1. Assume that u satisfies a general homogeneous linear partial differential equation of second order
$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + f\,u = 0 \tag{1}$$
with *constant* coefficients a, b, c, d, e, and f. Show that the functions u_x, u_y, u_{xx}, u_{xy}, and u_{yy} are also solutions of equation (1).

2. Verify that a standing wave of the form
$$u(x,t) = A\,\sin(kx)\,\cos(\omega t) \tag{2}$$
with constant amplitude A, wavenumber k, and angular frequency ω satisfies the 1-D wave equation
$$u_{xx} - \frac{1}{c^2} u_{tt} = 0 \tag{3}$$
provided that a certain condition holds for k, ω, and the phase velocity c.
   (a) Determine that condition.
   (b) Write the function (2) in the form of the general d'Alembert solution of the wave equation considered in exercise 2(b) of week 46, i.e. find functions f and b of one variable, such that
$$u(x,t) = f(kx - \omega t) + b(kx + \omega t)\,. \tag{4}$$

3. Show that the function
$$u(x,t) = \frac{1}{\sqrt{t}}\,\mathrm{e}^{-x^2/(4kt)} \tag{5}$$
satisfies the 1-D heat equation $u_t = k\,u_{xx}$ for positive times $t > 0$. For $k = 1$, graph the solution u qualitatively at times $t = 1$ and $t = 4$.

4. Consider the Cauchy problem for the partial differential equation
$$u_{xx} - 4\,u_{xy} + y\,u_{yy} + u_x + y\,u_y = 0 \tag{6}$$
with Cauchy data $u(0,y) = y^3$ and $u_x(0,y) = 4y$ given on the y-axis.
Use the Cauchy data and the PDE to compute the first four terms of the Taylor expansion of the solution of the Cauchy problem around the y-axis. You should find the expression
$$u(x,y) = y^3 + 4y\,x + \frac{1}{2}(16 - 4y - 6y^2 - 3y^3)x^2 + \frac{1}{6}(-32 - 48y - 30y^2 + 3y^3)x^3 \pm \ldots\,. \tag{7}$$

5. Consider the Cauchy problem for the partial differential equation
$$u_{xx} - 6\,u_{yy} - 2\,u_x = 0 \tag{8}$$
with Cauchy data on the line Γ given by $y = 5x$:
$$u(x,5x) = \sin(x)\,,\quad (\boldsymbol{n}\cdot\boldsymbol{\nabla})\,u|_{(x,5x)} = \frac{1}{\sqrt{26}}(5\,u_x(x,5x) - u_y(x,5x)) = \frac{1}{\sqrt{26}}\,x^2\,. \tag{9}$$
Here $\boldsymbol{n} = \frac{1}{\sqrt{26}}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ is a normal vector perpendicular to Γ.
Use the Cauchy data and the PDE to compute a Taylor approximation of the solution around the origin up to second order, i.e. fill in the coefficients in the expansion
$$u(x,y) = u(0,0) + u_x(0,0)\,x + u_y(0,0)\,y + \frac{1}{2}\left(u_{xx}(0,0)\,x^2 + 2\,u_{xy}(0,0)\,xy + u_{yy}(0,0)\,y^2\right) \pm \ldots \tag{10}$$

Good luck!