1. Starting from the representation theorem for functions in $\mathbb{R}^2$ (cf. homework exercise IV.2),

$$u(\boldsymbol{x}) = \frac{1}{2\pi}\oint_{\partial\Omega}\left\{\frac{\partial u}{\partial n}(\boldsymbol{y})\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} - u(\boldsymbol{y})\frac{\partial}{\partial n}\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right\}\mathrm{d}s_y - \frac{1}{2\pi}\iint\limits_{\Omega}\Delta u(\boldsymbol{y})\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\mathrm{d}A_y\,, \quad (1)$$

derive the mean value property for harmonic functions in $\mathbb{R}^2$:

$$u(\boldsymbol{x}) = \frac{1}{2\pi r}\oint_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{y})\,\mathrm{d}s_y\,. \quad (2)$$

Here u is a function that is harmonic in a bounded domain $\Gamma \subset \mathbb{R}^2$. The integral extends over the boundary $\partial B(\boldsymbol{x},r)$ of a disk $B(\boldsymbol{x},r)$ centered at $\boldsymbol{x} \in \Gamma$ with a sufficiently small radius r, such that $B(\boldsymbol{x},r) \subset \Gamma$.

2. Consider the Dirichlet problem for the Poisson equation on a bounded domain Ω in $\mathbb{R}^2$, with inhomogeneous boundary conditions:

$$\Delta u(\boldsymbol{x}) = F(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\Omega, \quad u(\boldsymbol{x}) = f(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\partial\Omega\,, \quad (3)$$

where F and f are given continuous functions. Show that a formal solution of the problem (3) can be stated as

$$u(\boldsymbol{x}) = -\iint\limits_{\Omega} F(\boldsymbol{y})\,G(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}A_y - \oint_{\partial\Omega} f(\boldsymbol{y})\,\frac{\partial}{\partial n}G(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}s_y\,. \quad (4)$$

Here G is the Greens function for the Dirichlet problem of the Laplace equation on Ω, i.e. G is given as

$$G(\boldsymbol{x},\boldsymbol{y}) = v(\boldsymbol{x},\boldsymbol{y}) + \frac{1}{2\pi}\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} \quad (5)$$

where for each $\boldsymbol{x}\in\Omega$ the function v satisfies the conditions

$$\Delta_y v(\boldsymbol{x},\boldsymbol{y}) = 0 \;\text{ for }\; \boldsymbol{y}\in\Omega\,, \quad v(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{2\pi}\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} \;\text{ for }\; \boldsymbol{y}\in\partial\Omega\,. \quad (6)$$

Program: Assume that for the respective domain the functions v and consequently also G are known. Start with the representation theorem (1), together with Greens second identity for $\mathbb{R}^2$ (homework exercise IV.2), evaluated for u and v as above, where integrals and derivatives are meant with respect to $\boldsymbol{y}$, for fixed $\boldsymbol{x}$. Then apply a reasoning analogous to the $\mathbb{R}^3$-case (lecture) to derive the solution (4).

3. Consider the Dirichlet problem for a disk $B(\mathbf{0},\rho)\subset\mathbb{R}^2$ of radius ρ around the origin:

$$\Delta u(\boldsymbol{x}) = 0 \;\text{ for }\; \boldsymbol{x}\in B(\mathbf{0},\rho)\,, \quad u(\boldsymbol{x}) = f(\boldsymbol{x}) \;\text{ for }\; \boldsymbol{x}\in\partial B(\mathbf{0},\rho)\,, \quad (7)$$

with f a given function on the boundary of the disk. Show that the solution can be written in the form

$$u(\boldsymbol{x}) = -\oint_{\partial B(\mathbf{0},\rho)} f(\boldsymbol{y})\,\frac{\partial}{\partial n}G(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}s_y \quad (8)$$

where G is the Greens function for the disk:

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2\pi}\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} - \frac{1}{2\pi}\ln\frac{1}{\left|\dfrac{\rho}{|\boldsymbol{y}|}\boldsymbol{y} - \dfrac{|\boldsymbol{y}|}{\rho}\boldsymbol{x}\right|}\,. \quad (9)$$

In order to show that the v-part of G (cf. equation (5)) satisfies the Laplace equation inside the disk, you might wish to rewrite certain terms by using the identity $(\rho\boldsymbol{y}/|\boldsymbol{y}| - |\boldsymbol{y}|\boldsymbol{x}/\rho)^2 = (\rho\boldsymbol{x}/|\boldsymbol{x}| - |\boldsymbol{x}|\boldsymbol{y}/\rho)^2$.

Good luck !