## Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for  $\mathbf{E}, \mathbf{H}$ .)


 $\dots$  (  $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)  
 $\dots$  (Apply identity  $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}$ .)  
 $\dots$  (  $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)  
 $\dots$  (Manipulate, arrange terms, tidy up.)

$$\sum_m o_{lm} \partial_z C_m = -i \sum_m k_{lm} C_m \quad \forall l, \quad \text{coupled mode equations.}$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$


## Coupled mode theory (CMT)

(Variational derivation of CMT equations.)

$$\mathcal{F}(\mathbf{E}, \mathbf{H}) = \iiint \left\{ \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E}^* \cdot (\nabla \times \mathbf{H}) + i\omega\mu_0 \mathbf{H}^* \cdot \mathbf{H} + i\omega\epsilon_0 \mathbf{E}^* \cdot \hat{\epsilon} \mathbf{E} \right\} dx dy dz,$$

$$\delta\mathcal{F} = 0 \quad \forall \delta\mathbf{E}, \delta\mathbf{H} \quad \longleftrightarrow \quad \nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\epsilon_0 \hat{\epsilon} \mathbf{E}.$$

(Restrict  $\mathcal{F}$  to the CMT ansatz for  $\mathbf{E}, \mathbf{H} \rightsquigarrow \mathcal{F}_c(\mathbf{C})$ , require  $\delta\mathcal{F}_c = 0 \quad \forall \delta\mathbf{C}$ .)


 $\dots$  (  $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}$ ,  $\iint dx dy$ ,  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)  
 $\dots$  (Manipulate, arrange terms, tidy up.)

$$\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K}\mathbf{C}, \quad \text{coupled mode equations.}$$

$$\mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$


$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

## Coupled mode theory (CMT)

(Starting point: a "reciprocity identity".)

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}_l^* - \mathbf{E} \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) \mathbf{E}.$$

(Insert CMT ansatz for  $\mathbf{E}, \mathbf{H}$ .)


 $\dots$  (  $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)  
 $\dots$  (Apply identity  $\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}$ .)  
 $\dots$  (  $\iint dx dy$ , assume  $\mathbf{E}_m, \mathbf{H}_m \rightarrow 0$  for  $x, y \rightarrow \pm\infty$ .)  
 $\dots$  (Manipulate, arrange terms, tidy up.)


$$\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K}\mathbf{C}, \quad \text{coupled mode equations.}$$

$$\mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

## Coupled mode equations

$\dots$   

 $\mathbf{O} \partial_z \mathbf{C} = -i\mathbf{K}\mathbf{C}, \quad \mathbf{C} = (C_m), \quad \mathbf{O} = (o_{lm}), \quad \mathbf{K} = (k_{lm}).$

$$o_{lm} = \frac{1}{4} \iint (\mathbf{E}_l^* \times \mathbf{H}_m - \mathbf{H}_l^* \times \mathbf{E}_m)_z dx dy = (\mathbf{E}_l, \mathbf{H}_l; \mathbf{E}_m, \mathbf{H}_m),$$

$$k_{lm} = \frac{\omega\epsilon_0}{4} \iint \mathbf{E}_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) \mathbf{E}_m dx dy.$$

- A set of coupled *ordinary* linear differential equations, of first order. (Here.)
- $o_{lm}$ : **power coupling coefficients** (field overlaps). (No reason to assume  $o_{lm} = \delta_{lm}$ , in general.)
- $k_{lm}$ : **coupling coefficients**.
- $z$ -dependence of  $\hat{\epsilon}, \hat{\epsilon}_m, \mathbf{E}_m, \mathbf{H}_m \rightsquigarrow o_{lm}(z), k_{lm}(z), \mathbf{O}(z), \mathbf{K}(z)$ . (Compare the bend-straight couplers, Lecture H.)

$\dots$  to be solved by numerical procedures. (In general.)

## CMT for longitudinally homogeneous structures

$$\partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0,$$

$$\text{basis: guided modes} \quad \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) e^{-i\beta_m z},$$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \sum_m c_m(z) \begin{pmatrix} \mathbf{E}_m \\ \mathbf{H}_m \end{pmatrix} (x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y).$$

$(c_m(z) = C_m(z) \exp(-i\beta_m z), \text{ rewrite CMT equations for } c_m(z).)$



...



...

$(\nabla \cdot (\mathbf{H}_m \times \mathbf{E}_l^* - \mathbf{E}_m \times \mathbf{H}_l^*) = i\omega\epsilon_0 \mathbf{E}_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) \mathbf{E}, \text{ integrate, rewrite for } \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m.)$

(Symmetrize coefficients.)

$$\sum_m \sigma_{lm} \partial_z c_m = -i \sum_m (b_{lm} + \kappa_{lm}) c_m \quad \forall l,$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z \, dx \, dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

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## Longitudinally constant structures, coupled mode equations

...

