1. Show that the function

$$u(x, y) = \ln((x - x_0)^2 + (y - y_0)^2) \tag{1}$$

satisfies the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{2}$$

everywhere in the $x$-$y$-plane with the exception of the point $(x_0, y_0)$.

2. Consider the 1-D wave equation

$$u_{tt} = c^2 u_{xx}\,, \quad \text{for} \quad t \geq 0\,, \quad -\infty < x < \infty\,, \tag{3}$$

with initial conditions

$$u(x, 0) = \varphi(x)\,, \quad u_t(x, 0) = \psi(x)\,, \tag{4}$$

where $c$ is a positive constant and $\varphi$, $\psi$ are given (smooth) functions.

(a) Verify that the function

$$u(x, t) = \frac{1}{2}\varphi(x + ct) + \frac{1}{2}\varphi(x - ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s \tag{5}$$

satisfies the initial conditions (4).

(b) Show that, for two arbitrary smooth functions $f$ and $b$ of one variable, the expression

$$u(x, t) = f(x - ct) + b(x + ct) \tag{6}$$

constitutes a solution of the wave equation (3).

(c) Use the result 2b to show that the function (5) is a solution to equation (3).

3. Consider the first order partial differential equation

$$u_x + \mathrm{e}^x u_y = 1. \tag{7}$$

(a) Apply the transformation procedure for linear first order equations as outlined in the lecture:

i. Write down the characteristic equation, and determine an explicit parameterized expression for the characteristic lines. Here “characteristics” are meant as curves in the $x$-$y$-plane.

ii. Apply a transformation induced by the characteristics, and try to find a general solution of the transformed equation that includes an arbitrary smooth function $f$. After transforming that result back to the original variables, verify explicitly that your solution satisfies equation (7).

(b) Investigate equation (7) a second time, now by what we called the *method of characteristics*.

i. Determine the characteristics of (7), here meant as curves in $x$-$y$-$u$-space.

ii. Solve the equation (7) around the $x$-axis, for Cauchy data $u(x, 0) = 1$ given on the $x$-axis. You might wish to use the results from 3(b)i.

iii. Find two different solutions of equation (7) for Cauchy data $u(x, \mathrm{e}^x) = x$ on the curve $y = \mathrm{e}^x$.

iv. Show that there is no solution of equation (7) for Cauchy data $u(x, \mathrm{e}^x) = 1$ (here it is sufficient to use directly the expression for the Cauchy data and the partial differential equation).

*Good luck !*