1. Consider the linear second order PDE with parameter α

$$u_{xx} + \alpha\, u_{xy} + 7\, u_{yy} = 0\,. \tag{1}$$

   (a) Depending on the value of α, determine the type of the PDE (hyperbolic, parabolic, or elliptic).

   (b) For $\alpha = -8$,

      i. state the characteristic equations of (1),

      ii. solve these for the characteristic curves, and sketch some of the characteristics,

      iii. apply a change of variables and transform the PDE (1) to its canonical form,

      iv. find a general solution of the transformed equation that includes two free functions,

      v. transform the solution back to the original variables x, y, and

      vi. verify that the general solution you've found in this way satisfies the original equation (1).

   (c) Follow the procedure of part 1b for equation Eq. (1) with $\alpha = \sqrt{28}$.

   (d) For $\alpha = \sqrt{3}$, devise a change of variables that transforms (1) to its canonical form.

2. An alternative canonical form for hyperbolic equations:

   (a) Find a change of variables $\{x, y;\ u(x, y)\} \longrightarrow \{\xi, \eta;\ w(\xi, \eta)\}$ that transforms the hyperbolic equation in standard form

$$u_{xy} + \Phi(u, u_x, u_y, x, y) = 0 \tag{2}$$

   into a PDE of the form

$$w_{\xi\xi} - w_{\eta\eta} + \Psi(w, w_\xi, w_\eta, \xi, \eta) = 0. \tag{3}$$

   Hint: Try linear superpositions with constant coefficients of the original coordinates for the transformation rules $\{x, y\} \longrightarrow \{\xi, \eta\}$.

   (b) Transform the equation

$$v_{rr} - 8\, v_{rs} + 7\, v_{ss} = 0 \tag{4}$$

   to the alternative canonical form (3).

3. Consider a linear hyperbolic equation with constant coefficients in alternative canonical form, i.e. an equation

$$u_{xx} - u_{yy} + a\, u_x + b\, u_y + c\, u = 0\,, \tag{5}$$

   where a, b, and c are constants. Determine constants α, β, and h such that the function

$$v(x, y) = \mathrm{e}^{\alpha x + \beta y}\, u(x, y) \tag{6}$$

   satisfies the following PDE without first order derivatives:

$$v_{xx} - v_{yy} + h\, v = 0\,. \tag{7}$$

Good luck!