$(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$

$$\hookrightarrow \mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z \, dx \, dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- $\sigma_{ml}^* = \sigma_{lm}, \quad b_{ml}^* = b_{lm}; \quad \kappa_{ml}^* = \kappa_{lm}, \text{ if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m,$   
 $\mathbf{S}^\dagger = \mathbf{S}, \quad \mathbf{B}^\dagger = \mathbf{B}; \quad \mathbf{Q}^\dagger = \mathbf{Q}, \text{ if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m.$

- Power:  $P = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_{l,m} c_l^* (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) c_m = \mathbf{c}^* \cdot \mathbf{S} \mathbf{c}$

$$\hookrightarrow \partial_z P = i\mathbf{c}^* \cdot ((\mathbf{B} + \mathbf{Q})^\dagger - (\mathbf{B} + \mathbf{Q}))\mathbf{c}, \quad \partial_z P = 0 \text{ for } \mathbf{B}^\dagger = \mathbf{B}, \quad \mathbf{Q}^\dagger = \mathbf{Q}.$$

(For lossless waveguides the scheme is power conservative.)

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## Longitudinally constant structures, coupled mode equations

...

$(\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0)$

$$\hookrightarrow \mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}).$$

$$\sigma_{lm} = \frac{1}{4} \iint (\bar{\mathbf{E}}_l^* \times \bar{\mathbf{H}}_m - \bar{\mathbf{H}}_l^* \times \bar{\mathbf{E}}_m)_z \, dx \, dy = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m),$$

$$\kappa_{lm} = \frac{\omega\epsilon_0}{8} \iint \bar{\mathbf{E}}_l \cdot (\delta\hat{\epsilon}_l + \delta\hat{\epsilon}_m) \bar{\mathbf{E}}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}.$$

$$\delta\hat{\epsilon}_m = \hat{\epsilon} - \hat{\epsilon}_m,$$

- A set of coupled *ordinary* linear differential equations, of first order (Here.)
- $\sigma_{lm}$ : **power coupling coefficients** (field overlaps). (No reason to assume  $\sigma_{lm} = \delta_{lm}$ , in general.)
- $\kappa_{lm}$ : **coupling coefficients**.
- $\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0 \rightsquigarrow \partial_z \sigma_{lm} = \partial_z b_{lm} = \partial_z \kappa_{lm} = 0.$

(ODEs with constant coefficients.)

... quasi-analytical solutions.

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## Longitudinally constant structures, formal solution

$$\mathbf{S} \partial_z \mathbf{c} = -i(\mathbf{B} + \mathbf{Q})\mathbf{c},$$

$$\partial_z \mathbf{S} = \partial_z \mathbf{B} = \partial_z \mathbf{Q} = 0.$$

$$\text{Ansatz: } \mathbf{c}(z) = \mathbf{a} e^{-ibz},$$

$\mathbf{a}, b$  constants.

$$\hookrightarrow (\mathbf{B} + \mathbf{Q})\mathbf{a} = b \mathbf{S} \mathbf{a}, \quad \text{a generalized eigenvalue problem.}$$

(Dimension: number of basis modes included.)

Solutions:  $\{\mathbf{a}, b\}$ ,

$$\rightsquigarrow \text{“supermodes”} \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x, y, z) = \left( \sum_m a_m \begin{pmatrix} \bar{\mathbf{E}}_m \\ \bar{\mathbf{H}}_m \end{pmatrix} (x, y) \right) e^{-ibz}.$$

(Superpositions of the original mode profiles with constant coefficients.)

(As many supermodes as there are basis modes.)

(Formalism can be continued: power/orthogonality of supermodes ...)

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## Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes  $(\mathbf{E}_1, \mathbf{H}_1)$ ,  $(\mathbf{E}_2, \mathbf{H}_2)$ :

(Example: two modes supported by the same isotropic waveguide ( $\hat{\epsilon}_1 = \hat{\epsilon}_2$ ); interaction due to small anisotropy ( $\hat{\epsilon}$ ).  
(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0. \quad (\text{Orthogonal modes, uniform normalization } P_m = P_0.)$$

(Or: apply inverse of S to CM equations, continue with redefined expressions for  $\beta_m, \kappa_{lm}$ .)

