1. Consider the Cauchy problem for the 1-D wave equation:

$$u_{tt} = c^2\, u_{xx} \;\text{ for }\; -\infty < x < \infty,\; t > 0, \tag{1}$$

$$u(x,0) = \varphi(x),\;\; u_t(x,0) = \psi(x),\;\; \text{for }\; -\infty < x < \infty. \tag{2}$$

   (a) Evaluate the d'Alembert solution for $c = 7$ and given functions $\varphi(x) = \cos(3x),\; \psi(x) = x.$

   (b) Draw (qualitative) graphs of the solution for $c = 1$ and initial values $\psi = 0$,

$$\varphi(x) = \begin{cases} \sin(2x) & \text{for } -\pi < x < \pi\,, \\ 0 & \text{for } |x| > \pi\,, \end{cases} \tag{3}$$

   at times $t = 0$, $t = \pi/2$, $t = \pi$, and $t = 2\pi$.

   (c) Write the solution for $c = 2$ and initial data $\psi = 0$, $\varphi(x) = \sin(\pi x)$, and draw (qualitative) graphs of the solution at times $t = 0$, $t = 1/4$, $t = 1/2$, $t = 3/4$, $t = 1$, $t = 5/4$, and $t = 3/2$.

2. *Duhamel's principle*:
   Assume that the functions $w(x,t;T)$ are known solutions of the homogeneous 1-D wave equation, corresponding to initial values f given at time T:

$$w_{xx} - w_{tt} = 0,\;\; \text{for }\; -\infty < x < \infty,\;\; t > T,\quad w(x,T;T) = 0,\;\; w_t(x,T;T) = -f(x;T). \tag{4}$$

   Hence we have an entire series of solutions to initial value problems, with additional parameter T, where the functions $w(x,t;T)$ are available for all parameter values $T > 0$.

   Show that the function u defined by

$$u(x,t) = \int_0^t w(x,t;T)\,\mathrm{d}T \tag{5}$$

   solves the Cauchy problem for an inhomogeneous wave equation with homogeneous initial conditions:

$$u_{xx}(x,t) - u_{tt}(x,t) = f(x,t),\;\; \text{for }\; -\infty < x < \infty,\;\; t > 0,\quad u(x,0) = 0,\;\; u_t(x,0) = 0. \tag{6}$$

   Hint: It is not necessary to write out w explicitly; just exploit the properties of the functions as stated in (4).

3. Consider an initial-boundary-value problem for the 1-D wave equation with inhomogeneous boundary condition:

$$u_{tt} = c^2\, u_{xx}, \;\; u(x,0) = \varphi(x), \;\; u_t(x,0) = \psi(x), \;\; u(0,t) = \chi(t), \;\; \text{for } \; x > 0, \; t > 0. \tag{7}$$

Here φ, ψ, and χ are given functions. We'll proceed stepwise towards a solution.

(a) For given solutions v, w of the two problems with either inhomogeneous initial conditions

$$v_{tt} = c^2\, v_{xx}, \;\; v(x,0) = \varphi(x), \;\; v_t(x,0) = \psi(x), \;\; v(0,t) = 0, \;\; \text{for } \; x > 0, \; t > 0, \tag{8}$$

or an inhomogeneous boundary condition

$$w_{tt} = c^2\, w_{xx}, \;\; w(x,0) = 0, \;\; w_t(x,0) = 0, \;\; w(0,t) = \chi(t), \;\; \text{for } \; x > 0, \; t > 0, \tag{9}$$

check that the function $u = v + w$ satisfies all constraints of problem (7).

(b) Recall the solution of problem (8), as discussed in the lecture.

(c) To establish a solution of problem (9), try a reasoning in terms of the characteristic triangle: Nonzero contributions to the solution originate only from the source at $x = 0$; in propagating along the characteristics the perturbations can only reach points (x, t) in the first quadrant of the x-t-plane with $x < ct$. Clarify this by means of an appropriate sketch.

Then choose an ansatz w of the form

$$w(x,t) = f(x - ct)\,, \tag{10}$$

and fix the (piecewise defined) function f such that the solution satisfies the initial- and boundary conditions of problem (9).

(d) Assemble the solution u of problem (7), and compare with the expression given in the textbook (section 4.5).

Good luck !