$$\begin{aligned} \left( \begin{array}{c} \partial_z c_1 \\ \partial_z c_2 \end{array} \right) &= -i \left( \begin{array}{cc} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right), & \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ & & \kappa &= \kappa_{12}/P_0. \\ \dots & \\ \dots & \\ \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) (z) &= e^{-i \frac{(\beta'_1 + \beta'_2)}{2} z} \left( \begin{array}{cc} \cos \rho z - i \frac{\Delta \beta'}{2\rho} \sin \rho z & -i \frac{\kappa}{\rho} \sin \rho z \\ -i \frac{\kappa^*}{\rho} \sin \rho z & \cos \rho z + i \frac{\Delta \beta'}{2\rho} \sin \rho z \end{array} \right) \left( \begin{array}{c} c_{10} \\ c_{20} \end{array} \right), \\ \Delta \beta' &= \beta'_1 - \beta'_2, & \rho &= \sqrt{\left(\frac{\Delta \beta'}{2}\right)^2 + |\kappa|^2}. \end{aligned}$$

## Longitudinally constant structures, one “coupled” mode

CMT with one basis mode:  $\left( \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right) (x, y, z) = c_1(z) \left( \begin{array}{c} \bar{\mathbf{E}}_1 \\ \bar{\mathbf{H}}_1 \end{array} \right) (x, y)$

$$\begin{aligned} \partial_z c_1 &= -i \frac{b_{11} + \kappa_{11}}{\sigma_{11}} c_1, \\ \frac{b_{11}}{\sigma_{11}} &= \beta_1, & \frac{\kappa_{11}}{\sigma_{11}} &= \frac{\omega \epsilon_0 \iint \bar{\mathbf{E}}_1^* \cdot (\hat{\epsilon} - \hat{\epsilon}_1) \bar{\mathbf{E}}_1 \, dx \, dy}{2 \operatorname{Re} \iint (\bar{\mathbf{E}}_{1x}^* \bar{\mathbf{H}}_{1y} - \bar{\mathbf{E}}_{1y}^* \bar{\mathbf{H}}_{1x}) \, dx \, dy} =: \delta \beta_1, \end{aligned}$$

$$\partial_z c_1 = -i(\beta_1 + \delta \beta_1) c_1,$$

$$c_1(z) = c_1(0) e^{-i(\beta_1 + \delta \beta_1)z}.$$

↔ Theory of single mode perturbations.

## Longitudinally constant structures, two coupled modes

Two *orthogonal* coupled modes  $(\mathbf{E}_1, \mathbf{H}_1)$ ,  $(\mathbf{E}_2, \mathbf{H}_2)$ :

(Example: two modes supported by the same isotropic waveguide ( $\hat{\epsilon}_1 = \hat{\epsilon}_2$ ); interaction due to small anisotropy ( $\hat{\epsilon}$ ).  
(Or: non-orthogonality neglected as a further approximation.)

$$\sigma_{lm} = (\bar{\mathbf{E}}_l, \bar{\mathbf{H}}_l; \bar{\mathbf{E}}_m, \bar{\mathbf{H}}_m) = \delta_{lm} P_0. \quad (\text{Orthogonal modes, uniform normalization } P_m = P_0.)$$

(Or: apply inverse of S to CM equations, continue with redefined expressions for  $\beta_m, \kappa_{lm}$ .)

$$\left( \begin{array}{c} \partial_z c_1 \\ \partial_z c_2 \end{array} \right) = -i \left( \begin{array}{cc} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right), \quad \begin{aligned} \beta'_l &= \beta_l + \kappa_{ll}/P_0, \\ \kappa &= \kappa_{12}/P_0. \end{aligned}$$

- $c_{20} = 0 \rightsquigarrow \left| \frac{c_2(z)}{c_1(0)} \right|^2 = \eta_{\max} \sin^2(\rho z), \quad \eta_{\max} = \frac{|\kappa|^2}{|\kappa|^2 + (\Delta \beta'/2)^2}.$
- Maximum conversion**  $\eta_{\max}$  at  $z = L_c$  with  $\rho L_c = \pi/2$ ,  
coupling length  $L_c = \frac{\pi}{\sqrt{(\Delta \beta')^2 + 4|\kappa|^2}},$  (Conversion length, half-beat length.)
- In case of **phase matching**  $\Delta \beta' = \beta'_1 - \beta'_2 = 0$ :  $\eta_{\max} = 1, \quad L_c = \frac{\pi}{2|\kappa|}.$   
(Here the *phase-shifted* propagation constants are relevant.)  
(Small interaction (small maximum conversion) for out-of-phase modes, i.e. for  $|\Delta \beta'|^2 \gg |\kappa|^2$ .)

## Upcoming

Next lectures:

- Hybrid analytical / numerical coupled mode theory.
- A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.

